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HOMOLOGY 3-SPHERES BOUNDING ACYCLIC 4-MANIFOLDS

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Abstract. Let $\Sigma(a_1, a_2, \ldots, a_n)$ be a Seifert fibered homology 3-sphere with a_1 even. We show that if $\mu(\Sigma(a_1, a_2, \ldots, a_n)) = 1 \bmod 2$, then the class of $\Sigma(a_1, a_2, \ldots, a_n)$ has infinite order in the homology cobordism group of homology 3-spheres. In the proof we use Seiberg-Witten's monopole equation on four-dimensional V-manifolds.

1. Introduction

Let Θ^3_H be the homology cobordims group of oriented homology 3-spheres. Then a class $[\Sigma] \in \Theta^3_H$ has infinite order if and only if the connected sum of any number of copies of Σ cannot be the boundary of any acyclic 4-manifold. Our main theorem is:

Theorem 1. *Let $\Sigma(a_1, a_2, \ldots, a_n)$ be a Seifert fibered homology 3-sphere. We assume that one of the a_i's is even. If $\mu(\Sigma(a_1, a_2, \ldots, a_n)) = 1 \bmod 2$, then the homology cobordism class $[\Sigma(a_1, a_2, \ldots, a_n)] \in \Theta^3_H$ has infinite order.*

R. Stern [14], S. Akbulut and R. Kirby [1], and A. J. Casson and J. L. Harer [2] constructed examples of Seifert fibered homology 3-spheres whose classes in Θ^3_H are zero.

The μ-invariant gives a surjective homomorphism $\mu : \Theta^3_H \to \mathbf{Z}_2$. It implies that if $\mu(\Sigma) = 1 \bmod 2$, then the order of the class $[\Sigma]$ is even or infinite. R. Fintushel and R. Stern [3] used the Donaldson theory on V-manifolds to obtain a sufficient condition for $[\Sigma(a_1, a_2, \ldots, a_n)]$ to have infinite order. Froyshov [5] constructed homomorphisms from Θ^3_H to $\mathbf{Z}$ by using the Donaldson-Floer theory. N. Saveliev [12] also obtained such a sufficient condition by making use of a theorem proved in [8] by using Seiberg-Witten theory.

In this note we use the Seiberg-Witten theory on V-manifolds to investigate Seifert fibered homology 3-spheres.

We will define an integer $w(a_1, a_2, \ldots, a_n; m)$ for pairwise-coprime positive integers $a_1, a_2, \ldots, a_n$ and for each integer m, by making use of index formula of some elliptic operators on 4-V-manifolds.

The main theorem above follows from:

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Theorem 2. 1. *If the class* $[\Sigma(a_1, a_2, \ldots, a_n)]$ *has finite order, then for any* m *we have*

$$w(a_1, a_2, \ldots, a_n; m) \leq 0.$$

The same conclusion holds when the connected sum of some number of copies of $\Sigma(a_1, a_2, \ldots, a_n)$ *is the boundary of some positive definite 4-manifold.*

2. *Suppose one of the* a_i*'s is even. If the class* $[\Sigma(a_1, a_2, \ldots, a_n)]$ *has finite order, then*

$$w(a_1, a_2, \ldots, a_n; s) = 0, \quad \textit{for} \quad s = \frac{1}{2}\left\{\alpha\left(2 - \sum_{i=1}^{n}(1 - \frac{1}{a_i})\right) - 1\right\}.$$

When one of the a_i's is even, the integer $w(a_1, a_2, \ldots, a_n; s)$ turns out to be an integral lift of the μ-invariant of $\Sigma(a_1, a_2, \ldots, a_n)$.

The Casson invariant is an integral lift of the μ-invariant. The above integer, however, is not equal to the Casson invariant in general.

In Section 2 we consider almost definite closed 4-V-manifolds by using monopole equation. In Section 3 we consider 4-V-manifolds with boundaries by using the results of the previous section. In Section 4, we define some classes $\mathcal{S}(k^+, k^-)$ of oriented homology 3-spheres for non-negative integers k^+ and k^-. The integral lift of μ-invariant is constructed on $\mathcal{S}(k^+, k^-)$ when k^+ and k_- satisfy $k^+ + k^- \leq 2$. In Section 5 we show that the Seifert fibered homology 3-sphere $\Sigma(a_1, a_2, \ldots, a_n)$ is in the class $\mathcal{S}(0, 1)$, if one of a_i's is even. We also give some examples.

2. Closed V-4-manifolds

We refer the reader to [11] for general properties of V-manifold. Let X be an oriented closed V-4-manifold. Let $b_i(X)$ be the i-th Betti number of X and $b_2^+(X)$ (and $b_2^-(X)$) the dimension of maximal positive- (and negative-, respectively) definite subspace of $H^2(X, \mathbf{R})$.

We assume $b_1(X) = 0$. Suppose a V-spin^c-structure on X is given and fixed.

Fix a V-Riemannian metric on X. Let S^+, S^- be the associated positive and negative spinor bundles, and L the associated V-line bundle $\wedge^2 S^+ = \wedge^2 S^-$. From now on we omit the notation "V" in our terminology, since everything is defined in the category of V-manifolds.

Let $D_{\mathrm{ASD}}(X)$ be the elliptic operator defined to be:

$$D_{\mathrm{ASD}}(X) = d + d^* : \Omega^0(X) \oplus \Omega^+(X) \to \Omega^1(X),$$

where $\Omega^+(X)$ is the space of self-dual 2-forms. For a fixed connection on L, we can define the Dirac operator

$$D_{\mathrm{Dirac}}(X) : \Gamma(S^+) \to \Gamma(S^-)$$

associated with the spin^c-structure.

We denote by $\operatorname{ind} D_{\mathrm{ASD}}(X)$ and $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ the indices of these operators. Here the index is the difference between the real dimension of the kernel and that of the cokernel of the operator. Since the kernel and the cokernel of $D_{\mathrm{Dirac}}(X)$ have complex structures, $\operatorname{ind} D_{\mathrm{Dirac}}$ is even. These indices can be calculated by using the index theorem for V-manifold due to T. Kawasaki [10]. The index of D_{ASD} is, however, easily calculated by using harmonic V-forms and Hodge theory for V-manifold. It is straightforward to extend the usual de Rham theorem to the V-manifolds. This V-manifold version of de Rham theorem implies the following lemma.

Lemma 1. $\operatorname{ind} D_{\mathrm{ASD}}(X) = 1 + b_2^+(X)$.

Here we used our assumption $b_1(X) = 0$.

The formal dimension of the monopole moduli space for the spin^c-structure is equal to $\operatorname{ind} D_{\mathrm{Dirac}}(X) - \operatorname{ind} D_{\mathrm{ASD}}(X)$. P. Kronheimer pointed out that for negative definite closed 4-manifolds the formal dimension must be negative. The argument can be applied even for non-compact 4-manifolds as long as the monopole moduli space is compact [4]. We use V-manifold version of this statement. Since X is a closed V-manifold, the moduli space is compact. It implies:

Theorem 3. *If $b_2^+(X) = 0$, then we have* $\operatorname{ind} D_{\mathrm{Dirac}} \leq 0$.

The following theorem is proved in [8] for smooth closed spin 4-manifolds. This proof can immediately be extended to closed spin V-4-manifolds as well.

Theorem 4. *Suppose the spin^c-structure is induced by a spin structure. Then* $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ *is divisible by* 4. *If* $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ *is not zero, then we have*

$$1 + \frac{1}{2}\operatorname{ind} D_{\mathrm{Dirac}}(X) \leq b_2^+.$$

The right-hand side is equal to $\operatorname{ind} D_{\mathrm{ASD}}(X) - 1$ from Lemma 1. In the proof of the above theorem we use this equality.

When the spin^c-structure is induced by a spin structure, the kernel and the cokernel of D_{Dirac} have the structure of vector space over quaternions, and hence $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ is divisible by 4.

Let $-X$ denote X with reversed orientation. The spin structure of X canonically induces a spin structure of $-X$. The corresponding Dirac operator is the formal adjoint of the original one. Hence we have $\operatorname{ind} D_{\mathrm{Dirac}}(-X) = -\operatorname{ind} D_{\mathrm{Dirac}}(X)$ and, under the assumption of the above theorem, we obtain:

Corollary 1. *Suppose the spin^c-structure is induced by a spin structure.*

1. *If* $\operatorname{ind} D_{\mathrm{Dirac}}(X)$ *is not zero, then we have*

$$1 - \frac{1}{2}\operatorname{ind} D_{\mathrm{Dirac}}(X) \leq b_2^-.$$

2. *If $b_2^+(X), b_2^-(X) \leq 2$, then we have* $\operatorname{ind} D_{\mathrm{Dirac}} = 0$.

In particular, if X is a spin 4-V-manifold with $b_2 \leq 2$, then $\operatorname{ind} D_{\mathrm{Dirac}}$ must be zero for the spin structure.

3. V-4-manifolds with boundaries

In this section, suppose X is a compact oriented V-4-manifold with boundary Σ. We assume that the singularities of X are contained in its interior. Suppose Σ is a homology 3-sphere and X has a spin^c-structure c.

Let Y be a spin 4-manifold with boundary $-\Sigma$. Since $H^2(\Sigma, \mathbf{Z}) = 0$, the spin^c-structure on $\Sigma \times (0,1)$ is unique, and we have a spin^c-structure on $X \cup_\Sigma Y$ by patching the spin^c-structures on X and Y. Moreover, since $H^1(\Sigma, \mathbf{Z}) = 0$, the patching is unique up to homotopy.

Definition 1. *$w(\Sigma, X, c)$ is defined by:*

$$w(\Sigma, X, c) := \frac{1}{2} D_{\mathrm{Dirac}}(X \cup_\Sigma Y) + \frac{1}{8}(b_2^+(Y) - b_2^-(Y)).$$

Note that each term of the right-hand side is an integer.

Lemma 2. *The integer $w(\Sigma, X, c)$ is independent of the choice of Y and its spin structure.*

Proof. Since this kind of well-definedness is standard, we only give a sketch of the proof. Consider the elliptic operator $4D_{\mathrm{Dirac}} \oplus D_{\mathrm{sign}}$, where D_{sign} is the signature operator on $X \cup Y$, and the coefficient "4" indicates the direct sum of the four copies of D_{Dirac}. The Atiyah-Singer index theorem implies that the index of this operator vanishes for closed smooth 4-manifolds. Let Y' be another spin manifold with boundary $-\Sigma$. From an excision argument of the indices of the elliptic operators, we can show that the index for $X \cup_\Sigma Y$ is the sum of the indices of the operators for $X \cup_\Sigma Y'$ and $(-Y') \cup_\Sigma Y$. (One explicit way to show this equality is to use Kawasaki's index theorem for V-manifolds.) This implies that the indices for $X \cup_\Sigma Y$ and $X \cup_\Sigma Y'$ are the same, from which the lemma immediately follows. □

The same excision or localization argument also implies the following additivity.

Lemma 3. $w(\Sigma_0 \sharp \Sigma_1, X_0 \natural X_1, c_0 \natural c_1) = w(\Sigma_0, X_0, c_0) + w(\Sigma_1, X_1, c_1)$.

Here $\Sigma_0 \sharp \Sigma_1$ is the connected sum of Σ_0 and Σ_1, and $X_0 \natural X_1$ is the boundary connected sum of X_0 and X_1.

For $i = 1, 2, \ldots, m$, let X_i be a compact oriented V-4-manifold with boundary Σ_i such that its singularities are contained in its interiors. Suppose Σ_i is a homology 3-sphere and X_i has a spin^c-structure c_i for each i.

From Theorem 3 and Lemma 3 we obtain:

Theorem 5. *Suppose $X_1, X_2, \ldots, X_m$ are acyclic or negative definite. If the connected sum $\Sigma_1 \sharp \Sigma_2 \sharp \ldots \sharp \Sigma_m$ is the boundary of an acyclic 4-manifold, then we have*

$$\sum_{i=1}^{m} w(\Sigma_i, X_i, c_i) \leq 0$$

Proof. Let Y be the acyclic 4-manifold. Apply Theorem 3 to

$$(X_1 \natural X_2 \natural \ldots \natural X_m) \cup_{\Sigma_1 \sharp \Sigma_2 \sharp \ldots \sharp \Sigma_m} (-Y).$$

□

Before considering spin structure we note the following lemma.

Lemma 4. *If c is a spin structure of X, then we have*

$$\mu(\Sigma) = w(\Sigma, X, c) \bmod 2.$$

In particular, if Σ is the boundary of an acyclic 4-manifold, then $w(\Sigma, X, c)$ must be even.

Proof. Since $\mu(\Sigma) = (b_2^+(Y) - b_2^+(Y))/8 \bmod 2$, it suffices to show that the index of D_{Dirac} is divisible by 4. This follows from the fact that the kernel and the cokernel of the Dirac operator has the structure of vector bundle over quaternions when the spin^c-structure is indeced by a spin structure. □

From Theorem 4 and Lemma 3, by using a similar argument as in the proof of Theorem 5, we obtain:

Theorem 6. *Suppose all the c_i's are spin structures, and the connected sum $\Sigma_1 \sharp \Sigma_2 \sharp \ldots \sharp \Sigma_m$ is the boundary of an acyclic 4-manifold.*

1. *The sum $\sum_{i=1}^m w(\Sigma_i, X_i, c_i)$ is even.*
2. *If the above sum is not zero, then we have*

$$1 - \sum_{i=1}^m b_2^-(X_i) \leq \sum_{i=1}^m w(\Sigma_i, X_i, c_i) \leq -1 + \sum_{i=1}^m b_2^+(X_i).$$

Corollary 2. *Suppose c is a spin structure.*

1. *Suppose $b_2^+(X) \leq 2$, $b_2^-(X) \leq 2$ and $w(\Sigma, X, c) \neq 0$. Then the order of $[\Sigma] \in \Theta_H^3$ is even or infinite.*
2. *Suppose $b_2^+(X) \leq 1$, $b_2^-(X) \leq 1$ and $w(\Sigma, X, c) \neq 0$. Then the order of $[\Sigma] \in \Theta_H^3$ is infinite,*

4. An invariant for homology 3-spheres

We introduce the following notation.

Definition 2. 1. Let $\mathcal{X}$ be the set of isomorphism classes of the triples (Σ, X, c) that satisfy the following conditions.
   (a) X is a compact oriented spin V-4-manifold such that its singularities lies in its interior.
   (b) Σ is the boundary of X. We assume that Σ is a homology 3-sphere.
   (c) c is a spin structure of X.

2. $\mathcal{X}(k^+, k^-)$ is the subset of $\mathcal{X}$ given by

$$\mathcal{X}(k^+, k^-) := \{(\Sigma, X, c) \in \mathcal{X} \mid b_2^+(X) \leq k^+, b_2^-(X) \leq k^-\}.$$

3. $\mathcal{S}(k^+, k^-)$ is the set of the isomorphism classes of oriented homology 3-sphere Σ such that (Σ, X, c) is contained in $\mathcal{X}(k^+, k^-)$ for some X and c.

Note that the operation of connected sum induces the natural map:

$$\mathcal{S}(k_0^+, k_0^-) \times \mathcal{S}(k_1^+, k_1^-) \to \mathcal{S}(k_0^+ + k_1^+, k_0^- + k_1^-).$$

The next theorem is a corollary of Lemma 4 and Corollary 2.

Theorem 7. *Suppose Σ is in $\mathcal{S}(1,1)$. If $\mu(\Sigma) = 1 \bmod 2$, then the order of $[\Sigma] \in \Theta_H^3$ is infinite.*

Now we define an integral lift of μ-invariant on $\mathcal{S}(2,0)$, $\mathcal{S}(1,1)$ and $\mathcal{S}(0,2)$. The invariant w is defined on $\mathcal{X}$.

Theorem 8. *Suppose k^+ and k^- satisfy $k^+ + k^- \leq 2$. Let (Σ_0, X_0, c_0) and (Σ_1, X_1, c_1) be two triples in $\mathcal{X}(k^+, k^-)$. Then we have*

$$w(\Sigma_0, X_0, c_0) = w(\Sigma_1, X_1, c_1).$$

Proof. Recall that Σ_0 and Σ_1 are homology cobordant to each other if and only if the connected sum of Σ_0 and $-\Sigma_1$ is the boundary of an acyclic 4-manifold. Since $X_0 \natural - X_1$ satisfies the assumption of Corollary 2, we obtain

$$w(\Sigma_0, X_0, c_0) + w(-\Sigma_1, -X_1, -c_1) = 0.$$

In particular we have

$$w(\Sigma_1, X_1, c_1) + w(-\Sigma_1, -X_1, -c_1) = 0.$$

These two equality implies the required one. □

The above theorem implies that the map w descends to a map from $\mathcal{S}(k^+, k^-)$ to $\mathbf{Z}$ for k^+ and k^- satisfying $k^+ + k^- \leq 2$. We denote this map by $w(k^+, k^-)$:

$$w(k^+, k^-) : \mathcal{S}(k^+, k^-) \to \mathbf{Z}.$$

Then the above theorem also implies:

Theorem 9. *Suppose k^+ and k^- satisfy $k^+ + k^- \leq 2$. Then the map $w(k^+, k^-)$ is a homology cobordism invariant.*

Remark. The authors do not know if the invariant is a homology cobordism invariant on the union of $\mathcal{S}(2,0)$, $\mathcal{S}(1,1)$ and $\mathcal{S}(0,2)$.

The main theorem in this section is the following theorem, which is just a consequence of Lemma 4 and Corrollary 2

Theorem 10. *Let Σ be an oriented homology 3-sphere in $\mathcal{S}(k^+, k^-)$.*

1. *Suppose $k^+ + k^- \leq 2$. Then the μ-invariant $\mu(\Sigma)$ is given by:*

$$\mu(\Sigma) = w(k^+, k^-)(\Sigma) \bmod 2.$$

2. *Suppose $k^+ + k^- \leq 2$ and $w(k^+, k^-)(\Sigma) \neq 0$. Then the order of $[\Sigma] \in \Theta_H^3$ is even or infinite.*
3. *Suppose $k^+ + k^- \leq 1$ and $w(k^+, k^-)(\Sigma) \neq 0$. Then the order of $[\Sigma] \in \Theta_H^3$ is infinite.*

5. Seifert fibered homology 3-spheres

In this section we give some examples of Seifert fibered homology 3-spheres, which are classified as follows.

Let Z be a V-Riemann surface whose underlying space is $\mathbf{CP}^1$ with n marked points for $n \geq 3$. Let $a_1, a_2, \dots, a_n$ be the order of the cyclic isotropy groups at the singular points. Assume that a_i's are pairwise coprime to each other. Then the abelian group consisting of all the isomorphism classes of V-line bundles is an infinite order cyclic group [9]. The first Chern class of one of the two generators is equal to $1/\alpha$ for $\alpha = a_1 a_2 \dots a_n$. Let L_0 be a V-line bundle with $c_1(L_0) = -1/\alpha$. Then other V-line bundles are of the form L_0^k for some integer k.

We consider the disk bundle $X = D(L_0)$ as an oriented V-4-manifold. In general, when the isotropy representation at each singular point of Z acts freely on the fiber of a V-line bundle L except for the origin, the total space of the associated V-circle bundle $S(L)$ has natural smooth structure. The V-line bundle L_0 satisfies this condition. Hence we have a smooth oriented 3-manifold $S(L_0) = \partial X$. We write $\Sigma(a_1, a_2, \dots, a_n)$ for this oriented 3-manifold. It is easy to check that this is a homology 3-sphere, and that every oriented Seifert fibered homology 3-sphere is of the form $\pm\Sigma(a_1, a_2, \dots, a_n)$. See [9] for details. The canonical line bundle of the total space of L_0 is isomorphic to the dual of the pull-back of $L_0 \otimes TZ$. Note that TZ is isomorphic to L_0^k for some k. Then the canonical line bundle is isomorphic to the pullback of $L_0^{-(k+1)}$.

The total space of L_0 has a canonical spin^c-structure induced from its complex structure. As an open subset of L_0, we have a canonical spin^c-structure of X.

Theorem 11. *The V-4-manifold X has a spin structure if and only if one of the a_i's is even.*

Proof. Let G_n be the subgroup of $U(n) \times U(1)$ consisting of (g, z) satisfying $\det g = z^2$. Then G_n is isomorphic to the fiber product of $U(n) \to SO(2n)$ and $Spin(2n) \to SO(2n)$. It implies that a complex manifold has a spin structure if and only if its canonical line bundle is a square of a line bundle. This argument holds for complex V-manifolds as well. Since the canonical line bundle of the total space of L_0 is isomorphic to the pullback of $L_0^{-(k+1)}$, the total space is spin if and only if k is odd. We can calculate k by looking at the Euler characteristic number of Z [9]:

$$k = \frac{\chi(Z)}{-1/\alpha} = \alpha \left(-2 + \sum_{i=1}^{n} (1 - \frac{1}{a_i}) \right).$$

Since the right-hand side is odd if and only if one of the a_i's is even, the theorem follows. □

Corollary 3. *If one of the a_i's is even, then $\Sigma(a_1, a_2, \dots, a_n)$ is in $\mathcal{S}(0, 1)$.*

Let T^* denote the complex cotangent bundle. The Dirac operator for the canonical spin^c-structure is identified with the operator

$$\bar{\partial} + \bar{\partial}^* : \Gamma(\wedge^{0,0}T^*) \oplus \Gamma(\wedge^{0,2}T^*) \to \Gamma(\wedge^{0,1}T^*).$$

The determinant of the spinor bundle for the canonical spin^c-structure is isomorphic to $\wedge^2(\wedge^{0,1}T^*) \cong \wedge^2 T$, i.e., the dual of the canonical line bundle. We have to twist that spinor bundle by a line bundle to obtain the spinor bundle for the spin structure, Since the determinant of the spinor bundle for spin structure is trivial, the line bundle we have to use is the square root $L_0^{-(k+1)/2}$ of the canonical line bundle.

Let $c(m)$ be the spin^c-structure defined by the twisting of the canonical spin^c-structure by L_0^m. Then the spin structure is given by $c((k+1)/2)$.

Now the invariant $w(\Sigma(a_1, a_2, \dots, a_n), X, c(m))$ is calculated by using V-version of the Atiyah-Singer index theorem [10]. The details of the arguments and also a comparison with other invariants of plumbing 3-manifolds are given in [6]. In this note we only state the result of calculation.

Theorem 12.

$$\begin{aligned} w(\Sigma(a_1, a_2, \dots, a_n), X, c(m)) = \\ \frac{1}{8}\Bigg[-\frac{1}{\alpha}\left\{ \alpha\left(-2 + \sum_{i=1}^{n}(1 - \frac{1}{a_i})\right) + 1 + 2m \right\}^2 \\ + 1 - \sum_{i=1}^{n} \frac{1}{a_i} \sum_{l=1}^{a_i-1} \Bigg\{ \cot\left(\frac{\pi l}{a_i}\right) \cot\left(\frac{\pi b_i l}{a_i}\right) \\ + \cos\left(\frac{\pi(1 + b_i + 2mb_i)l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi b_i l}{a_i}\right) \Bigg\} \Bigg], \end{aligned}$$

where b_i's are integers which satisfy

$$\sum_{i=1}^{n} b_i \frac{\alpha}{a_i} = -1.$$

In the introduction we used the notation $w(a_1, a_2, \dots, a_n; m)$ to denote this number.

Theorem 13. *Suppose one of the a_i's is even. Then we have*

$$w(\Sigma(a_1, a_2, \dots, a_n)) = w(a_1, a_2, \dots, a_n; s),$$

for

$$s = \frac{1}{2}\left\{ \alpha\left(2 - \sum_{i=1}^{n}(1 - \frac{1}{a_i})\right) - 1 \right\}.$$

Here $w = w(1,1)$ or $w(0,2)$.

More explicitly we have

$$w(\Sigma(a_1, a_2, \dots, a_n)) = \frac{1}{8}\left[1 - \sum_{i=1}^{n} \frac{1}{a_i} \sum_{l=1}^{a_i-1} \left\{ \cot\left(\frac{\pi l}{a_i}\right) \cot\left(\frac{\pi b_i l}{a_i}\right) + 2\epsilon_i^l \operatorname{cosec}\left(\frac{\pi l}{a_i}\right) \operatorname{cosec}\left(\frac{\pi b_i l}{a_i}\right) \right\}\right],$$

where ϵ_i's are defined by

$$\epsilon_i = \begin{cases} (-1)^{1-b_i} & a_i \equiv 1 \pmod 2 \\ (-1)^{1-\Sigma_{j\neq i} b_j} & a_i \equiv 0 \pmod 2). \end{cases}$$

We give some examples of Brieskorn homology 3-spheres such that the connected sum of any number of the copies cannot be the boundary of an acyclic 4-manifold. The following is a list of Brieskorn homology 3-spheres for which Theorem 5 is not applied, but Theorem 6 can be applied.

Brieskorn	max{ w }	spin case	Casson's invariant
$\Sigma(2,3,7)$	0	$w(-1) = -1$	-1
$\Sigma(4,5,7)$	0	$w(-29) = -1$	-5
$\Sigma(7,9,10)$	0	$w(-204) = -1$	-25
$\Sigma(2,5,11)$	0	$w(-12) = -1$	-3
$\Sigma(5,8,11)$	0	$w(-129) = -1$	-17
$\Sigma(4,9,11)$	0	$w(-109) = -1$	-15
$\Sigma(5,7,12)$	0	$w(-121) = -2$	-16
$\Sigma(7,11,12)$	0	$w(-316) = -1$	-37
$\Sigma(3,4,13)$	0	$w(-27) = -1$	-5
$\Sigma(5,6,13)$	0	$w(-109) = -1$	-15
$\Sigma(3,8,13)$	0	$w(-73) = -1$	-11

In the above list, $\Sigma(5,7,12)$ has the trivial μ-invariant but non-trivial w-invariant.

Some other examples of plumbing type are given in [6].

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