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MOD 2 SEIBERG-WITTEN INVARIANTS OF HOMOLOGY TORI

Daniel Ruberman and Sašo Strle

1. Introduction

While the Seiberg-Witten equations are defined for any Spin^c structure on a smooth 4-manifold, there is particular interest in the Seiberg-Witten equations associated to a spin structure. One reason for this is that the 4-dimensional spin representation has a quaternionic structure, which gives rise to a large symmetry group of the 'trivial' reducible solution to the Seiberg-Witten equations. This group, denoted J, is generated by U(1) and the quaternion j. The interplay of this symmetry group and the deformation theory of the trivial solution is central to Furuta's proof of the '10/8' inequality, which constrains the homotopy type of smooth spin manifolds. A closely related argument (known to Kronheimer and Furuta as well) was used by Morgan-Szabó [12] to determine the mod 2 Seiberg-Witten invariant of homotopy K3 surfaces and other simply-connected spin manifolds.

In this paper we show that the mod 2 Seiberg-Witten invariant can be determined for a spin manifold X which has the same homology *groups* as the 4-torus T^4. The value depends on the structure of the cohomology *ring* of X, and in particular on the 4-fold cup product $\Lambda^4 H^1(X) \to H^4(X)$. For the rest of the paper, X will denote a (spin) homology torus, by which we mean an oriented spin 4-manifold with $H_1(X;\mathbf{Z}) \cong \mathbf{Z}^4$ and $H_2(X;\mathbf{Z}) \cong \mathbf{Z}^6$. The cup product on $H^2(X)$ is readily seen to be hyperbolic, but the cup product on $H^1(X)$ is not determined by the dimensions of these groups. Let us define $\det(X)$, the *determinant* of X, to be the absolute value of

$$< \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4, [X] >,$$

where $\{\alpha_j\}$ is a basis for $H^1(X;\mathbf{Z})$.

Theorem A. *The value of the Seiberg-Witten invariant for the* Spin^c *structure on X with trivial determinant line is congruent (mod 2) to the determinant of X.*

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Let X be a homology torus, and let $W \to X$ be a Spin^c bundle with trivial determinant line $L \to X$. We fix a square root $L^{1/2}$ of the determinant bundle, or equivalently a spin structure on X. A trivialization of $L^{1/2}$ provides us with a preferred origin in the space $\mathcal{A}$ of $U(1)$-connections on $L^{1/2}$, namely the (smooth) product connection A_0. Note that although there are many spin structures on X, they are all isomorphic as Spin^c structures; the choice of spin structure is reflected only in the (gauge equivalence class of the) above mentioned trivialization and hence in the choice of A_0, but does not affect the argument.

Recall that the configuration space for the Seiberg-Witten equations is $\mathcal{C} = \mathcal{A} \oplus \Gamma(W^+)$; however, we restrict the equations to the slice $\mathcal{C}' = \mathcal{K} \oplus \Gamma(W^+)$ where $\mathcal{K} = \{A \in \mathcal{A} | \, d^*(A - A_0) = 0\}$. With this restriction, the moduli space $\mathcal{M}$ of solutions to the Seiberg-Witten equations is the quotient of the space of solutions by the action of the group of harmonic gauge transformations; denote by $\mathcal{G}_0$ the *based* gauge group of harmonic gauge transformations. The formal dimension of the moduli space $\mathcal{M}$ for the trivial Spin^c structure on X is zero (as is the index of Dirac operator $D_A \colon \Gamma(W^+) \to \Gamma(W^-)$ for any Spin^c connection ∇_A on W). In the absence of any perturbation terms, however, the moduli space is not cut out transversally because it contains the 'dual' 4-torus $T^* \cong H^1(X; S^1)$ of reducible solutions $[A, 0]$ with A harmonic. This dual torus is covered (under the action of $\mathcal{G}_0$) by the space of harmonic connections $\widetilde{T}^* = A_0 + i\mathcal{H}^1(X) \subset \mathcal{K}$. In the case $X = T^4$ the dual torus coincides with the moduli space; moreover, the only point along T^* where the Dirac operator has nontrivial kernel is $[A_0, 0]$. From this, and the structure of the quadratic term of the equations, one can show that the Seiberg-Witten invariant (for the trivial Spin^c structure) of T^4 is ± 1; this of course implies Theorem A for $X = T^4$. To prove it in the general case, we perturb the equations along $\widetilde{T}^*$ so that they become as nondegenerate as possible; this is done in two stages - we deal with the linear perturbations in section 2, and with nonlinear ones in section 3. The invariant is then determined by the count of solutions which lie near T^*; as in [12], we use the involution j on the moduli space of solutions, induced by taking the dual Spin^c structure, to pair off the solutions away from T^*.

The spaces of sections are L^2_2 unless stated otherwise; in particular, this holds for the configuration space $\mathcal{C}'$. The gauge transformations are in the space L^2_3.

2. Dirac operators along T^*

Recall that the Seiberg-Witten equations are given by a map $SW \colon \mathcal{K} \oplus \Gamma(W^+) \to i\Omega^2_+(X) \oplus \Gamma(W^-)$, $(A, \psi) \mapsto (F_A^+ - q(\psi), D_A\psi)$, where F_A^+ is the self-dual part of the curvature and q is a quadratic map. Thus for any $A \in \widetilde{T}^*$, the linearization of the equations at $(A, 0)$ is (d^+, D_A). Since d^+ does not depend on A, the behavior of the linearizations of the Seiberg-Witten equations along $\widetilde{T}^*$ is described by the family of Dirac operators $\{D_A, \ A \in \widetilde{T}^*\}$. We think of this family as a morphism of trivial bundles $\widetilde{T}^* \times \Gamma(W^+) \to \widetilde{T}^* \times \Gamma(W^-)$. Note that the gauge group $\mathcal{G}_0$ acts freely on the base space $\widetilde{T}^*$ of these bundles, so dividing out by the action of $\mathcal{G}_0$ produces bundles $\mathbf{\Gamma}(W^{\pm}) \to T^*$ with fibres $\Gamma(W^{\pm})$. Each of these bundles supports a free J-action which is compatible with the J-action on the base T^*; we call any such bundle a *J-bundle*. The family of Dirac operators defines a family of Fredholm operators parametrized by T^*; we denote the resulting morphism by $\boldsymbol{D}\colon \mathbf{\Gamma}(W^+) \to \mathbf{\Gamma}(W^-)$. From above we know that the pointwise index of $\boldsymbol{D}$ is 0, but the index *bundle* $Ind(\boldsymbol{D})$ of $\boldsymbol{D}$ may be nontrivial. We will see that the latter is determined by the cup product on $H^1(X)$. It is, therefore, the index computation that links the cup product structure to the behavior of the linear part of the Seiberg-Witten equations along $\widetilde{T}^*$.

The calculation of the family index of $\boldsymbol{D}$ is similar to the one arising in the proof [8, 13] of the wall-crossing formula for 4-manifolds with $b^1 > 0$. The Chern character of the index bundle is given by $\mathrm{ch}(Ind(\boldsymbol{D})) = \mathrm{ch}(\mathbf{L})/[X]$, since the $\hat{A}$-genus of X is 1. Here $\mathbf{L} \to X \times T^*$ is the universal line bundle equipped with a connection $\mathbf{A}$ as follows. Let $\alpha_1, \ldots, \alpha_4 \in \mathcal{H}^1(X;\mathbf{Z})$ be a basis and let $t_k \mapsto 2\pi i t_k \alpha_k$ be coordinates on $T^* \cong \mathcal{H}^1(X; i\mathbf{R})/\mathcal{H}^1(X; 2\pi i\mathbf{Z})$. The connection 1-form of $\mathbf{A}$ is given by

$$2\pi i \sum_k t_k \alpha_k \,.$$

By Chern-Weil theory, the first Chern class of $\mathbf{L}$ is then represented by the 2-form

$$\Omega = \sum_k \alpha_k \wedge dt_k \,,$$

and therefore

$$\mathrm{ch}(\mathbf{L}) = 1 + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{6}\Omega^3 + \frac{1}{24}\Omega^4 \,.$$

It is at this point that the cup product structure of X shows up; the formula for the Chern character of the index bundle gives

$$\mathrm{ch}(Ind(\boldsymbol{D})) = \pm r[vol_{T^*}],$$

where r denotes the determinant of X. Consequently, $c_2(Ind(\boldsymbol{D})) = \pm r$ and $c_1(Ind(\boldsymbol{D})) = 0$; this suggests that there is a simple model for the index bundle and the construction of this model occupies the rest of the section.

In the proposition below we construct a generic model for the index bundle, realized by stabilizing the domain and the range of the operators. This corresponds to a stabilization of the Seiberg-Witten equations; we will define the stabilized equations via a map

$$\overline{SW}\colon \mathcal{K} \oplus \Gamma(W^+) \oplus \mathbf{H}^n \to i\Omega^2_+(X) \oplus \Gamma(W^-) \oplus \mathbf{H}^n \,.$$

Proposition 2.1. *Suppose that* $\det(X) = r$. *Then there is a* $J \times \mathcal{G}_0$*–equivariant stabilization of the Seiberg-Witten equations, with reducible solutions along* T^**, such that the corresponding family of Dirac operators has nontrivial kernel at exactly* r *points on* T^*.

Proof. As an element of the ordinary K-theory of T^*, the index bundle may be represented as a difference of complex vector bundles. We need to represent $Ind(\boldsymbol{D})$ as the difference of two genuine J-bundles over T^*, and so adapt the standard argument to the context of J-equivariant K-theory (compare [1, 5]) as follows. By standard arguments there exists a $\mathbf{C}$-linear morphism $\mathbf{G}_0\colon T^* \times \mathbf{C}^n \to \mathbf{\Gamma}(W^-)$ which is onto the cokernel of $\boldsymbol{D}$. This morphism extends to a J-equivariant morphism $\mathbf{G}\colon T^* \times \mathbf{H}^n \to \mathbf{\Gamma}(W^-)$, where the product bundle $T^* \times \mathbf{H}^n$ has the product action of j; the j-action on the space of quaternions $\mathbf{H} = \mathbf{C} \oplus j\mathbf{C}$ is via right quaternionic multiplication. Given any $[A, 0] \in T^*$ and $w \in \mathbf{C}^n$ set

$$\mathbf{G}([A,0], jw) := j \cdot \mathbf{G}_0([j(A), 0], \overline{w}),$$

and extend by linearity. Perturbing the family of Dirac operators $\boldsymbol{D}$ by the morphism $\mathbf{G}$ produces a J-equivariant epimorphism

$$\begin{aligned} \overline{\boldsymbol{D}}\colon \mathbf{\Gamma}(W^+) \oplus T^* \times \mathbf{H}^n &\to \mathbf{\Gamma}(W^-) \\ ([A,0], \psi, w) &\mapsto \boldsymbol{D}_{[A,0]}\psi + \mathbf{G}([A,0], w)\,. \end{aligned}$$

Through this we have represented $Ind(\boldsymbol{D})$ in J-equivariant K-theory as the difference of the kernel bundle of $\overline{\boldsymbol{D}}$ and the product J-bundle $T^* \times \mathbf{H}^n$.

Considered as a complex bundle, the kernel bundle $K := \ker \overline{\boldsymbol{D}}$ splits as a sum $K = K' \oplus K''$, for dimensional reasons, where K' is a trivial complex bundle, and K'' is a $\mathbf{C}^2$-bundle with $c_2(K'') = \pm r$ and $c_1(K'') = 0$. In fact, K splits in the category of J-bundles over T^*, in such a way that K' is a trivial $\mathbf{H}^{n-1}$-bundle. To construct this splitting note that any vector bundle over T^* with fibre dimension greater than 4 (over $\mathbf{R}$) admits a nowhere vanishing section s. On any J-bundle $M \to T^*$ such a section s gives rise to a trivial J-invariant subbundle $N \to T^*$ of complex rank 2, spanned by s and $\overline{s}([A,0]) = j \cdot s([j(A), 0])$. Moreover, N has a J-invariant complement in M; the latter can be taken to be perpendicular to N with respect to some (compatible) hermitian inner product on M. For the case at hand we choose the standard hermitian structure on $T^* \times \mathbf{H}^n$ and the L^2-inner product on the fibres of $\mathbf{\Gamma}(W^+)$. Denote the resulting J-equivariant isomorphism by $\mathbf{F}'\colon K' \to T^* \times \mathbf{H}^{n-1}$.

The bundle K'' admits a structure of a quaternionic line bundle; we use this to construct a J-equivariant morphism $\mathbf{F}''\colon K'' \to T^* \times \mathbf{H}$, injective everywhere except at r chosen points on T^*. Let $\mathcal{R} \subset T^*$ be a j-invariant subset with r elements; such exists for any r since the j-action on T^* has fixed points (for example $[A_0, 0]$). We choose a section s_0 of the bundle K'' which vanishes only at the points of $\mathcal{R}$ and intersects the zero section transversely. Then the sections s_0 and $\overline{s}_0$ (defined from s_0 as above) endow K'' with a structure of a quaternionic line bundle over the complement of $\mathcal{R}$. Dividing s_0 by the square of its (quaternionic) norm produces a nowhere vanishing section s of K'' over the complement of $\mathcal{R}$ and this section s induces the required bundle morphism $\mathbf{F}''$.

Note that close to any $[A_k, 0] \in \mathcal{R}$, the norms of linear maps $\mathbf{F}''_{[A,0]}$ are bounded below by some positive constant times the distance from $[A, 0]$ to $[A_k, 0]$.

The morphisms $\mathbf{F}'$ and $\mathbf{F}''$ together define a J-equivariant morphism $\mathbf{F}\colon K \to T^* \times \mathbf{H}^n$ which is injective on all the fibres except over the points of $\mathcal{R}$ where the kernels can be identified with a copy of $\mathbf{H}$. We think of the pair $(\overline{\boldsymbol{D}}, \mathbf{F})$ as the family of Dirac operators associated to the stabilized Seiberg-Witten equations (which are defined below). Note that by construction of $\overline{\boldsymbol{D}}$ and $\mathbf{F}$, the associated family of Dirac operators has nontrivial kernels only at the points of $\mathcal{R}$, thus proving the last statement of the proposition.

To finish the construction of the stabilized equations, we need to globalize the perturbation terms $\mathbf{G}$ and $\mathbf{F}$. Let $P\colon \mathcal{K} \to \widetilde{T}^*$ be the L^2-orthogonal projection (where we treat A_0 as the origin of the above affine spaces), $Q\colon \mathcal{K} \to (\widetilde{T}^*)^\perp$ the orthogonal projection to the complement, and $\Pi\colon \mathbf{\Gamma}(W^+) \oplus T^* \times \mathbf{H}^n \to K$ the orthogonal projection to the kernel of $\overline{\boldsymbol{D}}$. The morphism $\mathbf{F}$ defines a map

$$F\colon \mathcal{K} \oplus \Gamma(W^+) \oplus \mathbf{H}^n \to \mathbf{H}^n,$$

given by $F(A,\psi,w) = pr_2 \circ \mathbf{F}(\Pi([P(A),\psi,w]))$. Similarly, $\mathbf{G}$ gives rise to

$$G\colon \mathcal{K} \oplus \Gamma(W^+) \oplus \mathbf{H}^n \to \Gamma(W^-),$$

which is well defined up to gauge change by $[P(A), G(A,\psi,w)] = \mathbf{G}([P(A),0],w)$; it is completely determined by the appropriate choice of G_{A_0}.

We define the stabilized Seiberg-Witten equations via a map

$$\overline{SW}\colon \mathcal{K} \oplus \Gamma(W^+) \oplus \mathbf{H}^n \to i\Omega^2_+(X) \oplus \Gamma(W^-) \oplus \mathbf{H}^n,$$

which is the sum of the original Seiberg-Witten map and the stabilization term given by

$$(A,\psi,w) \mapsto \beta(Q(A),\psi,w)\cdot(0, G(A,\psi,w), F(A,\psi,w)) + \\ (1-\beta(Q(A),\psi,w))\cdot(0,0,w),$$

where β depends smoothly on the L^2_2-norms of A and ψ and on the norm of w in such a way that it is equal to 1 in a small neighborhood of $(0,0,0)$ and equal to 0 in a slightly bigger neighborhood; notice that $\beta(Q(-),-,-)$ is invariant under the action of the gauge group $\mathcal{G}_0$ as well as under the action of J. It is clear from the nature of the perturbation terms that the moduli space of solutions to the stabilized Seiberg-Witten equation $\overline{SW} = 0$ still contains the torus of reducibles T^*. This proves the proposition. □

Remark. The proof of the proposition implies not only that T^* is contained in the moduli space of solutions to $\overline{SW} = 0$, but also that it is isolated, at least away from the points of $\mathcal{R}$. More precisely, for any neighborhood U of $\mathcal{R}$, the complement $T^* \setminus U$ is isolated in the moduli space. This follows from the fact that $\mathbf{F}$ is injective on the kernels of the perturbed family of Dirac operators $\overline{\boldsymbol{D}}$ (along T^*) away from $\mathcal{R}$.

3. Kuranishi maps at the points of $\mathcal{R}$

In this section we will construct a further perturbation of the stabilized Seiberg-Witten map $\overline{SW}$ whose solution space has a particularly simple form in a neighborhood of T^*, as described in the proposition below. The perturbation is supported in a neighborhood of the set $\mathcal{R} \subset T^*$ and is constructed by modifying the Kuranishi maps at the points of $\mathcal{R}$; these are the only points on T^* at which the stabilized Dirac operators have nontrivial kernels.

Proposition 3.1. *There exists a $J \times \mathcal{G}_0$-equivariant perturbation of the stabilized Seiberg-Witten equations, such that the perturbed map $\overline{\overline{SW}}$ satisfies the following:*

1. *The torus of reducibles T^* is contained and isolated in the moduli space of solutions to the perturbed equations $\overline{\overline{SW}} = 0$.*
2. *Given a small generic $\omega \in i\Omega^2_+(X)$ there exists an invariant neighborhood $\mathcal{U}$ of $\widetilde{T}^*$, such that all the solutions to $\overline{\overline{SW}} = (\omega, 0, 0)$ that lie in $\mathcal{U}$ are smooth and irreducible. More precisely, every point in $\mathcal{R}$ gives rise to a smooth circle of solutions to $\overline{\overline{SW}} = (\omega, 0, 0)$ in $\mathcal{U}$, contributing ± 1 to the invariant, and there are no other solutions in $\mathcal{U}$.*

Proof. Points of $\mathcal{R}$ fall into two categories depending on whether they are j-fixed or not. We consider the former case first, making use of the j-equivariance of the Kuranishi map. Then we modify the argument to deal with the rest of the points in $\mathcal{R}$.

Suppose $[A_k, 0] \in \mathcal{R}$ is j-fixed. The Kuranishi model for the solutions to $\overline{SW} = 0$ around $(A_k, 0, 0)$ is given by a J-equivariant map $Q\colon \mathbf{R}^4 \oplus \mathbf{H} \to \mathbf{R}^3 \oplus \mathbf{H}$, where $\mathbf{R}^4$ corresponds to the harmonic 1-forms, $\mathbf{R}^3$ to the self-dual harmonic 2-forms, and the quaternions represent the kernel and the cokernel of the perturbed Dirac operator at A_k. Note that the leading term of Q is a quadratic polynomial map which we will make non-degenerate by a perturbation. Denote by $\overline{Q}_1$, $\overline{Q}_2$ the quadratic parts of the components of Q. In principle these maps from $\mathbf{R}^4 \oplus \mathbf{H}$ can contain three sorts of terms: quadratic in the first or the second variable, or bilinear. Which terms really appear is determined by the J-equivariance. Recall that j acts on $\mathbf{H}$ by right quaternionic multiplication and on the spaces of forms by multiplication by -1, whereas $U(1)$ acts by complex multiplication on $\mathbf{H}$ and trivially on the spaces of forms. This forces $\overline{Q}_1$ to be quadratic in the second (quaternionic) variable and $\overline{Q}_2$ to be bilinear. Note that j-equivariance imposes extra restrictions on these terms; clearly $\overline{Q}_1 \circ j = -\overline{Q}_1$. The second component satisfies $\overline{Q}_2 \circ j = -j \circ \overline{Q}_2$ if we think of $\overline{Q}_2$ as a linear map $\mathbf{R}^4 \to \operatorname{End}_{\mathbf{C}}(\mathbf{H})$.

We choose a non-degenerate J-invariant quadratic map $R_k\colon \mathbf{H} \to \mathbf{R}^3$ (with the associated linear map an isomorphism, cf. [12]) to perturb $\overline{Q}_1$. For all but finitely many τ, the map $\overline{Q}_1 + \tau R_k$ is non-degenerate in the above sense. Admissible perturbations of $\overline{Q}_2$ are of the form $(a, w) \mapsto L(a)w$, where $L(a)$ is a $\mathbf{C}$-linear map which anti-commutes with the j-action. The space $\mathcal{I}$ of such maps is 4-dimensional over $\mathbf{R}$ and its non-zero elements are isomorphisms. We choose the map $L_k\colon \mathbf{R}^4 \to \mathcal{I}$, $a \mapsto L_k(a)$ to be an isomorphism. Then for almost all τ

the map $\overline{Q}_2 + \tau L_k$, where we interpret $\overline{Q}_2$ as a linear map $\mathbf{R}^4 \to \mathcal{I}$, is an isomorphism. Notice that $\overline{Q}_2(a, -)$ is itself an isomorphism for $a \neq 0$; this follows from the construction of the linear perturbation $\mathbf{F}$. Moreover, the norms of these linear maps are bounded from below by $C||a||$ for some positive C. This means that we can choose τ small enough so that for $a \neq 0$ the perturbation term is dominated by the original (quadratic) map. The benefits of this perturbation are twofold; firstly, the only solutions to the perturbed equations close to $(A_k, 0, 0)$ are the reducible ones. Secondly, for a generic $h \in \mathcal{R}^3$, the preimage of $(h, 0, 0)$ under the perturbed Kuranishi map consists of exactly one circle of solutions, hence the point $(A_k, 0, 0)$ contributes ± 1 to the Seiberg-Witten invariant.

Consider now a point $(A_k, 0, 0)$ with $[A_k, 0] \in \mathcal{R}$ not j-fixed. Such a point has its j-image in $\mathcal{R}$; to make the perturbation term j-equivariant in this case, we construct a $U(1)$-equivariant perturbation at $(A_k, 0, 0)$ and use the j-action to define the perturbation at its j-image. Given only $U(1)$-equivariance for $\overline{Q}_1$ and $\overline{Q}_2$ in this case, the structure of these quadratic maps is not so restricted. Using additional properties of the Kuranishi map, we still conclude that $\overline{Q}_1$ is quadratic in the second variable and $\overline{Q}_2$ is bilinear. However, the space of $U(1)$-invariant quadratic polynomials $\mathbf{H} \to \mathbf{R}$ is four dimensional and $\overline{Q}_2(a, -)$ can be any $\mathbf{C}$-linear map, so there is no canonical choice of a good perturbation. To gain the same control over the solution space as for j-fixed points, we endow the kernel and the cokernel with a quaternionic structure. The perturbation terms can then be constructed as above, using right multiplication by the quaternion j in place of the j-action. For the perturbation term $R_k \colon \mathbf{H} \to \mathbf{R}^3$, the associated linear map is surjective and for all but finitely many τ, the map $\overline{Q}_1 + \tau R_k$ is an epimorphism in the above sense. The perturbation of the second component gives rise to an injective map $a \mapsto L_k(a)$; again, for all but finitely many τ the map $a \mapsto \overline{Q}_2(a, -) + \tau L_k(a)$ is a monomorphism. The remark about domination of the pertubation term $\tau L_k(a)$ by $\overline{Q}_2(a, -)$ holds as above, and so do the conclusions about the solution space.

We fix a small, generic τ and define the perturbation term as a sum of terms localized near the points of $\mathcal{R}$. For a point $[A_k, 0] \in \mathcal{R}$ define the perturbing map by

$$(A, \psi, w) \mapsto \tau\beta_k(A, \psi, w) \cdot (R_k(\Pi_k(\psi, w)), L_k(P(A))\Pi_k(\psi, w), 0),$$

where $\Pi_k \colon \Gamma(W^+) \oplus \mathbf{H}^n \to K''_{A_k} = \mathbf{H}$ is the L^2-orthogonal projection and β_k is a $[0, 1]$-valued function depending smoothly on the norms of the arguments (using $A_k \equiv 0$), that has support inside a small neighborhood of $(A_k, 0, 0)$ (the projection of which by (P, Π_k) is contained in the domain of the Kuranishi map) and is equal to 1 on a smaller neighborhood. The moduli space of solutions to the perturbed equations $\overline{\overline{SW}} = 0$ still contains the torus of reducibles T^* and it is clear from above that this torus is isolated. $\square$

4. Completion of the argument

First we observe, following the line of argument in [12], that the moduli space of solutions to the perturbed equations $\overline{\overline{SW}} = 0$ is compact. Moreover, because the perturbed equations we use can be connected to the unperturbed equations by a 1-parameter family, the count of solutions we obtain coincides with the Seiberg-Witten invariant. In the complement of $\widetilde{T}^*$, the action of J is free, and so we can choose a small J-equivariant perturbation with support away from $\widetilde{T}^*$ such that the corresponding moduli space is smooth away from $\widetilde{T}^*$. (This fits into the general scheme laid down in §4.3.6 of [2] because the perturbation is simply a small Fredholm section of a bundle over $(\mathcal{C}' - \widetilde{T}^*)/J$, pulled back to $\mathcal{C}'$.) Because j acts freely, the solutions in the complement of $\widetilde{T}^*$ are paired up, and this part of the moduli space contributes an even number to the Seiberg-Witten invariant.

Along the space of reducible solutions we proceed by choosing a small generic self-dual 2-form ω which has a nonzero harmonic projection. If ω is small enough, the solutions to $\overline{\overline{SW}} = (\omega, 0, 0)$ in an invariant neighborhood $\mathcal{U}$ of $\widetilde{T}^*$ are described as follows. For every point in $\mathcal{R}$, there is a circle of solutions corresponding to the $U(1)$ orbit. There are r such circles, each of which contributes ± 1 to the invariant. All the rest of the solutions are paired by the j action, hence the statement of the theorem follows.

5. Some homology tori

There are a number of examples of homology tori whose Seiberg-Witten invariants one can compute directly; it is interesting to see how these are consistent with our theorem. The simplest are the torus T^4, whose Seiberg-Witten invariant is ± 1, and the connected sum

$$\#_4 S^1 \times S^3 \# \#_3 S^2 \times S^2,$$

whose Seiberg-Witten invariant vanishes. These manifolds have determinant 1 and 0, respectively.

A more interesting class of examples is the set of manifolds of the form $X = S^1 \times M^3$, where M is an orientable 3-manifold with the homology of a torus. Work of Meng and Taubes shows how to compute the invariant of X, in terms of the Alexander polynomial of M. There are two parts to the computation. First, there is an identification of the Seiberg-Witten invariant of X with the 3-dimensional Seiberg-Witten invariant of M^3. This is proved by a variant of the argument proving proposition 5.1 of [11]. In particular, the Spin^c structures on X with non-vanishing Seiberg-Witten invariant all pull back from M. The main theorem of [10] shows that the Seiberg-Witten invariant of M (and therefore of X) has for generating function the multivariable Alexander polynomial of M. In light of Theorem A, we explain how the determinant of X is related to the Alexander polynomial of M.

We define the determinant $\det(M)$ analogously to that of X, using the 3-fold cup product in $H^1(M)$. Note that the determinant of $S^1 \times M$ coincides with that of M. The Alexander polynomial of M, Δ_M, is a Laurent polynomial in variables $t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}$ which is defined up to multiplication by $\pm t_i$. The relation we need is the following:

Lemma 5.1. *If M is a homology torus, then*

$$\Delta_M(1,1,1) = \pm \det(M)^2.$$

The Lemma may be deduced from work of L. Traldi [14] and J. Levine [6]. Those authors treat the Alexander polynomial Δ_L of an n-component link L in a homology sphere; in our situation the homology sphere is obtained by doing surgery on a set of circles representing a basis of $H_1(M)$ and L consists of the meridians of those circles. If the linking numbers between the components are all 0, as is the case for us, they show that

$$\frac{\Delta_L}{(t_1 - 1)\cdots(t_n - 1)} = d_0 + \text{higher order terms in } t_i - 1, \tag{1}$$

where d_0 may be evaluated as a determinant involving the $\overline{\mu}$-invariants of L. (Compare [6, Corollary 1.6] and the proof of [14, Theorem 5.3].) When there are only 3 components, the determinant works out to be $\overline{\mu}_{123}(L)^2$. Now the quotient on the left-hand side of equation (1) is the Alexander polynomial of M, and it has been known for a long time [9] that the invariant $\overline{\mu}_{123}(L)$ coincides with the 3-fold Massey product.

In terms of Seiberg-Witten theory, the evaluation $\Delta_M(1,1,1)$ is the sum of the Seiberg-Witten invariants of all of the Spin^c structures on M. Recall that there is an involution on the set of Spin^c structures, whose only fixed point is the Spin^c structure S_0 with trivial determinant, i.e. the one we have been studying. Hence we have the chain of equalities and congruences

$$\mathrm{SW}_X(S_0) = \mathrm{SW}_M(S_0) \equiv \Delta_M(1,1,1) = \det(M)^2 \equiv \det(M) \pmod 2,$$

which is consistent with our main theorem since $\det(M) = \det(X)$.

It is not hard to find 3-manifolds with arbitrary determinant $\det(M)$; a simple construction is to take 0-framed surgery on the n-fold band sum of the Borromean rings. The case $n = 2$ is illustrated below in Figure 1. If each copy of the Borromean rings is oriented so that the triple Massey product is $+1$, then $\det(M) = n$.

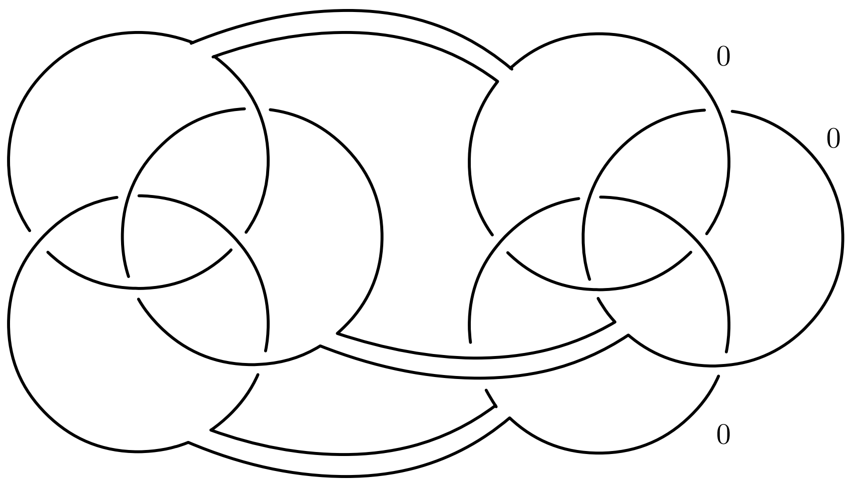

Figure 1

Remark. The calculations assembled above give rise to a curious criterion for a homology torus X^4 to be diffeomorphic to the product of S^1 and a 3-manifold. Namely, the sum of its Seiberg-Witten invariants should be a square (up to sign). It would be of interest to find an example where this criterion does not hold, but where X is homeomorphic (or perhaps homotopy equivalent) to a product.

One last class of examples is obtained via the 'knot-surgery' construction of Fintushel and Stern. Following [3], let K be a knot in S^3, with exterior E_K. Remove a copy of $T^2 \times D^2$ from T^4, and glue in $S^1 \times E_K$, resulting in a new manifold X_K with the same cohomology as T^4. It is not hard to see that X_K is in fact $S^1 \times M$, where M is gotten by replacing a copy of $S^1 \times D^2 \subset T^3$ with E_K. From this, or from gluing theorems (cf. [3, Theorem 1.5]), it follows that the Seiberg-Witten invariant of X_K is $\Delta_K(T^2)$. To make a manifold which is not a product, perform this construction on three disjoint tori T_1, T_2, T_3 (using knots K_1, K_2, K_3) in different (non-zero) homology classes, as in [4], to get a manifold X_{K_1,K_2,K_3}. Suppose that the knot-surgery is performed so that the circle factor in each $S^1 \times E_{K_i}$ is glued to the same circle factor in T^4. The result is a product of S^1 with the manifold obtained by 0-surgery on the Borromean rings with the knots K_i tied in the three rings. If the circle factors in each $S^1 \times E_{K_i}$ are glued to different circles in T^4 (and the knots are non-trivial) then X_{K_1,K_2,K_3} cannot be written as S^1 times any 3-manifold. This is verified by a fundamental group calculation; on the other hand the Seiberg-Witten invariants are independent of the gluing and are given by

$$\Delta_{K_1}(T_1^2)\,\Delta_{K_2}(T_2^2)\,\Delta_{K_3}(T_3^2).$$

References

[1] M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. **17** (1966), 367–386.
[2] S. Donaldson and P. Kronheimer, *The Geometry of Four-Manifolds*, Clarendon Press, Oxford, 1990.

[3] R. Fintushel and R. Stern, *Knots, links, and* 4*-manifolds*, Invent. Math. **134** (1998), 363–400.
[4] R. Gompf and T. Mrowka, *Irreducible 4-manifolds need not be complex*, Ann. of Math. **138** (1993), 61–111.
[5] K. Iriye, M. Mimura, K. Shimakawa, and M. Yasuo, *A quaternionic analogue of Atiyah's real K-theory*, Transformation group theory (Taejŏn, 1996), Korea Adv. Inst. Sci. Tech., Taejŏn, 51–61.
[6] J. Levine, *A factorization of the Conway polynomial*, Comment. Math. Helv., **74** (1999), 27–53.
[7] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), 119–174.
[8] T. J. Li and A. Liu, *General wall crossing formula*, Math. Res. Lett. **2** (1995), 797–810.
[9] W. S. Massey, *Higher order linking numbers*, J. Knot Theory Ramifications **7** (1998), 393–414.
[10] G. Meng and C. H. Taubes, $\underline{\mathrm{SW}}$ = *Milnor torsion*, Math. Res. Lett. **3** (1996), 661–674.
[11] J. Morgan, Z. Szabó, and C. Taubes, *A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture*, J. Differential Geom. **44** (1996), 706–788.
[12] J. W. Morgan and Z. Szabó, *Homotopy $K3$ surfaces and mod 2 Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), 17–21.
[13] D. Salamon, *Spin Geometry and Seiberg–Witten Invariants*, 1996. Birkhauser Verlag, to appear (2000).
[14] L. Traldi, *Milnor's invariants and the completions of link modules*, Trans. Amer. Math. Soc. **284** (1984), 401–424.

Department of Mathematics, MS 050, Brandeis University, Waltham MA 02454.
E-mail address: ruberman@brandeis.edu
E-mail address: strle@brandeis.edu