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A COUNTEREXAMPLE IN UNIQUE CONTINUATION

Carlos E. Kenig and Nikolai Nadirashvili

1. Introduction

In 1939, T. Carleman [Car39] showed that if $\Delta u - Vu = 0$ in $\mathbb{R}^2$, $V \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, and u vanishes of infinite order at $x_0 \in \mathbb{R}^2$, then $u = 0$. This was extended to $n \geq 3$ by C. Müller [Mül54]. In the late 70's and early 80's, there was considerable interest, in view of applications to the absence of embedded eigenvalues, in extending the above result to $V \in \mathrm{L}^p_{\mathrm{loc}}$, $p < \infty$ (see the surveys [Ken87] and [Ken89] and [Wol95]). In this direction, we want to recall the result in [JK85], where it is shown that, if $n > 2$ and $V \in \mathrm{L}^{\frac{n}{2}}_{\mathrm{loc}}$, an analogous conclusion can be obtained, and if $n = 2$, $V \in \mathrm{L}^p_{\mathrm{loc}}$, $p > 1$, the same is true. Moreover, in [Ste85], it is shown that it $n > 2$, the same conclusion can be reached if $V \in \mathrm{L}^{\frac{n}{2},\infty}$, the 'weak-type' Lorentz space, provided that the $\mathrm{L}^{\frac{n}{2},\infty}$ norm is small enough.

From several points of view, these results are optimal. Easy examples can be obtained (see [JK85]) for which, for $n > 2$, $V \in \mathrm{L}^p_{\mathrm{loc}}$, for all $p < \frac{n}{2}$, u vanishes of infinite order at x_0, but u is not identically zero. More subtle examples are due to T. Wolff [Wol92b], who shows that the smallness condition on the $\mathrm{L}^{\frac{n}{2},\infty}$-norm, $n > 2$ cannot be removed, and that when $n = 2$, there are $V \in \mathrm{L}^1$, and u vanishing of infinite order at x_0, for which u is not identically zero. Nevertheless, for the applications mentioned above, it would suffice to know that, if $\Delta u - Vu = 0$, and u has compact support, then $u \equiv 0$. Up to now, as was mentioned in [Ken87], [Ken89] and [Wol92a], it was not known if there are examples of $V \in \mathrm{L}^1$, with non-zero u of compact support, verifying this equation. In this note we close this gap in our knowledge, producing such an example, in all dimensions $n \geq 2$. The L^1-norm of the potential V can be taken as small as one likes.

Remark. After this paper was written, T. Wolff informed us of related work by Niculae Mandache [Man], for equations of the form $\Delta u = \vec{V} \cdot \nabla u$.

2. Main theorem

Theorem 1. *There are measurable functions u, V defined on $\mathbb{R}^2$, both supported in $\overline{B}_1$, where B_1 is the open unit disc, which are smooth in B_1, such that u, V, $Vu \in L^1(\mathbb{R}^2)$, and such that*

$$\Delta u - Vu = 0 \text{ in } \mathcal{D}'.$$

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In order to prove the theorem, we will need an inductive construction. Let

$$\begin{aligned} r_k^0 &= 1 - \tfrac{1}{5k}\,, & r_k^1 &= 1 - \tfrac{1}{5k+1}\,, \\ r_k^2 &= 1 - \tfrac{1}{5k+2}\,, & r_k^3 &= 1 - \tfrac{1}{5k+3}\,, \\ r_k^4 &= 1 - \tfrac{1}{5k+4}\,, \end{aligned}$$

so that

$$r_k^0 < r_k^1 < r_k^2 < r_k^3 < r_k^4 < r_{k+1}^0\,, \text{ for } k = 1, 2, \dots .$$

Let

$$\begin{aligned} B_k^4 &= \{x : |x| < r_k^4\}, \\ B_k^3 &= \{x : |x| < r_k^3\}, \\ B_k^2 &= \{x : |x| < r_k^2\}, \\ B_k^1 &= \{x : |x| < r_k^1\}, \\ A_k &= \{x : r_k^0 < |x| < r_k^2\}, \\ D_k &= \{x : r_k^3 < |x| < r_k^4\}. \end{aligned}$$

Finally, let $\phi_k \in \mathcal{C}_o^\infty(B_1)$, $0 \le \phi_k \le 1$, with $\phi_k = 1$ in B_k^3, $\operatorname{supp}\phi_k \subset B_k^4$. Note that $\operatorname{supp}\nabla\phi_k \subset D_k$, $\operatorname{supp}\Delta\phi_k \subset D_k$. We make a few remarks about these sets:

$$\begin{aligned} \operatorname{dist}(A_k, \partial B_1) &\simeq \tfrac{1}{k}\,, & \operatorname{dist}(A_k, D_k) &\simeq \tfrac{1}{k}\,, \\ \operatorname{dist}(\partial A_k, \partial B_k^1) &\simeq \tfrac{1}{k}\,, & \operatorname{dist}(D_k, \partial B_1) &\simeq \tfrac{1}{k}\,, \\ \operatorname{dist}(D_k, A_{k+1}) &\simeq \tfrac{1}{k}\,, & \operatorname{dist}(D_k, D_{k+1}) &\simeq \tfrac{1}{k}\,. \end{aligned}$$

3. The construction

We define $u_1 \equiv 1$ and now, for $k = 1, 2, \dots$, we define u_k inductively. Thus, assume that u_k has been defined, and we proceed to construct u_{k+1}.

Let $v_k = \phi_k u_k$, $f_k = \Delta(\phi_k u_k)$, so that v_k solves

$$\begin{cases} \Delta v_k = f_k \text{ in } B_1 \\ v_{k|\partial B_1} \equiv 0. \end{cases}$$

Let now α_n, $n = 1, 2, \dots$ be a sequence of distributions of the form

$$\alpha_n = \sum_{i=1}^{i_n} a_i \delta_{x_i^n}\,,$$

where $\delta_{x_i^n}$ is the delta mass at $x_i^n \in D_k$, and chosen so that

$$\alpha_n \to f_k \text{ weakly in } \overline{D}_k \text{ as } n \to \infty.$$

For fixed n, set

$$\alpha_n^\epsilon = \sum_{i=1}^{i_n} a_i \delta_{x_i^n}^\epsilon\,,$$

where $\delta^{\epsilon}_{x_i^n}$ is a smoothing of $\delta_{x_i^n}$, by a non-negative smooth function, supported in an ϵ neighborhood of x_i^n. We will always choose ϵ small so that

$$\mathrm{supp}\alpha_n^\epsilon \subset D_k\,.$$

Let now v_n^ϵ solve

$$\begin{cases} \Delta v_n^\epsilon = \begin{cases} f_k \text{ on } B_1 \backslash \overline{D}_k \\ \alpha_n^\epsilon \text{ on } \overline{D}_k \end{cases} \\ v^\epsilon_{n|\partial B_1} \equiv 0 \end{cases}$$

Note that as $n \to \infty$, and then $\epsilon \to 0$, $v_n^\epsilon \to v_k$. Now, choose first n_0 so large, and then ϵ_0 so small that

$$|v_{n_0}^{\epsilon_0} - v_k| \le \frac{1}{8^k} \text{ on } B_k^2 \cup A_{k+1}$$

and so that

$$\|\Delta(\phi_{k+1} v_{n_0}^{\epsilon_0})\|_{L^1(D_{k+1})} \le \frac{1}{2^{k+3}}.$$

The first condition is a direct consequence of th weak convergence of α_n. For the second one, note that on D_{k+1}, $f_k \equiv 0$, and $v_k \equiv 0$, $\nabla v_k \equiv 0$.

$$\begin{aligned} \Delta(\phi_{k+1} v_{n_0}^{\epsilon_0}) = \phi_{k+1}\Delta v_{n_0}^{\epsilon_0} + 2\nabla\phi_{k+1}\nabla v_{n_0}^{\epsilon_0} + (\Delta\phi_{k+1})v_{n_0}^{\epsilon_0} \\ = 2\nabla\phi_{k+1}\nabla v_{n_0}^{\epsilon_0} + \Delta\phi_{k+1} v_{n_0}^{\epsilon_0}\,, \end{aligned}$$

and so the second condition also follows from the weak convergence.

We may also assume, without loss of generality, that

$$\|\alpha_{n_0}^{\epsilon_0}\|_{L^1(D_k)} \le \|f_k\|_{L^1(D_k)}\,,$$

and since $|v_{n_0}^{\epsilon_0}| \to \infty$ on $\mathrm{supp}\alpha_n^{\epsilon_0}$, as $\epsilon_0 \to 0$, we may assume that

$$|v_{n_0}^{\epsilon_0}| \ge 1 \text{ on } \mathrm{supp}\alpha_{n_0}^{\epsilon_0}\,.$$

We will now define $u_{k+1} = v_{n_0}^{\epsilon_0}$.

We will next deduce a few properties of u_k.

4. Properties of u_k

(P1) $u_{k+1} \in \mathcal{C}^\infty(B_1)$. Moreover, $\mathrm{supp}\Delta u_{k+1} \subset \cup_{j=1}^k D_k$.

Proof. We will prove the two statements inductively. For $k = 1$, recall that

$$\Delta u_2 = \begin{cases} f_1 \text{ in } B_1 \backslash \overline{D}_1 \\ \alpha_{n_0}^{\epsilon_0} \text{ in } \overline{D}_1. \end{cases}$$

But, $f_1 = \Delta(u_1\phi_1) = \Delta(\phi_1)$, and since $\mathrm{supp}\Delta(\phi_1) \subset D_1$, f_1 is 0 in $B_1 \backslash \overline{D}_1$. Moreover, $\mathrm{supp}\alpha_{n_0}^{\epsilon_0} \subset D_1$, and so, clearly, Δu_2 is supported in D_1, and is smooth. But then u_2 is also smooth in B_1. Assume that both statements hold up to k.

$$\Delta u_{k+1} = \begin{cases} f_k \text{ on } B_1 \backslash \overline{D}_k \\ \alpha_{n_0}^{\epsilon_0} \text{ on } \overline{D}_k. \end{cases}$$

In $B_1\backslash\overline{B_k^4}$, $\phi_k \equiv 0$, and so $f_k \equiv 0$. In B_k^3, $\phi_k \equiv 1$, and so $f_k = \Delta u_k$. Hence, both statements hold up to $k+1$. □

$$\text{(P2)} \qquad |u_{k+1}| \le \frac{1}{8^k} \text{ on } A_{k+1}.$$

Proof. On A_{k+1}, $\phi_k \equiv 0$ and so,

$$|u_{k+1}| = |u_{+1} - \phi_k u_k| \le \frac{1}{8^k}.$$

□

$$\text{(P3)} \qquad \int_{B_1} |\Delta(\phi_{k+1}u_{k+1})| \le C, \text{ for all } k.$$

Proof. We know that

$$\|\Delta(\phi_{k+1}u_{k+1}\|_{L^1(D_{k+1})} \le \frac{1}{2^{k+3}}.$$

Moreover, in $B_1\backslash\overline{B_{k+1}^4}$, $\phi_{k+1} \equiv 0$, so $\Delta(\phi_{k+1}u_{k+1}) = 0$. By construction, inside B_{k+1}^3, $\phi_{k+1} \equiv 1$ and so $\Delta(\phi_{k+1}u_{k+1}) = \Delta(u_{k+1})$. But in $B_{k+1}^3\backslash B_k^4$, $\phi_k \equiv 0$, and so $\Delta(u_{k+1}) = \Delta(\phi_k u_k) = 0$ there. In D_k, $\Delta u_{k+1} = \alpha_{n_0}^{\epsilon_0}$, and so,

$$\int_{D_k} |\Delta u_{k+1}| \le 2\int_{D_k} |\Delta(\phi_k u_k)| \le \frac{2}{2^{(k-1)+3}} \le \frac{1}{2^k}.$$

Gathering the information, we obtain

$$\|\Delta\phi_{k+1}u_{k+1}\|_{L^1(B_1)} \le \|\Delta\phi_k u_k\|_{L^1(B_1)} + \frac{1}{2^{k+3}} + \frac{1}{2^k},$$

and (P3) follows. □

$$\text{(P4)} \qquad \int_{B_1} |\phi_{k+1}u_{k+1}| \le C \text{ for all } k.$$

This is immediate from (P3).

Proof of the theorem. We first claim that $\{u_k\}$ converges uniformly on compact subsets of B_1, to a function u, which is smooth in B_1 and for which $\text{supp}\Delta u \subset \cup_{k=1}^{\infty} D_k$, and such that $|u| > \frac{1}{2}$ on $\text{supp}\Delta u$.

Proof of claim. Fix $r < 1$, and choose k_0 so that $\overline{B_r} \subset B_{k_0}^2$, and hence, $\overline{B_r} \subset B_k^2$ for all $k \ge k_0$. For $n, m \ge k_0$, $n > m$, we have that $\phi_j \equiv 1$ on $\overline{B_r}$, $j = m, \dots, n-1$, and so

$$|u_m - u_n| \le \sum_{k=m}^{\infty} \frac{1}{8^k},$$

and thus we have the uniform convergence. Note also that (P1) implies that all the u_k's are harmonic outside of $\cup_{j=1}^{\infty} D_j$, and hence, so is u. Next, note that $\Delta u_k = \Delta u_{k_0}$ in $\overline{B_r}$, for $k \ge k_0$. This is because, for $k > k_0$, $D_k \subset B_1\backslash\overline{B_r}$, and

$\phi_{k-1} \equiv 1$ on $\overline{B_r}$. From this it follows that $\Delta u = \Delta u_{k_0}$ in $\overline{B_r}$, and hence, by (P1), Δu is smooth in B_r, and hence so is u.

We finally need to check that $|u| > \frac{1}{2}$ on $\text{supp}\Delta u$. It is enough to do it on $\text{supp}\Delta u \cap D_k$, for each k. Fix such a k, and note that, as before, we have for $j > k$, $\Delta u_j = \Delta u_{k+1}$ on D_k: since $D_k \subset B_j^3$, and so $\Delta u_j = \Delta(\phi_{j-1} u_{j-1}) = \Delta u_{j-1}$, where the last equality holds as long as $D_k \subset B_{j-1}^3$, or $k < j-1$. The last valid case is when $j - 1 = k + 1$, as claimed. On D_k, $\Delta u_{k+1} = \alpha_{n_0}^{\epsilon_0}$, and so, on $D_k \cap \text{supp}\Delta u = D_k \cap \text{supp}\Delta u_{k+1}$, we have that $|v_{n_0}^{\epsilon_0}| > 1$, i.e., $|u_{k+1}| > 1$. If $j > k+1$, $D_k \subset B_j^2$, $D_k \subset B_j^3$, and so $|u_j - u_{j-1}| < \frac{1}{8^j}$. Thus, if $j > k+1$, $|u_j - u_{k+1}| \le \sum_{j=k+2}^{\infty} \frac{1}{8^j} \le \frac{1}{2}$, and the last claim follows. Next, we claim that

$$\int_{B_1} |\Delta u| \le C,$$

$$\int_{B_1} |u| \le C.$$

These are immediate consequences of (P3) and (P4).

Finally, we define $u = 0$ outside B_1. We let $V = \Delta u / u$ in $\text{supp}\Delta u \cap B_1$, and 0 elsewhere. Note that, since $|u| > \frac{1}{2}$ on $\text{supp}\Delta u \cap B_1$, V is well defined, and $\Delta u = Vu$ pointwise in B_1. Note also that since $\Delta u \in L^1(B_1)$, $|V| \le 2|\Delta u|$, we have that $V \in L^1(B_1)$, $Vu \in L^1(B_1)$. Finally, we will check that $\Delta u - Vu = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. In order to check this, we first note that $|u| < \frac{1}{4^k}$ on A_{k+1}. Indeed, by (P2), $|u_{k+1}| \le \frac{1}{8^k}$ on A_{k+1}, and if $j > k+1$, $A_{k+1} \subset B_j^2$, and hence $|u_j - \phi_{j-1} u_{j-1}| < \frac{1}{8^j}$, and also $A_{k+1} \subset B_{j-1}^3$, and so $\phi_{j-1} \equiv 1$ there.

Note also that u is harmonic in A_{k+1}, and hence, by interior estimates we have $|\nabla u| \le \frac{C}{2^k}$ in ∂B_{k+1}^1. Let $\psi \in \mathcal{C}_o^\infty(\mathbb{R}^2)$. We need to check that

$$\int_{\mathbb{R}^2} [u\Delta\psi - Vu\psi] = 0.$$

The above integral equals

$$\int_{B_1} [u\Delta\psi - Vu\psi] = \lim_{k\to\infty} \int_{B_{k+1}^1} [u\Delta\psi - Vu\psi],$$

since $u \in L^1(B_1)$, $Vu \in L^1(B_1)$, $\psi \in \mathcal{C}_o^\infty(\mathbb{R}^2)$. Now,

$$\begin{aligned} \int_{B_{k+1}^1} [u\Delta\psi - Vu\psi] &= \int_{B_{k+1}^1} [u\Delta\psi - \Delta u\psi] \\ &= \int_{\partial B_{k+1}^1} \left[u\frac{\partial\psi}{\partial n} - \frac{\partial u}{\partial n}\psi \right], \end{aligned}$$

and so

$$\left| \int_{B_{k+1}^1} [u\Delta\psi - Vu\psi] \right| \le \frac{C}{4^k} + \frac{C}{2^k},$$

and the desired result follows.

Remark. Since we can make $v_{n_0}^{\epsilon_0}$ as large as we please on $\text{supp}\alpha_{n_0}^{\epsilon_0}$, we can take the L^1 norm of V as small as we like.

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Department of Mathematics, University of Chicago, Chicago, IL 60637.
E-mail address: `cek@math.uchicago.edu, nicholas@math.uchicago.edu`