COMPACT SINGULARITIES OF MEROMORPHIC MAPPINGS BETWEEN COMPLEX 3-DIMENSIONAL MANIFOLDS

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ABSTRACT. We prove that a meromorphic map defined on the complement of a compact subset of a three-dimensional Stein manifold M and with values in a compact complex three-fold X extends to the complement of a finite set of points. If X is simply connected, then the map extends to all of M.

1. Introduction

The study of the extendibility of holomorphic and meromorphic mappings began with the classical theorem of Hartogs [Ha] (see [Si]).

> Let K be a compact subset of a domain $M \subset \mathbb{C}^n$, $n \geq 2$, such that $M \setminus K$ is connected, and let $f : M \setminus K \to \mathbb{C}$ be a holomorphic function. Then there exists a holomorphic function $\hat{f} : M \to \mathbb{C}$ extending f, i.e., $\hat{f} \mid_{M \setminus K} = f$.

Shortly after Hartogs proved his theorem, E. E. Levi [Le] discovered that this extension result holds true also for meromorphic functions.

A natural problem is to understand under what conditions Hartogs' Theorem (respectively Levi's Theorem) holds when the mapping f takes values in a general complex manifold X rather than $\mathbb{C}$ (respectively $\mathbb{CP}^1$). Of course it is immediate that Hartogs' Theorem remains valid for holomorphic mappings with values in a Stein manifold X, since such a manifold X can be embedded into $\mathbb{C}^N$. It similarly follows that Levi's Theorem also remains valid for meromorphic mappings with values in compact projective manifolds.

In 1971, Griffiths [Gr] and the second author [Sh] independently showed that Hartogs' Theorem is valid for holomorphic mappings into manifolds X carrying a complete Hermitian metric with non-positive holomorphic scalar curvature, answering a question was asked by Chern in [Che]. Concerning the meromorphic mapping problem, the first author [Iv1] proved that Hartogs extension holds for meromorphic maps into compact Kähler manifolds.

We recall two more results here due to K. Stein and M. Chazal. Stein proved in [St] that Hartogs' Theorem holds for holomorphic maps if $\dim X \leq n - 2$. Recently Chazal [Cha] relaxed this condition to $\dim X \leq n-1$ and more generally f can be meromorphic. The next case of interest is the equidimensional case

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$\dim X = n$. It is well known that one doesn't always have meromorphic extension in this case, as is illustrated by the (holomorphic) projection $f : \mathbb{C}^n \setminus \{0\} \to X = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ to the Hopf manifold. (The $\mathbb{Z}$-action is given by $z \overset{n}{\mapsto} 2^n z$.) The goal of this paper is to show that, at least for dimension ≤ 3, the singularity at 0 of the Hopf map f is the only type of singularity that can occur for equidimensional meromorphic maps:

Theorem 1. *Let K be a compact set with connected complement in a Stein manifold M of dimension 3, let X be a compact complex manifold of the same dimension and let $f : M \setminus K \to X$ be a meromorphic map. Then there exists a finite set $A = \{a_1, \dots, a_d\} \subset K$ such that:*

1) *f has a meromorphic extension $\hat{f} : M \setminus A \to X$;*
2) *for every coordinate ball $B(a_j)$ about a point $a_j \in A$ such that $\partial B(a_j) \cap A = \emptyset$, $\hat{f}(\partial B(a_j))$ is not homologous to zero in X.*

The same result when both M and X have dimension two was proved by the first author in [Iv3, Corollary 4(b)]. It is open whether this result if valid for equidimensional maps of dimension greater than 3. Of course, one cannot expect to obtain such results when the dimension of X is greater then the dimension of M; see the remark in §1 below.

In the case of the Hopf map $f : \mathbb{C}^3 \setminus \{0\} \to (\mathbb{C}^3 \setminus \{0\})/\mathbb{Z}$ mentioned above, $A = \{0\}$. Of course, the elements $\hat{f}(\partial B(a_j))$ of the fifth homology group are very special; they are often called spherical shells. If, for example, $\hat{f}$ is a holomorphic embedding in a neighborhood of $\partial B(a_j)$, then X is of a very restricted type: it is a deformation of the Hopf 3-fold (see [Ka1]).

In particular, if the singular set (possibly blown up at finitely many points) A is nonempty, then $H^5(X, \mathbb{R}) \neq 0$. Poincaré duality then implies the following:

Corollary 2. *If under the conditions of Theorem 1, $H^1(X, \mathbb{R}) = 0$, then f extends meromorphically to all of M.*

2. Reductions

For degenerate mappings, the result is known and is due to F. Chazal [Cha]. Hence, in the sequel, we always suppose that f is nondegenerate; i.e., rank $f = 3$.

We let $\Delta(r) = \{z \in \mathbb{C} : |z| < r\}$ denote the disk or radius r about 0, and we write $\Delta = \Delta(1)$. We consider the polydisk $\Delta^n(r) = \Delta(r)^n$ and "annulus" $A^n(r, 1) = \Delta^n \setminus \overline{\Delta^n(r)}$. We shall make frequent use of the following Hartogs figure in $\mathbb{C}^3$:

$$H_1^2(r) = \left[\Delta(1 - r) \times \Delta^2\right] \cup \left[\Delta \times A^2(r, 1)\right].$$

By the standard method of extending analytic objects (see for example [Si]), it suffices to prove either of the following two equivalent results:

Proposition 3. *Let X be a compact complex 3-fold and let $f : H_1^2(r) \to X$ be a nondegenerate meromorphic map. Then there is a discrete set $\{a_j\} \subset \Delta^3 \setminus H_1^2(r)$*

and a meromorphic extension $\hat{f} : \Delta^3 \setminus \{a_j\} \to X$ such that if $B(a_j)$ are disjoint balls in Δ^3 centered at a_j, then $\hat{f}(\partial B(a_j))$ is not homologous to zero in X.

Proposition 4. *Let M, W be open sets in $\mathbb{C}^3$, and suppose $p \in M \cap \partial W$ such that W is smooth and strictly pseudoconvex at p. Let $U = M \setminus \overline{W}$. Suppose $f : U \to X$ is a nondegenerate meromorphic map to a compact 3-fold X. Then there is an open set $\widetilde{U} \supset U \cup \{p\}$ such that f has a meromorphic extension $\hat{f}$ to $\widetilde{U} \setminus \{p\}$, and either $\hat{f}$ is meromorphic at p or $\hat{f}(\partial B(p))$ is not homologous to zero in X, where $B(p)$ is a ball about p contained in $\widetilde{U}$.*

We note that Proposition 4 follows from Proposition 3 with the additional simplifying assumption that f is *holomorphic* on $\Delta \times A^2(r, 1)$. To see this, we observe that the set of points of indeterminacy I_f of our meromorphic map f has codimension at least two, i.e., is a curve together with a discrete set of points. Let M, W, p be as in Proposition 4 and let $\Delta_{p_1}^2 = \{p_1\} \times \Delta^2$ denote the vertical bidisk passing through $p = (p_1, p_2, p_3)$. We can assume, after making a quadratic change of coordinates, that $\Delta_{p_1}^2 \cap I_f$ contains no curves and $\Delta_{p_1}^2 \cap \overline{W} = \{p\}$. After translating and stretching coordinates, we then obtain a Hartogs figure $H_1^2(r)$ contained in U, with p in the corresponding polydisk Δ^3, such that $[\Delta \times A^2(r, 1)] \cap I_f = \emptyset$, so we can apply the modified Proposition 3 to obtain the conclusion of Proposition 4 with $\widetilde{U} = \Delta^3$.

Proposition 3 follows from Proposition 4, since the Hartogs figure can be exhausted by a family of strictly pseudoconcave hypersurfaces and this family can be continued to exhaust Δ^3. Thus, when proving Proposition 3, we may assume that f is holomorphic on $\Delta \times A^2(r, 1)$.

Remark: The reader may observe that these results involves extension from a "1-concave" 3-dimensional domain. It is worthwhile to note that in general there is no extension of meromorphic maps with values in compact 3-dimensional manifolds from 2-concave domains, such as the classical Hartogs figure

$$H_2^1(r) := \left[\Delta^2(1 - r) \times \Delta\right] \cup \left[\Delta^2 \times A^1(r)\right].$$

Namely, M. Kato constructed in [Ka2] an example of a compact complex three-fold X and a holomorphic mapping $f : \mathbb{C}^2 \setminus \bar{B} \to X$ defined on the complement of a ball $B \subset \mathbb{C}^2$, such that every point of the sphere ∂B is an essential singular point of f.

We shall prove Proposition 3 in §2 after we make the following reductions:

a) First of all, as was already explained, we can assume that $f : H_1^2(r) \to X$ is nondegenerate and holomorphic on $\Delta \times A^2(r, 1)$.

b) We can further assume that there is no hypersurface in $H_1^2(r)$ which f sends to a point. If such a hypersurface exists, then by shrinking $H_1^2(r)$ a little bit, we can suppose that there are finitely many of them. Then by blowing up the image points sufficiently many times, we obtain a modification $\widehat{X}$ of X together with a lift $\hat{f}$ of f to a meromorphic map $\hat{f} : H_1^2(r) \to \widehat{X}$

having the desired property. After extending $\hat{f}$, we can push it down to
extend f itself.

c) We write
$$A_s^2 = \{s\} \times A^2(r,1), \quad s \in \Delta.$$
After again shrinking $H_1^2(r)$ a little, we can suppose that A_s^2 contains no
curves contracted by f to a point, for all $s \in \Delta$. Indeed, since rank $f = 3$,
there are at most 1-parameter families of contracted curves. We consider
small quadratic changes of the z_1 coordinate: $\tilde{z}_1 = z_1 + Q(z_2, z_3)$, where Q
is a polynomial of degree 2 with small coefficients. The set of such Q such
that $\tilde{z}_1$ is constant on a fixed holomorphic curve is of codimension at least
2. Whence, for an open dense set of such Q, the coordinate function $\tilde{z}_1$ is
nonconstant on each contracted curve; i.e., for all $s \in \Delta$, $\tilde{z}_1^{-1}(s)$ contains
no contracted curves.

d) By the above argument, we can also assume that none of the A_s^2 are con-
tained in the critical set C_f of f.

e) By the argument below, we can also assume that for all $s \in \Delta$, there do
not exist nonempty disjoint open subsets V_1, V_2 of A_s^2 with $f(V_1) = f(V_2)$.

To show that we can realize property (e) after a change of coordinates, we let
$$U = \left[\Delta \times A^2(r,1)\right] \setminus (I_f \cup C_f)$$
and we consider the set
$$D = \{(z,w) \in U \times U : z \neq w, \ f(z) = f(w)\},$$
which is an analytic subvariety of $U \times U$ minus the diagonal. Note that D
is locally given as the graph of a biholomorphic map (and thus is a smooth
3-dimensional submanifold). It suffices to show that we can make a small per-
turbation of coordinates so that

(1) $$\dim D \cap (\Delta_s^2 \times \Delta_s^2) \leq 1$$

for all $s \in \Delta$.

To show (1), we let $\mathcal{P}_n^l$ denote the vector space of polynomials of degree $\leq l$
on $\mathbb{C}^n$. Note that

(2) $$\dim \mathcal{P}_n^l = \binom{l+n}{n}.$$

We also let $\mathcal{J}_a^l(g) \in \mathcal{P}_n^l$ denote the l-jet of a germ $g \in {}_n\mathcal{O}_a$ ($a \in \mathbb{C}^n$). We shall
use the following lemma:

Lemma 5. *Let $\varphi : \Delta^m \to \mathbb{C}^n$ be a holomorphic map such that $\varphi(0) = a$ and*
rank$_0\varphi = m$. *Then*

$$\text{codim}_{\mathcal{P}_n^l}\{f \in \mathcal{P}_n^l : \mathcal{J}_0^l(f \circ \varphi) = 0\} = \binom{l+m}{m}.$$

Proof. By a change of coordinates, we can assume without loss of generality that
$\varphi(z_1, \ldots, z_m) = (z_1, \ldots, z_m, 0, \ldots, 0)$. The result then follows immediately from
(1).   □

We let $\mathcal{Q}^l$ denote the set of polynomials g in $\mathcal{P}_3^l$ such that dg does not vanish on $\Delta^3(2)$. (Note that small polynomial perturbations of the coordinate function z_1 are in $\mathcal{Q}^l$.) Let $a = (z_0, w_0) \in D \cap \Delta^6(2)$ be arbitrary, and let $\mathcal{B}_a$ denote the set of polynomials g in $\mathcal{Q}^5$ with

$$\dim_a \{(z, w) \in D : g(z) = g(z_0), g(w) = g(w_0)\} > 1 \,.$$

We shall show that

$$(3) \qquad\qquad\qquad \operatorname{codim}_{\mathcal{Q}^5} \mathcal{B}_a \geq 6 \,.$$

Since $\dim D = 3$, (3) implies that we can choose $g \in \mathcal{P}_3^5$ such that g is a small deformation of the coordinate function z_1 and

$$\dim D \cap \left(g^{-1}(s) \times g^{-1}(t) \right) \leq 1 \quad \text{for all } s, t \in \Delta \,.$$

If we then replace z_1 with $\tilde{z}_1 = g$, (1) will be satisfied.

To verify (3), we first consider an arbitrary quadratic polynomial $g_1 \in \mathcal{Q}^2$, and we let

$$E = \{(z, w) \in D : g_1(z) = g_1(z_0)\} \,.$$

Since D is locally given as a graph and $dg_1(z_0) \neq 0$, E is smooth at z_0. Now let $\varphi = (\varphi_1, \varphi_2) : \Delta^2 \to E$ with $\varphi(0) = a$ and $\operatorname{rank}_0 \varphi = 2$. This implies that $\operatorname{rank}_0 \varphi_1 = \operatorname{rank}_0 \varphi_2 = 2$. Let $\mathcal{B}'(g_1)$ denote the set of $g_2 \in \mathcal{Q}^2$ such that $\mathcal{J}_0^2(g_2 \circ \varphi_2) = 0$. By Lemma 5, $\operatorname{codim} \mathcal{B}'(g_1) \geq \binom{4}{2} = 6$. Furthermore, we note that if we replace g_1 with a germ $\tilde{g}_1 \in {}_3\mathcal{O}_{z_0}$ with the same 2-jet at z_0, then we can choose $\widetilde{\varphi}_2$ with the same 2-jet (at 0) as φ_2 so that $(\varphi_1, \widetilde{\varphi}_2) : \Delta^2 \to E$ has the same 2-jet (at 0) as φ. Thus if $g_2 \in \mathcal{Q}^2 \setminus \mathcal{B}'(g_1)$, we have $\mathcal{J}_0^2(g_2 \circ \widetilde{\varphi}_2) = \mathcal{J}_0^2(g_2 \circ \varphi_2) \neq 0$. Furthermore, if we also replace this g_2 with $\tilde{g}_2 \in {}_3\mathcal{O}_{w_0}$ with the same 2-jet at w_0, then $\mathcal{J}_0^2(\tilde{g}_2 \circ \widetilde{\varphi}_2) \neq 0$, and hence

$$\dim_a \{(z, w) \in D : \tilde{g}_1(z) = g_1(z_0),\ \tilde{g}_2(w) = g_2(w_0)\} = 1 \,.$$

Now consider the linear map

$$\tau_a : \mathcal{P}_3^5 \to \mathcal{P}_3^2 \times \mathcal{P}_3^2 \,, \quad g \mapsto \left(\mathcal{J}_{z_0}^2(g), \mathcal{J}_{w_0}^2(g) \right) \,.$$

By the above discussion, $\mathcal{B}_a \subset \tau_a^{-1}(\mathcal{B}'_a)$, where

$$\mathcal{B}'_a = \{(g_1, g_2) : g_1 \in \mathcal{Q}^2,\ g_2 \in \mathcal{B}'(g_1)\} \,.$$

Since τ is surjective, it follows that

$$\operatorname{codim}_{\mathcal{Q}^5} \mathcal{B}_a \geq \operatorname{codim}_{\mathcal{Q}^2 \times \mathcal{Q}^2} \mathcal{B}'_a \geq 6 \,.$$

This completes the verification of (1), and hence conditions (a)–(e) above can be satisfied.

3. Proof of Proposition 3

We are now prepared to prove the Hartogs extension property. By our construction above, we may assume that the map $f : H_1^2(r) \to X$ of Proposition 3 possesses the following properties:

i) f is non-degenerate and holomorphic on a neighborhood of $\overline{\Delta \times A^2(r,1)}$;

ii) for all $s \in \Delta$, the set A_s^2 contains no curves contracted by f to a point;

iii) for all $s \in \Delta$, there do not exist nonempty disjoint open subsets V_1, V_2 of A_s^2 with $f(V_1) = f(V_2)$;

We must show that f extends meromorphically to Δ^3 minus a discrete set of points. Denote by W some open subset of Δ such that f can be meromorphically extended onto the Hartogs domain

$$H_W(r) := \left[W \times \Delta^2\right] \cup \left[\Delta \times A^2(r,1)\right].$$

Let Ω be a strictly positive $(2,2)$-form on X with $dd^c\Omega = 0$. Existence of such a form on the compact 3-dimensional manifold X follows from the absence of nonconstant plurisubharmonic functions on X via duality and the Hahn-Banach theorem. In fact even more is true. Every Hermitian metric on X is conformally equivalent to a metric such that the square of its associated $(1,1)$-form ω is dd^c-closed, see [Ga]. In the sequel, we shall take $\Omega = \omega^2$, where ω is such a Gauduchon form. Denote by T the pull-back of Ω by f, i.e. $T = f^*\Omega$. More accurately, $f^*\Omega$ is defined in the case of meromorphic f as follows. Let $\widetilde{\Gamma}_f$ denote the desingularization of the graph $\Gamma_f \subset H_W(r) \times X$ of f and let $\pi_1 : \widetilde{\Gamma}_f \to H_W(r)$ and $\pi_2 : \widetilde{\Gamma}_f \to X$ be the natural projections. Note that π_1 is proper by the very definition of meromorphic map. Define

(4) $$f^*\Omega := \pi_{1*}\pi_2^*\Omega.$$

The current $T = f^*\Omega$ is a positive bidegree $(2,2)$ current on $H_W(r)$. Being the push-forward of a smooth form (on a desingularization of Γ_f), T has coefficients in $\mathcal{L}_{loc}^1(H_W(r))$.

To see that the push-forward of a smooth form η by a modification $\pi : \widetilde{M} \to M$ has coefficients in $\mathcal{L}_{loc}^1$, it suffices to show that $\pi_*\eta$ has no mass on the center C of π. (In our case $C = I_f$.) But for any test form φ on M and any sequence $\rho_n \to \chi_C$ with $0 \le \rho_n \le 1$, we have $(\pi_*\eta, \rho_n\varphi) = \int_{\widetilde{M}} \eta \wedge \pi^*(\rho_n\varphi) \to 0$. Hence $\|\pi_*\eta\|(E) = 0$. (In fact, this holds when π is any surjective holomorphic map that is proper on $\text{Supp}\,\eta$.)

It follows immediately from (4) that $dd^cT = 0$. Moreover, T is smooth on $H_W(r) \setminus I_f$, since outside the set I_f of indeterminacy points of f, it is the usual pull-back of the smooth form Ω.

We write $\Delta_s^2 := \{s\} \times \Delta^2$, for $s \in \Delta$. The function

$$\mu(s) := \int_{\Delta_s^2} f^*\Omega$$

is well defined for all $s \in W$, since by the above, $(f^*\Omega)|_{\Delta_s^2} = (f_{\Delta_s^2})_*\Omega$ is a positive, bidegree $(2,2)$-current on a neighborhood of $\overline{\Delta_s^2}$ and is in $\mathcal{L}_{\mathrm{loc}}^1$. We remark that $\mu(s)$ is nothing but the volume of $f(\Delta_s^2)$ with respect to Ω counted with multiplicities.

Lemma 6. *The function μ is positive and smooth on W, and its Laplacian $\Delta\mu$ smoothly extends onto the whole unit disk Δ.*

Proof. (We follow the method of proof of [Iv2, Lemma 3.1].) The positivity of μ follows from the positivity of $f^*\Omega$ and property (ii) above. To show that $\Delta\mu$ extends to the unit disk, we begin by writing

$$T = \sum_{\alpha,\beta=1}^{3} t_{\alpha\bar\beta} dz^{\overset{\alpha}{\vee}} \wedge d\bar z^{\overset{\beta}{\vee}},$$

where $dz^{\overset{1}{\vee}} = dz_2 \wedge dz_3$, $dz^{\overset{2}{\vee}} = -dz_1 \wedge dz_3$, $dz^{\overset{3}{\vee}} = dz_1 \wedge dz_2$. The function μ is then given by

$$\mu(z_1) = \int_{\Delta^2} t_{1\bar 1}(z_1, z_2, z_3) dz_2 \wedge d\bar z_2 \wedge dz_3 \wedge d\bar z_3.$$

Let

$$T^\varepsilon = \sum_{\alpha,\beta=1}^{3} t_{\alpha\bar\beta}^\varepsilon dz^{\overset{\alpha}{\vee}} \wedge d\bar z^{\overset{\beta}{\vee}}$$

be the smoothing of T by convolution; the T^ε are smooth forms converging to T in $\mathcal{L}^1$ as $\varepsilon \to 0$. On $H_W(r) \setminus I_f$ the convergence is in the $\mathcal{C}^\infty$ topology. The functions

$$\mu^\varepsilon(z_1) := \int_{\Delta_{z_1}^2} T^\varepsilon = \int_{\Delta_{z_1}^2} t_{1\bar 1}^\varepsilon(z_1, z_2, z_3) dz_2 \wedge d\bar z_2 \wedge dz_3 \wedge d\bar z_3$$

are smooth in W.

The condition $dd^c T = 0$ reads as

$$\sum_{\alpha,\beta} \frac{\partial^2 t_{\alpha\bar\beta}}{\partial z_\alpha \partial \bar z_\beta} = 0.$$

So,

$$
\begin{aligned}
\Delta\mu^\varepsilon(z_1) &= 4\int_{\Delta^2_{z_1}} \frac{\partial^2 t^\varepsilon_{1\bar1}}{\partial z_1 \partial \bar z_1} dz_2 \wedge dz_3 \wedge d\bar z_2 \wedge d\bar z_3 \\
&= -4\int_{\Delta^2_{z_1}} \sum_{(\alpha,\beta)\neq(1,1)} \frac{\partial^2 t^\varepsilon_{\alpha\bar\beta}}{\partial z_\alpha \partial \bar z_\beta} dz_2 \wedge d\bar z_2 \wedge dz_3 \wedge d\bar z_3 \\
&= -4\int_{\partial\Delta^2_{z_1}} \left[\sum_{\alpha=2}^{3} \pm \frac{\partial t^\varepsilon_{\alpha\bar1}}{\partial \bar z_1} dz_{5-\alpha} \wedge d\bar z_2 \wedge d\bar z_3 \right. \\
&\qquad\qquad \left. + \sum_{\alpha=1}^{3}\sum_{\beta=2}^{3} \pm \frac{\partial t^\varepsilon_{\alpha\bar\beta}}{\partial z_\alpha} dz_2 \wedge dz_3 \wedge d\bar z_{5-\beta} \right].
\end{aligned}
$$

Since f is holomorphic on a neighborhood of $\Delta \times \partial\Delta^2$, the current T is smooth on $\Delta \times \partial\Delta^2$ and thus $\Delta\mu^\varepsilon$ converges smoothly to the function ψ given by

$$
(5) \quad \psi(z_1) = -4\int_{\partial\Delta^2_{z_1}} \left[\sum_{\alpha=2}^{3} \pm \frac{\partial t_{\alpha\bar1}}{\partial \bar z_1} dz_{5-\alpha} \wedge d\bar z_2 \wedge d\bar z_3 \right.
$$

$$
\left. + \sum_{\alpha=1}^{3}\sum_{\beta=2}^{3} \pm \frac{\partial t_{\alpha\bar\beta}}{\partial z_\alpha} dz_2 \wedge dz_3 \wedge d\bar z_{5-\beta} \right].
$$

But (5) defines a smooth function on all of Δ. While $\mu^\varepsilon \to \mu$ in $\mathcal{L}^1$ on W, so $\Delta\mu^\varepsilon \to \Delta\mu$ on W. This shows that $\Delta\mu = \psi$ is smooth and smoothly extends onto the disk Δ. Thus μ is also smooth on W. $\qquad\square$

Lemma 7. *Suppose that f is non-degenerate and that there exists a sequence $\{s_n\} \in W$ converging to $s_0 \in \Delta$ such that $\mu(s_n)$ is bounded. Then:*
    1) $f_0 := f|A^2_{s_0}$ meromorphically extends onto $\Delta^2_{s_0}$;
    2) the volumes of the graphs $\Gamma_{f_{s_n}}$ are uniformly bounded in n;
    3) f meromorphically extends onto $U_0 \times \Delta^2$ for some neighborhood U_0 of s_0.

Proof. 1) We let $f_n = f|\Delta^2_{s_n}$ and we write $F_n = f_n(\Delta^2_{s_n})$. We further write $\Sigma_n = f_n(\partial\Delta^2_{s_n})$, $\Sigma_0 = f_0(\partial\Delta^2_{s_0})$. Since the volumes $\mu(s_n)$ of the F_n are uniformly bounded, by Bishop's theorem (see for example [HS]) we can assume, after passing to a subsequence, that F_n converges to a pure 2-dimensional analytic subset F of $X \setminus \Sigma_0$. Note that $\bar F = F \cup \Sigma_0$.

Case 1. $\bar F$ is a subvariety of X.
    In this case

$$
f_0|_{A^2_{s_0}} : A^2_{s_0} \to F^0
$$

is a holomorphic map into an irreducible component F^0 of $\bar F$. If $\widehat{F}^0$ denotes a desingularization of F^0, then f_0 lifts to a meromorphic map $\hat f_0$ from $A^2_{s_0}$ to $\widehat{F}^0$. By the 2-dimensional version of Theorem 1 proved in [Iv3, Cor. 4(b)], $\hat f_0$ extends meromorphically onto $\Delta^2_{s_0}$ minus a finite set of points. If this set is nonempty,

then $\hat{f}_0(\partial\Delta^2_{s_0})$ would not be homologous to zero in $\widehat{F}^0$, and hence $\Sigma_0 = f_0(\partial\Delta^2_{s_0})$ would not be homologous to zero in F^0. But $\Sigma_0 = \lim \Sigma_n = \lim \partial F_n = \partial \lim F_n$ in the sense of currents. Since $\operatorname{Supp} \lim F_n \subset \bar{F}$, Σ_0 is homologous to zero in $\bar{F}$ and therefore in F^0, a contradiction. So f_0 extends onto all of $\Delta^2_{s_0}$.

Case 2. $\bar{F}$ is not a subvariety of X.

Let F^0 be the irreducible component of F containing $f_0(A^2_{s_0}) \setminus \Sigma_0$. Define the analytic space $E = F_0 \cup A^2_{s_0}/\sim$, where the equivalence relation is defined as follows: The points $a \in F^0$ and $b \in A^2_{s_0}$ are equivalent iff $a = f_0(b)$, and necessarily $b' \in A^2_{s_0}$ is equivalent to b iff $f_0(b) = f_0(b')$. By property (ii) above, this is a proper equivalence relation and hence E is a complex space. Let $\pi : E \to \bar{F}^0 \subset X$ be the projection defined by $\pi(a) = a$ for $a \in F^0$ and $\pi(b) = f_0(b)$ for $b \in A^2_{s_0}$. Let $\widehat{E}\overset{\eta}{\to}E$ denote the normalization. By property (iii), the map $f_0 : A^2_{s_0} \to E$ is generically one-to-one and thus is a normalization of its image. By the uniqueness of the normalization, f_0 lifts to a map $\hat{f}_0 : A^2_{s_0} \to \widehat{E}$, i.e., $\eta \circ \hat{f}_0 = f_0$. The map $\hat{f}_0$ is a biholomorphism onto its image.

The boundary $\partial\widehat{E}$, being biholomorphic to $\partial\Delta^2$, is strictly pseudoconvex after shrinking slightly, so by Grauert's theorem, $\widehat{E}$ can be blown down to a normal Stein space. This easily yields an extension of f_0 onto $\Delta^2_{s_0}$.

2) We denote the extension of f_0 onto $\Delta^2_{s_0}$ also by f_0. Let F' be the maximal compact pure 2-dimensional variety contained in $\bar{F}$. (In Case 1 above, $F' = \bar{F}$, whereas in Case 2, $F' = \overline{F \setminus F^0}$.) We consider the pure two-dimensional analytic set

$$\Gamma = \Gamma_{f_0} \cup (I_{f_0} \times F').$$

in $(\Delta \times \Delta^2) \times X$, where I_{f_0} is the (finite) set of points of indeterminacy of f_0.

Step 1. We claim that for all $\varepsilon > 0$ the graph Γ_{f_n} belongs to the ε-neighborhood of Γ, for $n \gg 0$.

Neighborhoods are taken with respect to the Euclidean metric on $\mathbb{C}^3$ and Gauduchon metric on X. (In fact, any choice of metric works as well as this one.) This claim follows immediately from Lemma 8 below.

We say that a sequence of meromorphic maps $f_n : U \to X$ converges to a holomorphic map f_0 on a domain U if for all compact subsets $K \subset U$, $I_{f_n} \cap K = \emptyset$ for $n \gg 0$ and $f_n \to f_0$ uniformly on K.

Lemma 8. *Let $f_n : \bar{\Delta}^2 \to X$ be a sequence of meromorphic maps, where X is a compact complex manifold. Suppose that f_n is holomorphic on A, where $A = A^2(r, 1)$. If there exists a meromorphic map $f_0 : \Delta^2 \to X$ such that $f_n|_A \to f_0|_A$, then $f_n \to f_0$ on $\Delta^2 \setminus I_{f_0}$.*

Lemma 8 is a special case of Proposition 1.1.1 in [Iv4]. (Proposition 1.1.1 in [Iv4] is stated in terms of "strong convergence" of meromorphic maps. However, if $\{f_n\}$ strongly converges to a holomorphic map, then the sequence converges in the above sense. This is the content of the "Rouché principle" of [Iv4, Theorem 1].)

To complete the proof of (2), we consider a point $p \in \Gamma$ and take any open $W \ni p$ adapted to Γ, i.e. biholomorphic to $\Delta^2 \times \Delta^4 = U \times B$ in such a way that $(\bar{U} \times \partial B) \cap \Gamma = \emptyset$. Then for $n \gg 1$, we have $\Gamma_{f_n} \cap (\bar{U} \times \partial B) = \emptyset$ and thus $p \mid_{\Gamma_{f_n}} : \Gamma_{f_n} \cap (U \times B) \to U$ is a d_n-sheeted analytic covering, where $p : U \times B \to U$ is a natural projection.

Step 2. The number $\{d_n\}$ of sheets is uniformly bounded.

Consider the following two cases. *Case 1.* $p \in (I_{f_0} \times F') \setminus \Gamma_{f_0}$. In this case as $W \ni p$ we can take the following neighborhood. Let $p = (a, b)$, where $a \in I_{f_0} \subset \mathbb{C}^3$ and $b \in F' \subset X$. Take a neighborhood of b in X of the form $\Delta^2 \times \Delta$ such that $F' \cap (\bar{\Delta}^2 \times \partial \Delta) = \emptyset$. Then take some small $\Delta^3 \ni a$ in $\mathbb{C}^3$ and put $U = \Delta^2$ and $B = \Delta \times \Delta^3$. If the number d_n of sheets of the analytic cover $\pi_U : \Gamma_{f_n} \cap (U \times B) \to U$ is not bounded, it will contradict the fact that $f_n(\Delta_{s_n}^2) \cap (\bar{\Delta}^2 \times \Delta) = F_n \cap (\Delta^2 \times \Delta)$ has uniformly bounded volume (counted with multiplicities).

One should remark now that boundedness of the number of sheets does not depend on the particular choice of the adapted neighborhood of p.

Case 2. $p \in \Gamma_{f_0}$. Let $W = U \times B \ni p$ be some adapted neighborhood. Find a point $q \in U$ such that all its pre-images $\{q_1, ..., q_N\} = \pi_U^{-1}(q) \cap \Gamma$ are smooth points of Γ and π_U is a biholomorphism between neighborhoods $V_j \ni q_j$ on Γ and V on U. Denote by b_j the projection of q_j into B. Take mutually disjoint polydisks $B_j \subset B$ with centers b_j. Consider $W_j := V_j \times B_j$ as adapted neighborhoods of Γ in q_j. They are adapted also for $\Gamma_{f_n}, n >> 0$. Denote by d_n^j the corresponding number of sheets. If d_n is not bounded then at least one sequence d_n^j is also unbounded. Fix j with d_j^n unbounded.

If $q_j \in (I_{f_0} \times F') \setminus \Gamma_{f_0}$, then everything reduces to Case 1. So let $q_j \in \Gamma_{f_0}$. Perturbing q and thus q_j if necessary, we can suppose that q_j is a point where our map f is holomorphic. More precisely $q_j = (a, f(a))$ for some $a \in \Delta \times \Delta^2 \subset \mathbb{C}^3$. Now the contradiction is immediate, because the graphs Γ_{f_n} uniformly approach Γ_{f_0} while f_n converges to f in a neighborhood of a.

3) We are exactly under the assumptions of Proposition 1.3 of [Iv3], i.e., we can apply the "Continuity Principle." (The condition of boundedness of the cycle geometry is insured by Proposition 1.4 from [Iv3].) This gives us an extension of f onto $U_{s_0} \times \Delta^2$.   □

Let us proceed further with the proof of the theorem. Let W be the maximal open subset of the disc Δ such that f meromorphically extends onto $H_W(r)$.

Lemma 9. $\Delta \setminus W$ *is a closed complete polar set in* Δ.

The proof is the same as that of Lemma 2.4 from [Iv3] and will be omitted.

It suffices to show that there exists $\hat{f} : \Delta^3(1 - \delta) \setminus \{a_j\} \to X$ satisfying the conclusion of Proposition 3 for arbitrary $\delta > 0$. We now repeat the above arguments using two slightly deformed coordinate systems (z_1', z_2, z_3) and (z_1'', z_2, z_3),

where

$$z_1' = z_1 + \varepsilon z_2 + O(|z|^2), \quad z_1'' = z_1 + \varepsilon z_3 + O(|z|^2).$$

Here the $O(|z|^2)$ terms are chosen so that conditions (i)–(iii) at the beginning of this section are satisfied for each of the two coordinate systems, after shrinking r if necessary. (As was shown in §1, these terms can be taken to be polynomials consisting of terms of degrees 2 through 5.) We choose ε and the $O(|z|^2)$ terms to be small enough so that $\Delta^3(1-\delta) \subset \Delta'^3 \subset \Delta^3$, $\Delta^3(1-\delta) \subset \Delta''^3 \subset \Delta^3$, where Δ'^3 and Δ''^3 are the polydisks of radius $1 - \frac{\delta}{2}$ in the new coordinates.

Applying the above argument to the new coordinate systems, we obtain maximal open W', W'' in $\Delta^\sim := \Delta(1 - \frac{\delta}{2})$ such that f extends meromorphically to the Hartogs domains $H_{W'}'(r)$, $H_{W''}''(r)$. We let $S_1 = \Delta^\sim \setminus W$, $S_2 = \Delta^\sim \setminus W'$, $S_3 = \Delta^\sim \setminus W''$. Now consider the coordinates

$$w_1 = z_1, \quad w_2 = z_1', \quad w_3 = z_1''$$

and let U denote the image of $\Delta^3(1-\delta)$ under the coordinate map (w_1, w_2, w_3). (We may assume that z_1', z_1'' are chosen so that the w_j indeed provide coordinates on Δ^3.)

In terms of the w-coordinates, f then extends to a meromorphic map $\hat{f}$ on $U \setminus (S_1 \times S_2 \times S_3)$. Now let s_0 be an arbitrary point in $S := S_1 \times S_2 \times S_3$. We must show that s_0 is an isolated point of S and that $\hat{f}(\partial B_{s_0}(r))$ is not homologous to zero in X, for any ball $B_{s_0}(r)$ centered at s_0 such that $B_{s_0}(r) \cap S = \{s_0\}$.

Since polar sets in $\mathbb{C}$ are of Hausdorff dimension zero, we can choose a polydisk Δ_0^3 about s_0 such that the set $K := S \cap \Delta_0^3$ is compact. An identical proof to that of Lemma 3.3 from [Iv2] now shows that the current $T = f^*\Omega$ has locally summable coefficients on all of Δ_0^3. Hence T extends to a unique current $\tilde{T}$ on Δ_0^3 with $\mathcal{L}_{\mathrm{loc}}^1$ coefficients. The following lemma then tells us that $dd^c\tilde{T}$ is of order 0:

Lemma 10. [Iv2, Proposition 2.3] *Let K be a complete pluripolar, compact set in a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ and T a positive, bidimension $(1,1)$ current in $D \setminus K$. Suppose that:*

1) *$dd^cT \leq 0$ in $D \setminus K$,*
2) *T has locally finite mass in a neighborhood of K,*
3) *dT and d^cT have measure coefficients on $D \setminus K$.*

Then the current $dd^c\tilde{T}$ has measure coefficients in D.

(Condition (3) on dT and d^cT was omitted in [Iv2], but is used in the proof. In our case $T = \pi_{1*}\pi_2^*\Omega$, so this condition follows from the fact that $dT = \pi_{1*}\pi_2^*d\Omega$ is the push-forward of a smooth form by a proper map, and similarly for d^cT.)

Since $dd^cT = 0$, the support of the current $dd^c\tilde{T}$ must be contained in K. We also conclude from the Lemma 2.6 in [Iv3] that $dd^c\tilde{T} \leq 0$. Thus we can write $dd^c\tilde{T} = \nu\omega_e^3$, where ω_e is the Euclidean Kähler form on $\mathbb{C}^3$. Then for any ball $B_{s_0}(r) \subset\subset \Delta_0^3$ about s_0 with $\partial B_{s_0}(r) \cap K = \emptyset$, we have that either

$$(6) \qquad\qquad\qquad \nu(K \cap B_{s_0}(r)) = 0,$$

or

$$0 \; > \; \nu(K \cap B_{s_0}(r)) \; = \; \int_{B_{s_0}(r)} dd^c \tilde{T} \; = \; \lim_{\varepsilon \to 0} \int_{B_{s_0}(r)} dd^c \tilde{T}_\varepsilon$$

$$(7)$$

$$= \; \lim_{\varepsilon \to 0} \int_{\partial B_{s_0}(r)} d^c \tilde{T}_\varepsilon \; = \; \int_{\partial B_{s_0}(r)} d^c T_\varepsilon \; = \; \int_{f(\partial B_{s_0}(r))} d^c \Omega \, .$$

Case 1. $\nu(K \cap B_{s_0}(r)) = 0$.

In this case the negativity of $\tilde{T}$ implies that $dd^c \tilde{T} = 0$. Therefore we can find a polydisk neighborhood $\Delta^3 \subset B_{s_0}(r)$ of s_0 and a $(2,1)$-form Γ in Δ^3 such that:

1) f is holomorphic in a neighborhood of $\Delta \times \partial\Delta^2$ (and therefore $\tilde{T}$ is smooth there);
2) $\tilde{T} = i(\partial\bar{\Gamma} - \bar{\partial}\Gamma)$ in a neighborhood of $\bar{\Delta}^3$;
3) Γ is smooth in a neighborhood of $\Delta \times \partial\Delta^2$.

The zero-dimensionality of K implies that there exists a nonempty open $W \subset \Delta$ such that f is defined and meromorphic on $W \times \Delta^2$ and that $s_0^1 \in \partial W \cap \Delta$, where s_0^1 is the first coordinate of s_0. As before we let

$$\mu(z_1) = \int_{\Delta^2_{z_1}} \tilde{T} = i \int_{\partial\Delta^2_{z_1}} (\bar{\Gamma} - \Gamma) \, .$$

By the smoothness of Γ, the function μ is bounded. Therefore by Lemma 7, f extends meromorphically to a neighborhood of s_0.

Case 2. $\nu(S \cap B_{s_0}(r)) < 0$.

By (7), the 5-cycle $f(\partial B_{s_0}(r))$ is not homologous to zero in X. Furthermore, $\int_{f(\partial B_{s_0}(r))} d^c \Omega$ depends only on the integer homology class of $f(\partial B_{s_0}(r))$, since $dd^c \Omega = 0$. Hence,

$$\int_{f(\partial B_{s_0}(r))} d^c \Omega \leq -\delta < 0 \, ,$$

where δ is independent of s_0 and r (and depends only on X and Ω). This shows that K is finite, and completes the proof. $\qquad\square$

We remark that our proof gives more. Namely, if $D \subset\subset M$ is a bounded domain with smooth boundary Σ such that $D \cap A \neq \emptyset$ and $\Sigma \cap A = \emptyset$, then $f(\Sigma)$ is not homologous to zero in X.

4. Generalizations and open questions

In [Iv3], the classes $\mathcal{P}_k^-$ and $\mathcal{G}_k$ of complex spaces were introduced. Recall that $\mathcal{P}_k^-$ is the class of normal complex spaces which carry a strictly positive (k,k)-form $\Omega^{k,k}$ with $dd^c\Omega^{k,k} \leq 0$, and $\mathcal{G}_k$ is the subclass of $\mathcal{P}_k^-$ which consists of complex spaces carrying a strictly positive (k,k)-form $\Omega^{k,k}$ with $dd^c\Omega^{k,k} = 0$. Note that $\mathcal{G}_k$ contains all compact complex manifolds of dimension $k+1$.

It is easy to observe that our above proof gives the following more general statement of Proposition 3:

Proposition 11. *Let X be a compact complex manifold in the class $\mathcal{P}_2^-$. Then every meromorphic map $f : H_1^2(r) \to X$ extends meromorphically onto $\Delta^3 \setminus A$, where A is a closed, complete pluripolar subset of Hausdorff dimension zero. If moreover $X \in \mathcal{G}_2$ then A is discrete and for every ball B with center $a \in A$ such that $\partial B \cap A = \emptyset$, $f(\partial B)$ is not homologous to zero in X.*

To consider the extension of mappings from higher dimensional domains, we introduce the Hartogs figures

$$H_d^k(r) := \left[\Delta^d(1-r) \times \Delta^k\right] \cup \left[\Delta^d \times A^k(r)\right] \subset \mathbb{C}^{d+k} \,.$$

We conjecture that the analogous result should hold for meromorphic mappings from $H_d^k(r)$ to compact manifolds (and spaces) in the classes $\mathcal{P}_k^-$ and $\mathcal{G}_k$. In particular, Theorem 1 should be true for meromorphic mappings between equidimensional manifolds in all dimensions. The main difficulty lies in the fact that it is impossible in general to make the reductions (a)–(c) of §1. (Note that reductions (d)–(e) can be achieved in all dimensions.) However, these reductions are unnecessary in the case when our map is locally biholomorphic, as we state below.

Proposition 12. *Let X be a compact complex space of dimension $k+1$. Then every holomorphic map $f : H_1^k(r) \to X$ with zero-dimensional fibers extends meromorphically onto $\Delta^{k+1} \setminus A$, where A is discrete, and for every ball B with center $a \in A$ such that $\partial B \cap A = \emptyset$, $f(\partial B)$ is not homologous to zero in X.*

The proof is by induction on the dimension $n = k+1$. For the inductive step, the function μ is defined in terms of the push-forward of a dd^c-closed, positive (k,k)-form Ω on a desingularization of X.

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