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STRONGLY ASYMPTOTICALLY HYPERBOLIC SPIN MANIFOLDS

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1. Introduction

In this note, we shall generalize Min-Oo's theorem of scalar curvature rigidity of strongly asymptotically hyperbolic spin manifolds.

Inspired by Witten's proof of the positive energy theorem for asymptotically flat manifolds, Min-Oo introduced strongly asymptotically hyperbolic manifolds and proved that any strongly asymptotically hyperbolic spin manifold of dimension $n \geq 3$ with scalar curvature $R \geq -n(n-1)$ is isometric to hyperbolic space [3]. This rigid theorem was extended later to locally asymptotically hyperbolic spin manifolds by Andersson and Dahl [1], and to odd-dimensional complex hyperbolic manifold by Herzlich [2].

Recall the hyperbolic space (H^n, g_0) is R^n endowed with the metric

$$g_0 = dr^2 + \sinh^2(r) d\Omega^2_{n-1} \tag{1.1}$$

in polar coordinates, which has constant sectional curvature $K_0 = -1$ and scalar curvature $R_0 = -n(n-1)$. For $r > 0$, we denote by H_r the complement of a closed ball of radius r around the origin, i.e., $H_r = H^n - \bar{B}_r(0)$. A smooth Riemannian manifold (M^n, g) is said to be strongly asymptotically hyperbolic (with one end) if there exists a compact subset $K \subset M$ and a diffeomorphism $f : M_\infty \equiv M - K \to H_r$ for some $r > 0$, such that, on M_∞, the metric g can be written as

$$\Big(AX, AY\Big)_g = \Big(X, Y\Big)_{g_0} \tag{1.2}$$

for some (symmetric, positive definite) gauge transformation A of the tangent bundle on M_∞, where we identify the hyperbolic end M_∞ with H_r. Setting $r = d_{g_0}(x_0, \cdot)$, then there exists an $\varepsilon > 0$ such that

(i) A is uniformly bounded from below and above;
(ii) $|\nabla^{g_0} A| + |A - Id| = O(e^{-(n+\varepsilon)r})$ at infinity.

Theorem 1.1. *If there exists a real function H on strongly asymptotically hyperbolic spin manifold (M, g) and $\varepsilon' > 0$,*

$$H = O(e^{-(n+\varepsilon')r})$$

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at infinity such that

$$R \geq -n(n-1) - n(H^2 + 2H) + 2n|dH|, \tag{1.3}$$

then (M^n, g) *is isometric to* (H^n, g_0)*. In particular,* $H \equiv 0$*.*

This rigid theorem is true also for local and odd-dimensional complex hyperbolic spaces, as well as the multi-end case, this theorem of course holds for the assumption of strongly asymptotic hyperbolic via integral conditions on A and H, for instance,

$$e^r(A - Id) \in L^{1,p}, \quad e^r H \in L^p$$

for $p \geq 1$.

The positive energy theorem for asymptotically flat spin manifolds whose scalar curvature satisfies

$$R \geq -n(H^2 + 2H) + 2n|dH|$$

for some real function

$$H = O(r^{-(1+\varepsilon')})$$

at infinity was considered in [9].

The positive energy theorem was first proved by Schoen and Yau via the geometric analysis method [5, 6, 7]. So it is interesting to use their method to prove these rigid theorems.

2. Preliminaries

Let M be an (oriented) Riemannian spin manifold of dimension $n \geq 3$ with spinor bundle S. Let g, g_0 be Riemannian metrics on M. Let $A \in End(TM)$ be the symmetric, positive definite 'gauge transformation' such that

$$\Big(AX, AY\Big)_g = \Big(X, Y\Big)_{g_0}.$$

The gauge transformation A induces a map from $SO(M, g_0)$ to $SO(M, g)$. Therefore it induces a map from $Spin(n, g_0)$ to $Spin(n, g)$ and hence a map from the spinor bundles $S(M, g_0)$ to $S(M, g)$ by

$$A\Big(X \cdot \phi\Big) = \Big(AX\Big) \cdot \Big(A\phi\Big).$$

Let ∇, ∇^{g_0} be the Levi-Civita connections of g, g_0 respectively. We extend them to the spinor bundle S and denote as ∇, ∇^{g_0} also. To compare the two connections, we define a connection $\bar{\nabla}$ with respect to the metric g by

$$\bar{\nabla} X = A\Big(\nabla^{g_0}(A^{-1}X)\Big). \tag{2.1}$$

This connection has torsion

$$\begin{aligned} \bar{T}(X, Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \\ &= -\Big((\nabla^{g_0}_X A)A^{-1}Y - (\nabla^{g_0}_Y A)A^{-1}X\Big). \end{aligned} \tag{2.2}$$

Since the connection ∇ is torsionless with respect to the metric g, we obtain

$$
(2.3)\quad 2\Big(\bar\nabla_X Y - \nabla_X Y, Z\Big)_g = \Big(\bar T(X,Y),Z\Big)_g - \Big(\bar T(X,Z),Y\Big)_g - \Big(\bar T(Y,Z),X\Big)_g .
$$

Now we compare ∇, $\bar\nabla$ on spinor bundle S. Let $\{e_i\}$ be a local orthonormal frame for g, and $\{e^i\}$ be the local orthonormal coframe. Let $\{\sigma^\alpha\}$ be the local orthonormal frame of the spinor bundle. Denote by ω_{ij}, $\bar\omega_{ij}$ the connection 1-forms for ∇, $\bar\nabla$ defined by

$$
\omega_{ij} = (\nabla e_i, e_j)_g\,, \qquad \bar\omega_{ij} = (\bar\nabla e_i, e_j)_g\,.
$$

For spinor $\phi = \phi^\alpha \sigma_\alpha$, we have

$$
\begin{aligned}
\nabla\phi &= d\phi^\alpha \otimes \sigma_\alpha + \frac{1}{4}\sum_{i,j} \omega_{ij} \otimes e_i \cdot e_j \cdot \phi,\\
\bar\nabla\phi &= d\phi^\alpha \otimes \sigma_\alpha + \frac{1}{4}\sum_{i,j} \bar\omega_{ij} \otimes e_i \cdot e_j \cdot \phi.
\end{aligned}
$$

Therefore,

$$
(2.4)\qquad \nabla\phi - \bar\nabla\phi = \frac{1}{4}\sum_{i,j}\Big(\omega_{ij} - \bar\omega_{ij}\Big) \otimes e_i \cdot e_j \cdot \phi.
$$

By (2.2) and (2.3), we obtain

$$
\Big|\Big(\omega_{ij} - \bar\omega_{ij}\Big)(e_k)\Big| \le C\Big|A^{-1}\Big|\Big|\nabla^{g_0} A\Big|,
$$

for some $C > 0$. Therefore,

$$
\begin{aligned}
(2.5)\qquad \Big|\nabla X - \bar\nabla X\Big| &\le C\Big|A^{-1}\Big|\Big|\nabla^{g_0} A\Big||X|,\\
(2.6)\qquad \Big|\nabla\phi - \bar\nabla\phi\Big| &\le C\Big|A^{-1}\Big|\Big|\nabla^{g_0} A\Big||\phi|,\\
(2.7)\qquad \Big|D\phi - \bar D\phi\Big| &\le C\Big|A^{-1}\Big|\Big|\nabla^{g_0} A\Big||\phi|,
\end{aligned}
$$

where X is a vector, ϕ is a spinor and D, $\bar D$ are the Dirac operators with respect to the connections ∇, $\bar\nabla$.

3. The Killing connections and Dirac operators

The Killing connection on spinor bundle S is defined by

$$
(3.1)\qquad \hat\nabla_X = \nabla_X + \frac{\mathbf{i}}{2} X\cdot,
$$

spinors vanish with respect to this connection are called imaginary Killing spinors. The curvature of connection $\hat{\nabla}$ is

$$\begin{aligned}\hat{R}(X,Y)\phi &= \Big(\hat{\nabla}_X\hat{\nabla}_Y - \hat{\nabla}_Y\hat{\nabla}_X - \hat{\nabla}_{[X,Y]}\Big)\phi \\ &= R(X,Y)\phi - \frac{1}{4}(X\cdot Y\cdot - Y\cdot X\cdot)\phi.\end{aligned}$$

Hence if a manifold M has a local basis of imaginary Killing spinors, then $\hat{R}$ vanishes, so that M has constant sectional curvature -1 and is locally isometric to hyperbolic space. If a manifold M has an imaginary Killing spinor ϕ, then

$$0 = \sum_i e_i \hat{R}(X,e_i)\phi = -\frac{1}{2}\Big(Ric(X) + (n-1)X\Big)\cdot\phi\,,$$

therefore M has constant Ricci curvature $-(n-1)$.

The hyperbolic space H^n has a full set of imaginary Killing spinors $\{\phi_0\}$, see [1, 3]. Moreover, there is constant $C > 0$ such that

$$C^{-1}e^{-r} \leq |\phi_0|^2 \leq Ce^r \tag{3.2}$$

as $r \to \infty$ by (3.1). This fact is very important in the proof of rigid theorems for strongly asymptotically hyperbolic manifolds.

Now we define a generalized Killing connection

$$\tilde{\nabla}_X = \nabla_X + \frac{\mathbf{i}}{2}(1+H)X\cdot \tag{3.3}$$

for some real function H, and a Dirac operator

$$\tilde{D} = e_i\cdot\tilde{\nabla}_i = D - \frac{\mathbf{i}}{2}n(1+H) \tag{3.4}$$

with respect to this generalized Killing connection. Since

$$\begin{aligned}d(\phi,\psi)*e^i &= (\nabla_i\phi,\psi)*1 + (\phi,\nabla_i\psi)*1 \\ &= \Big(\tilde{\nabla}_i\phi - \frac{\mathbf{i}}{2}(1+H)e_i\cdot\phi,\psi\Big)*1 \\ &\quad + \Big(\phi,\tilde{\nabla}_i\psi - \frac{\mathbf{i}}{2}(1+H)e_i\cdot\psi\Big)*1 \\ &= \Big(\tilde{\nabla}_i\phi,\psi\Big)*1 + \Big(\phi,\tilde{\nabla}_i\psi - \mathbf{i}(1+H)e_i\cdot\psi\Big)*1,\end{aligned}$$

$$\begin{aligned}d(e_i\cdot\phi,\psi)*e^i &= (D\phi,\psi)*1 - (\phi,D\psi)*1 \\ &= \Big(\tilde{D}\phi + \frac{\mathbf{i}}{2}n(1+H)\phi,\psi\Big)*1 \\ &\quad -\Big(\phi,\tilde{D}\psi + \frac{\mathbf{i}}{2}n(1+H)\psi\Big)*1 \\ &= \Big(\tilde{D}\phi,\psi\Big)*1 - \Big(\phi,\tilde{D}\psi + \mathbf{i}n(1+H)\psi\Big)*1,\end{aligned}$$

we have

$$\tilde{\nabla}_i^* = -\tilde{\nabla}_i + \mathbf{i}(1+H)e_i\cdot \ = \ -\nabla_i + \frac{\mathbf{i}}{2}(1+H)e_i\cdot, \tag{3.5}$$

$$\tilde{D}^* = \tilde{D} + \mathbf{i}n(1+H) \ = \ D + \frac{\mathbf{i}}{2}n(1+H). \tag{3.6}$$

Note that

$$\hat{\nabla}^*\hat{\nabla} = \hat{\nabla}\hat{\nabla}^* = \nabla^*\nabla + \frac{n}{4}. \tag{3.7}$$

Denote

$$\hat{R} = R + n(n-1).$$

By (3.7), we have the following Weitzenböck formulas

$$\begin{aligned}\tilde{D}^*\tilde{D} &= \Big(D + \frac{\mathbf{i}}{2}n(1+H)\Big)\Big(D - \frac{\mathbf{i}}{2}n(1+H)\Big)\\ &= \nabla^*\nabla + \frac{1}{4}\Big(R + n^2(1+H)^2\Big) - \frac{\mathbf{i}}{2}ndH\,\cdot\\ &= \hat{\nabla}^*\hat{\nabla} + \frac{1}{4}\Big(\hat{R} + n^2(H^2+2H)\Big) - \frac{\mathbf{i}}{2}ndH\cdot\,,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\tilde{D}\tilde{D}^* &= \Big(D - \frac{\mathbf{i}}{2}n(1+H)\Big)\Big(D + \frac{\mathbf{i}}{2}n(1+H)\Big)\\ &= \nabla^*\nabla + \frac{1}{4}\Big(R + n^2(1+H)^2\Big) + \frac{\mathbf{i}}{2}ndH\,\cdot\\ &= \hat{\nabla}^*\hat{\nabla} + \frac{1}{4}\Big(\hat{R} + n^2(H^2+2H)\Big) + \frac{\mathbf{i}}{2}ndH\cdot\end{aligned} \tag{3.9}$$

Now we derive the integral forms of the Weitzenböck formulas. Since

$$\begin{aligned}d(e_i\cdot\phi, \tilde{D}\phi) * e^i &= \Big((\tilde{D}\phi, \tilde{D}\phi) - (\phi, \tilde{D}^*\tilde{D}\phi)\Big) * 1,\\ d(\phi, \hat{\nabla}_i\phi) * e^i &= \Big((\hat{\nabla}\phi, \hat{\nabla}\phi) - (\phi, \hat{\nabla}^*\hat{\nabla}\phi)\Big) * 1,\end{aligned}$$

then

$$\begin{aligned}& d\Big(\phi, \nabla_i\phi + e_i\cdot D\phi - \frac{\mathbf{i}}{2}[n(1+H)-1]e_i\cdot\phi\Big) * e^i\\ =\ & d\Big(\phi, \hat{\nabla}_i\phi + e_i\cdot\tilde{D}\phi\Big) * e^i\\ =\ & \Bigg(|\hat{\nabla}\phi|^2 - |\tilde{D}\phi|^2 + \frac{1}{4}\Big(\hat{R} + n^2(H^2+2H)\Big)|\phi|^2 - \Big(\phi, \frac{\mathbf{i}}{2}ndH\cdot\phi\Big)\Bigg) * 1.\end{aligned}$$

Similarly,

$$\begin{aligned} & d\Big(\nabla_i\psi + e_i\cdot D\psi + \frac{\mathbf{i}}{2}[n(1+H)-1]e_i\cdot\psi,\psi\Big) * e^i \\ = \;& d\Big(-\hat{\nabla}_i^*\psi + e_i\cdot\tilde{D}^*\psi,\psi\Big) * e^i \\ = \;& \left(|\hat{\nabla}^*\psi|^2 - |\tilde{D}^*\psi|^2 + \frac{1}{4}\Big(\hat{R} + n^2(H^2+2H)\Big)|\psi|^2 + \Big(\frac{\mathbf{i}}{2}ndH\cdot\psi,\psi\Big)\right) * 1. \end{aligned}$$

Let

$$\hat{D} = e_i\cdot\hat{\nabla}_i = D - \frac{\mathbf{i}}{2}n$$

and

$$\tilde{R}_\pm = \hat{R} + n^2(H^2+2H) \pm 2\mathbf{i}ndH\cdot\,.$$

Note that (1.3) ensures that $\tilde{R}_\pm$ is nonnegative. Now we obtain

$$\begin{aligned} & \int_{\partial M}\Big(\phi,\nabla_i\phi + e_i\cdot D\phi\Big) * e^i - \int_{\partial M}\Big(\phi,\frac{\mathbf{i}}{2}[n(1+H)-1]e_i\cdot\phi\Big) * e^i \\ = \;& \int_{\partial M}\Big(\phi,\hat{\nabla}_i\phi + e_i\cdot\hat{D}\phi\Big) * e^i - \int_{\partial M}\Big(\phi,\frac{\mathbf{i}}{2}nHe_i\cdot\phi\Big) * e^i \\ (3.10) = \;& \int_M |\hat{\nabla}\phi|^2 - |\tilde{D}\phi|^2 + \frac{1}{4}\Big(\phi,\tilde{R}_-\cdot\phi\Big), \end{aligned}$$

and

$$\begin{aligned} & \int_{\partial M}\Big(\nabla_i\psi + e_i\cdot D\psi,\psi\Big) * e^i + \int_{\partial M}\Big(\frac{\mathbf{i}}{2}[n(1+H)-1]e_i\cdot\psi,\psi\Big) * e^i \\ = \;& \int_{\partial M}\Big(-\hat{\nabla}_i^*\psi + e_i\cdot\hat{D}^*\psi,\psi\Big) * e^i + \int_{\partial M}\Big(\frac{\mathbf{i}}{2}nHe_i\cdot\psi,\psi\Big) * e^i \\ (3.11) = \;& \int_M |\hat{\nabla}^*\psi|^2 - |\tilde{D}^*\psi|^2 + \frac{1}{4}\Big(\tilde{R}_+\cdot\psi,\psi\Big). \end{aligned}$$

4. Rigidity for strongly asymptotically hyperbolic manifolds

Let M be a complete spin manifold with $\hat{R}$ nonnegative and bounded. Let $C_0^\infty(S)$ be the space of smooth sections with compact support. Define an inner product on S by

$$\Big(\phi,\psi\Big)_1 = \int_M \Big(\nabla\phi,\nabla\psi\Big) + \frac{n}{4}\Big(\phi,\psi\Big)$$

and let $H^1(S)$ be the closure of $C_0^\infty(S)$ with respect to this inner product. Then $H^1(S)$ with the above inner product is a Hilbert space. Now define a bounded bilinear form B on $C_0^\infty(S)$ by

$$B\Big(\phi,\psi\Big) = \int_M \Big(\tilde{D}\phi,\tilde{D}\psi\Big).$$

By (3.10), we obtain

$$\begin{aligned} B\Big(\phi,\phi\Big) &= \int_M |\hat\nabla\phi|^2 + \frac{1}{4}\Big(\phi, \tilde R_- \cdot \phi\Big) \\ &= \int_M |\nabla\phi|^2 + \frac{n}{4}|\phi|^2 + \frac{1}{4}\Big(\phi, \tilde R_- \cdot \phi\Big). \end{aligned}$$

Since $\tilde R_-$ is bounded also, we can extend B to $H^1(S)$ as a coercive bilinear form if $\tilde R_-$ is nonnegative.

We extend the imaginary Killing spinors $\{\phi_0\}$ on M_∞ to the whole M. With respect to the metric g, these Killing spinors can be written as $\bar\phi_0 = A\phi_0$. Now we show that $\hat\nabla\bar\phi_0 \in L^2(S)$: On M_∞,

$$\begin{aligned} \hat\nabla_X\bar\phi_0 &= \Big(\nabla_X + \frac{\mathbf{i}}{2}X\,\cdot\Big)\Big(A\phi_0\Big) \\ &= \Big(\nabla_X - \bar\nabla_X\Big)\Big(A\phi_0\Big) + \bar\nabla_X\Big(A\phi_0\Big) + \frac{\mathbf{i}}{2}X\cdot\Big(A\phi_0\Big) \\ &= \Big(\nabla_X - \bar\nabla_X\Big)\Big(A\phi_0\Big) + A\Big(\nabla_X^{g_0}\phi_0\Big) + \frac{\mathbf{i}}{2}X\cdot\Big(A\phi_0\Big) \\ &= \Big(\nabla_X - \bar\nabla_X\Big)\Big(A\phi_0\Big) - \frac{\mathbf{i}}{2}\Big(AX\Big)\Big(A\phi_0\Big) + \frac{\mathbf{i}}{2}X\cdot\Big(A\phi_0\Big). \end{aligned}$$

Thus,

$$|\hat\nabla\bar\phi_0| \le C\Big(|\nabla^{g_0}A| + |A - Id|\Big)|\phi_0|$$

by (2.6). Therefore,

$$|\hat\nabla_X\bar\phi_0| \le C\Big(|\nabla^{g_0}A| + |A - Id|\Big)e^{\frac{r}{2}}|X|, \tag{4.1}$$

$$|\hat D\bar\phi_0| \le C\Big(|\nabla^{g_0}A| + |A - Id|\Big)e^{\frac{r}{2}} \tag{4.2}$$

as $r \to \infty$. Since the sphere S_r has area of the order $e^{(n-1)r}$, we know that

$$\hat\nabla\bar\phi_0 \in L^2(S),$$

and hence

$$\hat D\bar\phi_0 \in L^2(S)$$

by the assumption of asymptotic hyperbolic metric. Therefore

$$\tilde D\bar\phi_0 = \hat D\bar\phi_0 - \frac{\mathbf{i}}{2}nH\bar\phi_0 \in L^2(S)$$

by the assumption of behavior of H at infinity. Note that $\bar\phi_0$ is not L^2 because $|\bar\phi_0|^2 \ge C^{-1}e^{-r}$.

Lemma 4.1. *There exists a unique spinor ϕ_1 on $H^1(S)$ such that*

$$\tilde D\Big(\phi_1 + \bar\phi_0\Big) = 0$$

if (1.3) *holds.*

Proof. Since $B\Big(\cdot,\cdot\Big)$ is coercive on $H^1(S)$, and $\tilde{D}\bar{\phi}_0 \in L^2(S)$, $\hat{\nabla}\bar{\phi}_0 \in L^2(S)$, the Lax-Milgram lemma shows that there exists a spinor $\phi_1 \in H^1(S)$ such that

$$\tilde{D}^*\tilde{D}\phi_1 = -\tilde{D}^*\tilde{D}\phi_0$$

weakly. Let $\phi = \phi_1 + \bar{\phi}_0$ and $\psi = \tilde{D}\phi$. The elliptic regularity tells us that $\psi \in H^1(S)$, and

$$\tilde{D}^*\psi = 0$$

in the classical sense. Then (3.11) implies

$$\hat{\nabla}_i^*\psi = \nabla_i\psi - \frac{\mathbf{i}}{2}(1+H)e_i \cdot \psi = 0.$$

Therefore,

$$\Big|\partial_i|\psi|^2\Big| \leq (1+H)|\psi|^2.$$

Hence

$$|\partial_i \ln |\psi|^2| \leq 1+H$$

on the complement of the zero set of ψ on M. If there exists $x_0 \in M$ such that $|\psi(x_0)| \neq 0$, then integrate it along a path from $x_0 \in M$ gives

$$|\psi(x)|^2 \geq |\psi(x_0)|^2 e^{(1+H)(|x_0|-|x|)}.$$

Obviously, ψ is not in $L^2(S)$ which gives the contradiction. Hence $\psi \equiv 0$, and the proof of lemma is complete. □

Proof of Theorem 1.1. Let ϕ be the solution of $\tilde{D}\phi = 0$ corresponding to the imaginary Killing spinor ϕ_0 constructed in Lemma 4.1. Submit this ϕ into (3.10), we obtain

$$\begin{aligned} & \int_{S_\infty}\Big(\phi, \hat{\nabla}_i\phi + e_i \cdot \hat{D}\phi\Big) * e^i - \int_{S_\infty}\Big(\phi, \frac{\mathbf{i}}{2}nHe_i \cdot \phi\Big) * e^i \\ = & \int_M |\hat{\nabla}\phi|^2 + \frac{1}{4}\Big(\phi, \tilde{R}_- \cdot \phi\Big). \end{aligned} \tag{4.3}$$

Write $\phi = \phi_1 + \bar{\phi}_0$ where $\phi_1 \in H^1(S)$. Then

$$\begin{aligned} \left|\int_{S_\infty}\Big(\phi, \hat{\nabla}_i\phi + e_i \cdot \hat{D}\phi\Big) * e^i\right| &= \left|\int_{S_\infty}\Big(\bar{\phi}_0, \hat{\nabla}_i\bar{\phi}_0 + e_i \cdot \hat{D}\bar{\phi}_0\Big) * e^i\right| \\ &\leq C\int_{S_\infty}\Big(|\nabla^{g_0}A| + |A - Id|\Big)e^r, \end{aligned}$$

and

$$\left|\int_{S_\infty}\Big(\phi, \frac{\mathbf{i}}{2}nHe_i \cdot \phi\Big) * e^i\right| \leq C\int_{S_\infty}|H|e^r.$$

By the assumption on the asymptotic behaviors of A and H at infinity, we see that the left hand side of (4.3) vanishes, and hence

$$\hat{\nabla}\phi = 0.$$

Therefore

$$0 = \tilde{D}\phi = \hat{D}\phi - \frac{\mathbf{i}}{2}nH\phi = -\frac{\mathbf{i}}{2}nH\phi.$$

This implies $H \equiv 0$. Thus it reduces to the case of Min-Oo's theorem [3] and the proof of the theorem is complete. □

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