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EISENSTEIN SERIES TWISTED BY MODULAR SYMBOLS FOR THE GROUP SL_n

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Abstract. We define Eisenstein series twisted by modular symbols for the group SL_n, generalizing a construction of the first author [1, 2]. We show that, in the case of series attached to the minimal parabolic subgroup, our series converges for all points in a suitable cone. We conclude with examples for SL_2 and SL_3.

1. Introduction

1.1. Let Γ denote a finitely generated discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ that contains translations and acts on the upper halfplane $\mathfrak{h}$. An automorphic form of real weight r and multiplier $\psi\colon \Gamma \to \mathbf{U}$ (here $\mathbf{U} = \{w \in \mathbb{C} \mid |w| = 1\}$ is the unit circle) is a meromorphic function $G\colon \mathfrak{h} \to \mathbb{C}$ that satisfies

$$G(\gamma z) = \psi(\gamma) \cdot j(\gamma, z)^r \cdot G(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $j(\gamma, z) = cz + d$. For $r \geq 0$, an integer, and $F\colon \mathfrak{h} \to \mathbb{C}$ any function with sufficiently many derivatives, G. Bol [3] proved the identity

$$\frac{d^{r+1}}{dz^{r+1}}\Big((cz+d)^r F(\gamma z)\Big) = (cz+d)^{-r-2} F^{(r+1)}(\gamma z),$$

which holds for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$. It follows that if $f(z)$ is an automorphic form of weight $r+2$ and multiplier ψ, and if F is any $(r+1)$-fold indefinite integral of f, then F satisfies the functional equation

$$F(\gamma z) = \psi(\gamma)(cz+d)^{-r}\Big(F(z) \ + \ \phi(\gamma, z)\Big),$$

where $\phi(\gamma, z)$ is a polynomial in z of degree $\leq r$ satisfying the cocycle condition

$$\phi(\gamma_1\gamma_2, z) = \overline{\psi(\gamma_2)}\, j(\gamma_2, z)^r\, \phi(\gamma_1, \gamma_2 z) \quad + \quad \phi(\gamma_1, z).$$

Such a function F is called an *automorphic* (or *Eichler*) *integral*, and the corresponding polynomial $\phi(\gamma, z)$ is called a *period polynomial*.

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1.2. The Eisenstein series $E^*(z;r,\psi,\phi)$ (twisted by a period polynomial ϕ) is defined by the infinite series

$$E^*(z;r,\psi,\phi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\gamma)\,\phi(\gamma,z)\,j(\gamma,z)^{-r}.$$

The twisted Eisenstein series $E^*(z;r,\psi,\phi)$ was first introduced by Eichler [4] (1965). Automorphic integrals and Eisenstein series twisted by period polynomials were systematically studied by Knopp [5] (1974). More recently [1, 2] (1995) nonholomorphic Eisenstein series twisted by modular symbols (period polynomials of degree 0) were introduced (cf. §5.3). O'Sullivan [6] found (using Selberg's method) the functional equation of these twisted Eisenstein series; very recently O'Sullivan and Chinta [7] explicitly computed the scattering matrix occurring in the functional equation.

In this paper we show how to generalize the construction of Eisenstein series twisted by modular symbols to the group SL_n. The basic properties and region of absolute convergence of such series are obtained in the case of the minimal parabolic subgroup. We conjecture that these series satisfy functional equations.

2. Eisenstein series

2.1. In this section we recall the definition of cuspidal Eisenstein series following Langlands [8, Ch. 4]. We begin with some notation.

Let $G = \mathrm{SL}_n(\mathbb{R})$, let $K = \mathrm{SO}_n(\mathbb{R})$, and let $\Gamma \subset G(\mathbb{Z})$ be an arithmetic group. Let $P_0 \subset G$ be the subgroup of upper-triangular matrices, and let $A_0 \subset P_0$ be the subgroup of diagonal matrices with each entry positive. For each decomposition $n = n_1 + \cdots + n_k$ with $n_i > 0$, we have a standard parabolic subgroup

$$P = \left\{ \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_k \end{pmatrix} \;\middle|\; P_i \in \mathrm{GL}_{n_i}(\mathbb{R}), \prod \det(P_i) = 1 \right\}.$$

We fix a standard Langlands decomposition $P = M_P A_P N_P$ as follows: M_P is the subgroup of block diagonal matrices, with each block an element of $\mathrm{SL}^{\pm}_{n_i}(\mathbb{R})$; $A_P \subset P$ is the subgroup with the ith block of the form $a_i I_{n_i}$, where $a_i > 0$ and I_{n_i} is the $n_i \times n_i$ identity; and $N_P \subset P$ is the subgroup with the ith block equal to I_{n_i}. We transfer these decompositions to all rational parabolic subgroups by conjugation.

2.2. Let $\mathfrak{a}_0$, $\mathfrak{a}_P$ be the Lie algebras of the groups A_0, A_P. Let $\check{\mathfrak{a}}_0$, $\check{\mathfrak{a}}_P$ be their $\mathbb{R}$-duals, and denote the pairing by $\langle\ ,\ \rangle$. Let $R = R^+ \cup R^- \subset \check{\mathfrak{a}}_0$ be the roots of G, and let $\Delta \subset R^+$ be the standard set of simple roots. For any root α, let $\check{\alpha}$ be the corresponding coroot. For any parabolic subgroup P, let $\rho_P = 1/2 \sum_{\alpha\in R^+\cap\check{\mathfrak{a}}_P} \alpha$.

We recall the definition of the *height function* $H_P\colon P \to \mathfrak{a}_P$. Given $p \in P$, write $p = man$, where $m \in M_P$, $a \in A_P$, and $n \in N_P$. Then $H_P(p)$ is defined via

$$e^{\langle \chi, H_P(p)\rangle} = a^\chi, \quad \text{for all } \chi \in \check{\mathfrak{a}}_P.$$

Using an Iwasawa decomposition $G = PK$, we extend the height function to a map $H_P\colon G \to \mathfrak{a}_P$ by setting $H_P(g) = H_P(p)$, where $g = pk$, $p \in P$, $k \in K$.

2.3. Fix a parabolic subgroup P, and let $\Gamma_P = \Gamma \cap P$. Let $f \in C^\infty(A_P N_P \backslash G)$ be a Γ_P-invariant, K-finite function such that for each $g \in G$, the function $m \mapsto f(mg)$, $m \in M_P$, is a square-integrable automorphic form on M_P with respect to $\Gamma_P \cap M_P$. Let $\lambda \in (\check{\mathfrak{a}}_P) \otimes \mathbb{C}$, and let $g \in G$.

Definition 2.4. The *Eisenstein series* associated to the above data is

$$E_P(f, \lambda, g) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} e^{\langle \rho_P + \lambda, H_P(\gamma g)\rangle} f(\gamma g).$$

It is known [8, Lemma 4.1] that this series converges absolutely and uniformly on compact subsets of $G \times C$, where

$$C = \big\{\lambda \bigm| \langle \Re\lambda, \check{\alpha}\rangle > \langle \rho_P, \check{\alpha}\rangle, \text{for all } \alpha \in \Delta\big\}; \tag{1}$$

here $\Re$ denotes real part.

3. Modular symbols

3.1. We recall the definition of modular symbols. Our definition is equivalent to that of Ash [9] and Ash-Borel [10], but we need a slightly different formulation for our purposes.

Let $V = \mathbb{Q}^n$ with the canonical $G(\mathbb{Q})$-action. Let $\mathbf{w}$ be a tuple of subspaces $(W_1, \ldots, W_k)$, where $W_i \subset V$. The *type* of $\mathbf{w}$ is the tuple $(\dim W_1, \ldots, \dim W_k)$. The tuple $\mathbf{w}$ is called *full* if $\sum \dim W_i = n$, and is called a *splitting* if $V = \bigoplus_i W_i$. Any splitting determines a rational flag

$$F_{\mathbf{w}} = \big\{\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq V\big\}$$

by $F_j = \bigoplus_{i \le j} W_i$, and thus determines a rational parabolic subgroup $P_{\mathbf{w}}$, the stabilizer of $F_{\mathbf{w}}$. We let $P_{\mathbf{w}} = M_{\mathbf{w}} A_{\mathbf{w}} N_{\mathbf{w}}$ be a choice of Langlands decomposition such that $M_{\mathbf{w}}(\mathbb{Q}) A_{\mathbf{w}}(\mathbb{Q})$ preserves each W_i. Note that this is not the same as the standard Langlands decomposition from §2.1.

3.2. Let X be the symmetric space G/K, and let $\bar{X}$ be the bordification of X constructed by Borel-Serre [11]. Then the cohomology $H^i(\Gamma; \mathbb{C})$ may be identified with $H^i(\Gamma\backslash X; \mathbb{C})$ and $H^i(\Gamma\backslash\bar{X}; \mathbb{C})$.

Let $Y = \Gamma\backslash X$, $\bar{Y} = \Gamma\backslash\bar{X}$, $\partial\bar{Y} = \bar{Y} \smallsetminus Y$, and let $\pi\colon X \to Y$ be the canonical projection. Let $d = (n^2+n)/2 - 1$ be the dimension of Y. For all i, Lefschetz duality gives an isomorphism

$$H_{d-i}(\bar{Y}, \partial\bar{Y}; \mathbb{C}) \longrightarrow H^i(\Gamma; \mathbb{C}).$$

3.3. Let $\mathbf{w}$ be a splitting, and let $K_{\mathbf{w}}$ be $K \cap M_{\mathbf{w}} A_{\mathbf{w}}$. The inclusion $M_{\mathbf{w}} A_{\mathbf{w}} \to G$ induces a proper map

$$\iota \colon M_{\mathbf{w}} A_{\mathbf{w}} / K_{\mathbf{w}} \longrightarrow X.$$

Let $Y_{\mathbf{w}}$ be the closure of $(\pi \circ \iota)(M_{\mathbf{w}} A_{\mathbf{w}} / K_{\mathbf{w}})$, and let $d(\mathbf{w})$ be the dimension of $Y_{\mathbf{w}}$. The submanifold $Y_{\mathbf{w}}$ is independent of the choice of decomposition in §3.1.

Definition 3.4. Let $\mathbf{w} = (W_1, \ldots, W_k)$ be a full tuple of subspaces. Then the *modular symbol* $\Xi_{\mathbf{w}}$ associated to $\mathbf{w}$ is defined as follows:

1. If $\mathbf{w}$ is a splitting, then $\Xi_{\mathbf{w}} \in H_{d(\mathbf{w})}(\bar{Y}, \partial \bar{Y}; \mathbb{C})$ is the fundamental class of $Y_{\mathbf{w}}$.
2. Otherwise, $\Xi_{\mathbf{w}}$ is defined to be $0 \in H_{d(\mathbf{w})}(\bar{Y}, \partial \bar{Y}; \mathbb{C})$, where $d(\mathbf{w})$ is the homological degree determined by any splitting with the same type as $\mathbf{w}$.

3.5. We define a $G(\mathbb{Q})$-action on tuples as follows. Given a full tuple $\mathbf{w} = (W_1, \ldots, W_k)$, let $g \cdot \mathbf{w}$ be the tuple $(W_1, gW_2, \ldots, gW_k)$. By abuse of notation we write $g \cdot \Xi_{\mathbf{w}}$ for the modular symbol $\Xi_{g \cdot \mathbf{w}}$.

Note that this is not a $G(\mathbb{Q})$-action on modular symbols, since associativity does not hold. However, the definition $g \cdot \Xi_{\mathbf{w}}$ will suffice for our construction.

Note also that $g \cdot \Xi_{\mathbf{w}}$ is different from the modular symbol obtained via the natural $G(\mathbb{Q})$-action defined by left translation of all subspaces in a tuple. In particular, let $\gamma \in \Gamma$, and let $\mathbf{w}' = (\gamma W_1, \ldots, \gamma W_k)$. Then $\Xi_{\mathbf{w}} = \Xi_{\mathbf{w}'}$, but $\Xi_{\mathbf{w}} \neq \gamma \cdot \Xi_{\mathbf{w}}$ in general.

4. Eisenstein series twisted by modular symbols

4.1. Let $\mathbf{w} = (W_1, \ldots, W_k)$ be a full tuple of subspaces, and let P be a rational parabolic subgroup. We say that P and $\mathbf{w}$ are *compatible* if the following conditions hold: there is a splitting $\mathbf{w}' = (W'_1, \ldots, W'_k)$ such that $P = P_{\mathbf{w}'}$, the types of $\mathbf{w}$ and $\mathbf{w}'$ are equal, and $W_1 = W'_1$.

Fix a rational parabolic subgroup P and a compatible splitting $\mathbf{w}$. Let f, λ be as in §2.3, and let φ be a $\mathbb{C}$-valued linear form on $H_{d(\mathbf{w})}(\bar{Y}, \partial \bar{Y}; \mathbb{C})$.

Definition 4.2. The *twisted Eisenstein series* associated to the above data is

$$E^*_{P,\varphi} = E^*_{P,\varphi}(f, \lambda, g, \mathbf{w}) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \varphi(\gamma \cdot \Xi_{\mathbf{w}}) e^{\langle \rho_P + \lambda, H_P(\gamma g) \rangle} f(\gamma g). \tag{2}$$

We refer to §5 for examples of this series, and for a comparison with the construction in [1, 2].

Proposition 4.3. *The series in* (2) *is well-defined.*

Proof. Let $\mathbf{w} = (W_1, \ldots, W_k)$ and let $\gamma \in \Gamma$. We need to show that the modular symbol $\gamma \cdot \Xi_{\mathbf{w}}$ depends only on the coset $\Gamma_P \gamma$.

First we assume $\gamma \cdot \mathbf{w}$ is a splitting. By the remarks at the end of §3.5, if $\gamma \in \Gamma$ and $\mathbf{w}' = (\gamma W_1, \ldots, \gamma W_k)$ is the tuple obtained by left translation, then $\Xi_{\mathbf{w}} = \Xi_{\mathbf{w}'}$. From this it follows that if $\gamma_P \in \Gamma_P$, then $(\gamma_P \gamma) \cdot \Xi_{\mathbf{w}} = \Xi_{\mathbf{w}}$. Indeed,

$\gamma_P \cdot \Xi_{\mathbf{w}} = \Xi_{\mathbf{w}''}$, where $\mathbf{w}'' = (\gamma_P^{-1} W_1, W_2, \dots, W_k)$, and any element of Γ_P preserves W_1.

Now assume that $\gamma \cdot \mathbf{w}$ isn't a splitting. There are two possibilities: (1) $\gamma W_i \cap \gamma W_j \neq \{0\}$ for some $i, j > 1$; (2) $\gamma W_i \cap \gamma W_j = \{0\}$ for all $i, j > 1$ and $W_1 \cap \gamma W_j \neq \{0\}$ for some j. In the first case, we have $\gamma \cdot \Xi_{\mathbf{w}} = 0$ for all γ, so the Eisenstein series is identically 0. In the second case, we have $(\gamma_P \gamma) \cdot \Xi_w = 0$ for all $\gamma_P \in \Gamma_P$, since left translation of the tuple $\gamma_P \gamma \cdot \mathbf{w}$ by γ_P^{-1} preserves the incidence conditions satisfied by the W_i. This completes the proof.

□

4.4. For the rest of this note, we will assume that P is the minimal parabolic subgroup P_0, and will take $f \equiv 1$. Although the functions $E^*_{P,\varphi}$ are not automorphic, a certain sum of them is.

Proposition 4.5. *Let W_i, $i = 0, \dots, n$ be 1-dimensional subspaces of V, and let $\mathbf{w}(i)$ be the tuple $(W_0, \dots, \hat{W}_i, \dots, W_n)$, where $\hat{W}_i$ means delete W_i. Then*

$$\varphi(\Xi_{\mathbf{w}(0)}) E_P(f, \lambda, g) = \sum_{i=1}^{n} (-1)^{i+1} E^*_{P,\varphi}(f, \lambda, g, \mathbf{w}(i)).$$

Proof. First, the twisted series on the right are well-defined, since if P and $\mathbf{w}$ are compatible then so are P and $\mathbf{w}(i)$ for each $i \geq 1$. We have the following basic relation among modular symbols for the minimal parabolic subgroup, from [12, 13]:

$$\Xi_{\mathbf{w}(0)} = \sum_{i=1}^{n} (-1)^{i+1} \Xi_{\mathbf{w}(i)}.$$

Note that the relations in [12, 13] imply that this equality holds true in $H_{d(\mathbf{w})}(Y_{\mathbf{w}}, \partial Y_{\mathbf{w}}; \mathbb{C})$ for *any* collection of 1-dimensional rational subspaces $(W_0, \dots, W_n)$, even with the possibility that some $\mathbf{w}(i)$ aren't splittings. The result follows immediately from Definition 4.2 and the fact that if $\mathbf{w}' = (\gamma W_1, \dots, \gamma W_n)$ with $\gamma \in \Gamma$, then $\Xi_{\mathbf{w}'} = \Xi_{\mathbf{w}(0)}$ □

Theorem 4.6. *Let P be the minimal parabolic subgroup P_0, and let $\mathbf{w}$ be a compatible splitting. Let φ be a linear form on $H_{n-1}(\bar{Y}, \partial \bar{Y}; \mathbb{C})$. Then the series* (4.2) *converges uniformly on compact subsets of $G \times C$, where C is the cone* (1).

Proof. We begin by recalling some facts from the theory of modular symbols associated to the minimal parabolic subgroup. These facts are equivalent to results in [13], and are just reformulated in terms of tuples and splittings.

Let $\mathcal{W}$ be the set of all full tuples of 1-dimensional subspaces. We define a function $\| \ \| \colon \mathcal{W} \to \mathbb{Z}$ as follows. From each 1-dimensional subspace W, we choose and fix a primitive vector $v(W) \in \mathbb{Z}^n$. Then we set

$$\|\mathbf{w}\| = |\det(v(W_1), \dots, v(W_n))|.$$

Let $\mathcal{W}_u \subset \mathcal{W}$ be the subset of tuples for which $\|\mathbf{w}\| = 1$. The set $\Gamma\backslash\mathcal{W}_u$ is finite, where Γ acts by left translations. One can show that any modular symbol $\Xi_{\mathbf{w}}$ can be written as a sum

$$\Xi_{\mathbf{w}} = \sum_{\mathbf{w}' \in S} \Xi_{\mathbf{w}'}, \tag{3}$$

where S is a finite subset of $\mathcal{W}_u$ (depending on $\Xi_{\mathbf{w}}$). Moreover, the cardinality of S is bounded by $p(\log\|\mathbf{w}\|)$, where p is a polynomial depending only on n [14].

Let $\gamma \in \Gamma$ and consider the modular symbol $\gamma \cdot \Xi_{\mathbf{w}}$. Since $\mathbf{w}$ is compatible with P, the space W_1 is the span of the first basis element of V. Let us assume for the moment that for $i > 1$, W_i is the span of the ith standard basis element of V. This implies that $\|\gamma \cdot \mathbf{w}\|$ is the absolute value of the determinant of a fixed $(n-1) \times (n-1)$ minor of γ. Hence

$$\|\gamma \cdot \mathbf{w}\| << \max\bigl\{|\gamma_{ij}|^{n-1} \bigm| 1 \le i, j \le n\bigr\}, \tag{4}$$

where the implied constant depends only on n.

Let $M(\gamma)$ be the right hand side of (4). It follows that there is a polynomial p_1, depending only on n, such that

$$p(\log\|\gamma \cdot \mathbf{w}\|) < p_1(\log M(\gamma)).$$

Now consider the value $\varphi(\gamma \cdot \Xi_{\mathbf{w}})$. Since $\Gamma\backslash\mathcal{W}_u$ is finite, there is a maximum value $\varphi_{\max}$ that $|\varphi|$ attains on this set. Writing $I(\gamma,\lambda) = \exp(\langle \rho_P + \lambda, H_P(\gamma g)\rangle)$, we have

$$\sum_{\gamma \in \Gamma_P\backslash\Gamma} |\varphi(\gamma \cdot \Xi_{\mathbf{w}}) I(\gamma,\lambda)| << \sum_{\gamma \in \Gamma_P\backslash\Gamma} |p_1(\log M(\gamma)) I(\gamma,\lambda)|, \tag{5}$$

where the implied constant depends on n and $\varphi_{\max}$. The right of (5) has the same convergence properties as the usual Eisenstein series, and so the proof is complete under our assumption on $\mathbf{w}$.

Now assume W_i is a general 1-dimensional subspace of V for $i > 1$. Let $v(W_i)_j$ be the jth coordinate of $v(W_i)$, and let

$$M(\mathbf{w}) = \max\bigl\{|v(W_i)_j| \bigm| 1 \le i, j \le n\bigr\}.$$

Then

$$\|\gamma \cdot \mathbf{w}\| << M(\gamma),$$

where the implied constant depends on n and $M(\mathbf{w})$. The rest of the proof proceeds as above.

□

5. Examples

5.1. In this section we continue to assume that P is the minimal parabolic subgroup P_0. We begin by discussing the connection between the construction in this note and that of [1, 2].

Let ℓ be a positive integer, let $G = \mathrm{SL}_2(\mathbb{R})$, and let $\Gamma = \Gamma_0(\ell)$. The space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is the upper halfplane $\mathfrak{h}$, and we let $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ be the usual partial compactification obtained by adjoining cusps. Given a pair of cusps (q_1, q_2), we can determine a full tuple (W_1, W_2) by setting W_i to be the subspace of $\mathbb{Q}^2$ corresponding to the point $q_i \in \mathbb{P}^1(\mathbb{Q})$. Slightly abusing notation, we denote the corresponding modular symbol by $\Xi(q_1, q_2)$.

5.2. We can construct an interesting linear form on the modular symbols as follows. Let f be a fixed weight two holomorphic cuspform on Γ. Then we set

$$\varphi(\Xi(q_1, q_2)) = -2\pi i \int_{q_1}^{q_2} f(z)\, dz,$$

where the integration is taken along the ideal geodesic from q_1 to q_2. Note that if f is a newform, then $\varphi(\Xi(\infty, 0))$ is the special value $-L(1, f)$.

To compute the series (2), let $\Gamma_\infty = \Gamma \cap P$, and let $\Im\colon \mathfrak{h} \to \mathbb{R}$ be the imaginary part. Let $\alpha \in \check{\mathfrak{a}}_0$ be the standard positive root, so that $\rho_P = \alpha/2$. Write $\lambda = t\alpha$, where $t \in \mathbb{C}$. It is easy to check that $e^{\langle \lambda + \rho_P, H_P(g)\rangle} = \Im(z)^{t+1/2}$, where $z \in \mathfrak{h}$ is the point corresponding to g. Setting $(q_1, q_2) = (\infty, 0)$, we see that the corresponding tuple $\mathbf{w}$ is compatible with P. We obtain

$$E^*_{P,\varphi}(\lambda, g, \mathbf{w}) = E^*_{P,\varphi}(t, z, \mathbf{w}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\gamma \cdot \Xi_{\mathbf{w}}) \Im(\gamma z)^{t+1/2}, \quad t \in \mathbb{C}. \tag{6}$$

By Theorem 4.6, this converges for $\Re t > 1/2$.

5.3. To relate this to [1, 2], we recall the pairing between classical modular symbols and cuspforms. One fixes a point $z_0 \in \mathfrak{h}^*$, and defines a map

$$\begin{aligned} [\ ]_f \colon \Gamma &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto -2\pi i \int_{z_0}^{\gamma z_0} f(z)\, dz. \end{aligned}$$

(In [1, 2], this map is written as $\gamma \mapsto \langle \gamma, f \rangle$.) One can show that this map is independent of z_0, vanishes on Γ_∞, and satisfies

$$[\gamma\gamma']_f = [\gamma]_f + [\gamma']_f, \quad \text{for } \gamma, \gamma' \in \Gamma.$$

. Then the series in [1, 2] is defined by

$$E^*(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} [\gamma]_f \Im(\gamma z)^s, \quad s \in \mathbb{C}, \tag{7}$$

which converges for $\Re s > 1$.

To compare this with (6), let $q_1 = \infty$ and $q_2 = z_0 = 0$, and put $s = t + 1/2$. Since

$$\int_{q_1}^{q_2} f + \int_{q_2}^{\gamma q_2} f = \int_{q_1}^{\gamma q_2} f,$$

we find

$$\varphi(\Xi(q_1, q_2))E(z, s) + E^*(z, s) = E^*_{P,\varphi}(s - 1/2, z, \mathbf{w}),$$

where

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s$$

is the classical nonholomorphic Eisenstein series.

In [1, 2] it is shown that E^* is "automorphic up to a shift." Precisely, if $\gamma \in \Gamma$, then

$$E^*(\gamma z, s) = E^*(z, s) - [\gamma]_f E(z, s).$$

This is easily seen to be equivalent to Proposition 4.5 above.

5.4. Now let $G = \mathrm{SL}_3(\mathbb{R})$, and let $\Gamma = \Gamma_0(\ell)$. This is the arithmetic group defined to be the subgroup of $G(\mathbb{Z})$ consisting of matrices with bottom row congruent to $(0, 0, *)$ mod ℓ. The symmetric space $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$ is a 5-dimensional smooth noncompact manifold, and our modular symbols live in $H_2(\bar{Y}, \partial\bar{Y}; \mathbb{C})$.

To construct an interesting linear form on these modular symbols, we may use elements of the *cuspidal cohomology* $H^3_{\mathrm{cusp}}(\Gamma; \mathbb{C})$. These are classes that, via the de Rham isomorphism, correspond to Γ'-invariant differential forms $\omega = \sum_I f_I \, d\omega_I$, where the coefficients are cusp forms and $\Gamma' \subset \Gamma$ is a torsionfree subgroup of finite index. In this context, $H^3_{\mathrm{cusp}}(\Gamma; \mathbb{C})$ can alternatively be defined to be the kernel of the restriction map $H^3(\bar{Y}; \mathbb{C}) \to H^3(\partial\bar{Y}; \mathbb{C})$. We refer to [15] for details.

5.5. To explicitly construct classes in $H^3_{\mathrm{cusp}}(\Gamma; \mathbb{C})$ that can be paired with modular symbols, we may use techniques of [16]. There it is shown that $H^3_{\mathrm{cusp}}(\Gamma; \mathbb{C})$ is isomorphic to a space $W(\Gamma)$ of functions $f\colon \mathbb{P}^2(\mathbb{Z}/\ell\mathbb{Z}) \to \mathbb{C}$ satisfying certain relations [16, Summary 3.23]. A modular symbol $\Xi_{\mathbf{w}}$ modulo Γ gives rise to a point $p_{\mathbf{w}} \in \mathbb{P}^2(\mathbb{Z}/\ell\mathbb{Z})$ by taking the bottom row of the matrix $(v(W_1), v(W_2), v(W_3))$ [16, Prop. 3.12]. Hence given an element $\alpha \in H^3_{\mathrm{cusp}}(\Gamma; \mathbb{C})$ corresponding to a function $f_\alpha \in W(\Gamma)$, we obtain a linear form by setting

$$\varphi(\Xi_{\mathbf{w}}) = f_\alpha(p_{\mathbf{w}}).$$

This linear form is induced from the intersection pairing

$$H_3(\bar{Y}) \times H_2(\partial\bar{Y}) \longrightarrow \mathbb{C};$$

we refer to [16, Prop. 3.24] for details.

5.6. For an explicit example, we may take $\ell = 53$. This is the first level for which the cuspidal cohomology is nonzero; one finds that $\dim H^3_{\mathrm{cusp}}(\Gamma_0(53); \mathbb{C}) = 2$. A sample element is given as a function in $W(\Gamma)$ in Table II of [16].

To compute $E^*_{P,\varphi}$, we may take $\alpha \in H^3_{\mathrm{cusp}}(\Gamma_0(53); \mathbb{C})$ to be a Hecke eigenclass. For a prime p with $(p, 53) = 1$, the local L-factor of the representation corresponding to α has the form

$$(1 - a_p p^{-s} + \bar{a}_p p^{1-2s} - p^{3-3s})^{-1},$$

where $s \in \mathbb{C}$ and a_p is the eigenvalue of a certain Hecke operator. If we fix an algebraic integer ρ satisfying $\rho^2 = -11$, we find that for our Hecke eigenclass

$$a_2 = -2 - \rho, \quad a_3 = -1 + \rho, \quad a_5 = 1, \quad a_7 = -3, \quad \ldots$$

If we represent α using a function $f \in W(\Gamma)$, and apply the formulæ in [17, Ch. V and VII], we can obtain a very explicit expression for $E^*_{P,\varphi}$.

In contrast to the SL_2 case, the twisted Eisenstein series on SL_3 isn't simply automorphic up to a shift. If we consider the relation in Proposition 4.5, we see that a certain sum of *three* twisted Eisenstein series is equal to an automorphic function.

References

[1] D. Goldfeld, *The distribution of modular symbols*, Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), de Gruyter, 1999, pp. 849–865.

[2] ______, *Zeta functions formed with modular symbols*, Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), Proc. Sympos. Pure Math. **66**, Amer. Math. Soc., Providence, RI, 1999, pp. 111–121.

[3] G. Bol, *Invarianten linearer differentialgleichungen*, Abh. Math. Sem. Univ. Hamburg **16** (1949), 1–28.

[4] M. Eichler, *Grenzkreisgruppen und kettenbruchartige Algorithmen*, Acta Arith. **11** (1965), 169–180.

[5] Marvin I. Knopp, *Some new results on the Eichler cohomology of automorphic forms*, Bull. Amer. Math. Soc. **80** (1974), 607–632.

[6] Cormac O'Sullivan, *Properties of Eisenstein series formed with modular symbols*, J. Reine Angew. Math. **518** (2000), 163–186.

[7] G. Chinta and D. Goldfeld, *Grössencharakter L-functions of real quadratic fields twisted by modular symbols*, Inv. Math., to appear.

[8] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, Vol. 544, Springer-Verlag, 1976, v+337.

[9] A. Ash, *Nonminimal modular symbols for $GL(n)$*, Invent. Math., **91** (1988), 483–491.

[10] A. Ash and A. Borel, *Generalized modular symbols*, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), Lecture Notes in Math. **1447**, Springer, Berlin, 1990, pp. 57–75.

[11] A. Borel and J.-P. Serre, *Corners and arithmetic groups* (Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault), Comment. Math. Helv. **48** (1973), 436–491.

[12] A. Ash, *Unstable cohomology of $SL(n, \mathcal{O})$*, J. Algebra **167** (1994), 330–342.

[13] A. Ash and L. Rudolph, *The modular symbol and continued fractions in higher dimensions*, Invent. Math. **55** (1979), 241–250.

[14] A. I. Barvinok, *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Math. Oper. Res. **19** (1994), 769–779.
[15] R. Lee and J. Schwermer, *Cohomology of arithmetic subgroups of SL_3 at infinity*, J. Reine Angew. Math. **330** (1982), 100–131.
[16] A. Ash, D. Grayson, and P. Green, *Computations of cuspidal cohomology of congruence subgroups of $SL(3,\mathbb{Z})$*, J. Number Theory **19** (1984), 412–436.
[17] D. Bump, *Automorphic forms on $GL(3,\mathbb{R})$*, Lecture Notes in Mathematics **1083**, Springer-Verlag, 1984.

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