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ϵ-CONSTANTS AND ARAKELOV EULER CHARACTERISTICS

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1. Introduction

Let X be a regular scheme projective and flat over $\mathrm{Spec}(\mathbb{Z})$, equidimensional of relative dimension d. Consider the Hasse-Weil zeta function of X, $\zeta(X,s) = \prod_x (1 - N(x)^{-s})^{-1}$ where x ranges over the closed points of X and $N(x)$ is the order of the residue field of x. Denote by $L(X,s)$ the zeta function with Γ-factors $L(X,s) = \zeta(X,s)\Gamma(X,s)$. The L-function conjecturally satisfies a functional equation

$$L(X,s) = \epsilon(X)A(X)^{-s}L(X, d+1-s)$$

where $\epsilon(X)$ and $A(X)$ are real numbers defined independently of any conjectures (the "ϵ-constant" and the "conductor"). In fact, the unconditional definition of $\epsilon(X)$ and $A(X)$ involve choices of auxiliary primes l with embeddings $\mathbb{Q}_l \subset \mathbf{C}$ (see [De]). In this note, we will suppress any notation regarding these choices; this should not cause any confusion.

The purpose of this note is to explain a way to obtain the absolute value $|\epsilon(X)|$ as an "arithmetic" Euler de Rham characteristic in the framework of the higher dimensional Arakelov theory of Gillet and Soulé. Choose a hermitian metric on the tangent bundle of $X(\mathbf{C})$ which is Kähler; it gives a hermitian metric on $\Omega^1_{X_\mathbf{C}}$. Recall the definition of the arithmetic Grothendieck group $\widehat{\mathrm{K}}_0(X)$ of hermitian vector bundles of Gillet and Soulé ([GS1, II, §6]; all hermitian metrics are smooth and invariant under the complex conjugation on $X(\mathbf{C})$). There is an arithmetic Euler characteristic homomorphism

$$\chi_Q : \widehat{\mathrm{K}}_0(X) \longrightarrow \mathbb{R},$$

such that if $(\mathcal{F}, h)$ is a vector bundle on X with a hermitian metric on $\mathcal{F}_\mathbf{C}$, then $\chi_Q((\mathcal{F}, h))$ is the Arakelov degree of the hermitian line bundle on $\mathrm{Spec}(\mathbb{Z})$ formed by the determinant of the cohomology of $\mathcal{F}$ with its Quillen metric. The arithmetic Grothendieck group $\widehat{\mathrm{K}}_0(X)$ is a λ-ring with λ^i-operations defined in loc. cit. §7: If $(\mathcal{F}, h)$ is the class of a vector bundle with a hermitian metric on $\mathcal{F}_\mathbf{C}$ then $\lambda^i((\mathcal{F}, h))$ is the class of the vector bundle $\wedge^i\mathcal{F}$ with the exterior power metric on $\wedge^i\mathcal{F}_\mathbf{C}$ induced from h. Now consider the sheaf of differentials $\Omega^1_{X/\mathbb{Z}}$;

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this is a "hermitian coherent sheaf" in the terminology of [GS3, 2.5]. Since X is regular, by loc. cit. 2.5.2, $\Omega^1_{X/\mathbb{Z}}$ defines an element Ω in $\widehat{\mathrm{K}}_0(X)$ as follows: Each embedding of X into projective space over $\mathrm{Spec}(\mathbb{Z})$ gives a short exact sequence

$$\mathcal{E} \;:\; 0 \;\to\; N \;\to\; P \;\to\; \Omega^1_{X/\mathbb{Z}} \;\to\; 0$$

with P and N vector bundles on X (here P is the restriction of the relative differentials of the projective space to X and N is the conormal bundle of the embedding). Pick hermitian metrics h^P and h^N on $P_{\mathbf{C}}$ and $N_{\mathbf{C}}$ respectively and denote by $\widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}})$ the secondary Bott-Chern characteristic class of the exact sequence of hermitian vector bundles $\mathcal{E}_{\mathbf{C}}$ (as defined in [GS1]; there is a difference of a sign between this definition and the definition in [GS3, 2.5.2]). Then

$$\Omega = ((P, h^P), 0) - ((N, h^N), 0) + ((0,0), \widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}})) \in \widehat{\mathrm{K}}_0(X)$$

depends only on the original choice of Kähler metric.

For each $i \geq 0$ we can consider now the element $\lambda^i(\Omega)$ in $\widehat{\mathrm{K}}_0(X)$. Motivated by the "higher dimensional Fröhlich conjecture" of [CEPT], we conjecture that

$$-\log|\epsilon(X)| = \sum_{i=0}^{d}(-1)^i \chi_Q(\lambda^i(\Omega)). \tag{1.1}$$

Denote by X_S the disjoint union of the singular fibers of $f : X \to \mathrm{Spec}(\mathbb{Z})$. In [B], S. Bloch conjectures that the conductor $A(X)$ is given by

$$A(X) = \mathrm{ord}((-1)^d c^{X_S}_{d+1}(\Omega^1_{X/\mathbb{Z}})),$$

where $c^{X_S}_{d+1}(\Omega^1_{X/\mathbb{Z}}) := c^X_{d+1,X_S}(\Omega^1_{X/\mathbb{Z}}) \cap [X]$ is the localized $d+1$-st Chern class in $\mathrm{CH}_0(X_S)$ described in loc. cit. Here for a zero cycle $\sum_i n_i x_i$, $\mathrm{ord}(\sum_i n_i x_i) = \prod_i (\# k(x_i))^{n_i}$, with $k(x_i)$ the residue field of x_i. In this paper we show:

Theorem 1.2. *The equality* 1.1 *is equivalent to Bloch's conjecture.*

The main ingredients in the proof are the Arithmetic Riemann-Roch theorem of Gillet and Soulé and the fact (Proposition 3.1) that Bloch's localized Chern class agrees with the corresponding "arithmetic" Chern class of Gillet-Soulé.

Since Bloch has proven in [B] his conjecture for an arithmetic surface ($d = 1$) we see that 1.1 holds in this case. In this note we also show:

Theorem 1.3. *Bloch's conjecture, and therefore equality* 1.1*, holds when for all primes p, the fiber of $X \to \mathrm{Spec}(\mathbb{Z})$ over p is a divisor with strict normal crossings with multiplicities relative prime to p.*

In fact, under the hypothesis of the above theorem, we can show 1.1 directly by replacing the use of the arithmetic Riemann-Roch theorem by Serre duality and the fact, due to Ray and Singer ([RS], Theorem 3.1), that the analytic torsion of the de Rham complex is trivial. We are grateful to C. Soulé for pointing this out to us; this approach is explained in detail in [CPT2]. Also, as $\epsilon(X)^2 = A(X)^{d+1}$, we could have expressed 1.1 using the conductor $A(X)$. However, it seems that 1.1 is more canonical and it could generalize in a motivic framework (for example to varieties with a group action). Indeed, the inspiration for 1.1 comes from [CEPT], see also [CPT1], where we observed a close connection between an equivariant version of an Euler de Rham characteristic as above and ϵ-constants. Viewed this way, Theorem 1.2 also provides some indirect positive evidence for the general higher dimensional Fröhlich conjecture of [CEPT]. In [CPT2], we use the results of this note to obtain the actual ϵ-constant (not just its absolute value) of the Artin motive obtained from the pair (X, V) of an arithmetic variety X with an action of a finite group G and a symplectic character V of G.

We would like to express our thanks to C. Soulé; this note would not have existed without his advice. We would also like to thank T. Saito for useful conversations and B. Erez for pointing out the reference [A]. After a preliminary version of this note was completed we have learned that K. Kato and T. Saito have announced a proof of a stronger version of Theorem 1.3 in which the assumption on the multiplicities is dropped; their proof is significantly more involved than the proof of the tame case that we consider here. T. Saito informed us that a similar argument to ours for the proof of the tame case is given by K. Arai in his thesis, which is currently in preparation.

2. Arithmetic Riemann-Roch

The formulae of [De] imply that $\epsilon(X)^2 = A(X)^{d+1}$ (we can see that this also follows directly from the conjectural functional equation). Therefore, 1.1 translates to

$$\frac{d+1}{2} \cdot \log A(X) = -\sum_{i=0}^{d} (-1)^i \chi_Q(\lambda^i(\Omega)). \tag{2.1}$$

Denote by $\widehat{\mathrm{CH}}^{\cdot}(X)$, $\widehat{\mathrm{CH}}_{\cdot}(X)$ the arithmetic Chow groups of Gillet and Soulé ([GS1-2]), graded by codimension and dimension of cycles respectively. Since X_S has empty generic fiber, there is a natural homomorphism

$$z_S : \mathrm{CH}_0(X_S) \to \widehat{\mathrm{CH}}_0(X) = \widehat{\mathrm{CH}}^{d+1}(X).$$

The direct image homomorphism

$$f_* : \widehat{\mathrm{CH}}^{d+1}(X) \to \widehat{\mathrm{CH}}^{1}(\mathrm{Spec}(\mathbb{Z})) = \mathbb{R}$$

satisfies $f_*(z_S(a)) = \log(\mathrm{ord}(a))$ for $a \in \mathrm{CH}_0(X_S)$. Therefore, Theorem 1.2 will follow if we show:

Theorem 2.2. $$\sum_{i=0}^{d} (-1)^i \chi_Q(\lambda^i(\Omega)) = (-1)^{d+1} \frac{d+1}{2} f_*(z_S(c_{d+1}^{X_S}(\Omega^1_{X/\mathbb{Z}}))).$$

In what follows we will use heavily the notations and results of [GS1], [GS2] and [GS3].

First observe that from the definition of Ω, we obtain $\mathrm{ch}(\Omega) = \mathrm{ch}(\Omega^1_{X(\mathbf{C})})$, where ch denotes the Chern character form (its domain can be extended to $\widehat{\mathrm{K}}_0(X)$ as in [GS1]). By [GS1, Lemma 7.3.3], we have $\mathrm{ch}(\Omega^i_{X(\mathbf{C})}) = \lambda^i(\mathrm{ch}(\Omega^1_{X(\mathbf{C})}))$;

here $\Omega^i_{X(\mathbf{C})}$, $0 \le i \le d$, has the exterior power metric and the λ-ring structure on differential forms is given by the grading as in loc. cit. We obtain that $\mathrm{ch}(\lambda^i(\Omega)) = \lambda^i(\mathrm{ch}(\Omega)) = \lambda^i(\mathrm{ch}(\Omega^1_{X(\mathbf{C})})) = \mathrm{ch}(\Omega^i_{X(\mathbf{C})})$ where the first equality follows from the fact that $\mathrm{ch} = \omega \cdot \widehat{\mathrm{ch}} : \widehat{\mathrm{K}}_0(X) \to A(X_{\mathbb{R}})$ is a λ-ring homomorphism (see loc. cit.).

From the Arithmetic Riemann Roch theorem of Gillet and Soulé ([GS3], Theorem 7, see also 4.1.5 loc. cit.) we now have

$$\sum_{i=0}^{d}(-1)^i\chi_Q(\lambda^i(\Omega)) = f_*\left(\left(\widehat{\mathrm{ch}}\Big(\sum_{i=0}^{d}(-1)^i\lambda^i(\Omega)\Big)\cdot\widehat{\mathrm{Td}}(X)\right)^{(d+1)}\right) - \frac{1}{2}\int_{X(\mathbf{C})}\mathrm{ch}\Big(\sum_{i=0}^{d}(-1)^i\Omega^i_{X(\mathbf{C})}\Big)\mathrm{Td}(T_{X(\mathbf{C})})R(T_{X(\mathbf{C})}) \tag{2.3}$$

where the notations are as in loc. cit. and the factor of $1/2$ in front of the second term results from the normalization discussed after equation (15) in section 4.1.5. We first show:

Proposition 2.4. $$\int_{X(\mathbf{C})}\mathrm{ch}\Big(\sum_{i=0}^{d}(-1)^i\Omega^i_{X(\mathbf{C})}\Big)\mathrm{Td}(T_{X(\mathbf{C})})R(T_{X(\mathbf{C})}) = 0.$$

Proof. (Shown to us by C. Soulé.) By the classical identity applied on the level of Chern forms we obtain

$$\mathrm{ch}(\lambda_{-1}(\Omega^1_{X(\mathbf{C})}))\mathrm{Td}(T_{X(\mathbf{C})})) = c_d(T_{X(\mathbf{C})})$$

(see [R, 6.19]). Therefore the integral is equal to:

$$\int_{X(\mathbf{C})} c_d(T_{X(\mathbf{C})})R(T_{X(\mathbf{C})}).$$

But $R(T_{X(\mathbf{C})})$ is non-zero in positive degrees only; therefore the degree of the form $c_d(T_{X(\mathbf{C})})R(T_{X(\mathbf{C})})$ is at least $d+1$ and the integral vanishes. □

It remains to deal with the first term of the right hand side of 2.3. We will show:

Proposition 2.5.

$$\left(\widehat{\mathrm{ch}}\Big(\sum_{i=0}^{d}(-1)^i\lambda^i(\Omega)\Big)\cdot\widehat{\mathrm{Td}}(X)\right)^{(d+1)} = (-1)^{d+1}\frac{d+1}{2}\hat{c}_{d+1}(\Omega).$$

Proof. Recall the definition of $\widehat{\mathrm{Td}}(X)$ from [GS3]; we have an exact sequence

$$\mathcal{E}^*_{\mathbf{C}} : 0 \to T_{X_{\mathbf{C}}} = (\Omega^1_{X_{\mathbf{C}}})^* \to P^*_{\mathbf{C}} \to N^*_{\mathbf{C}} \to 0.$$

We set

$$\widehat{\mathrm{Td}}(X) := \widehat{\mathrm{Td}}(\bar{P}^*)\widehat{\mathrm{Td}}^{-1}(\bar{N}^*) + a(\widetilde{\mathrm{Td}}(\mathcal{E}^*_{\mathbf{C}})\mathrm{Td}(\bar{N}^*_{\mathbf{C}})^{-1}),$$

where $\widetilde{\mathrm{Td}}(\mathcal{E}^*_{\mathbf{C}})$ is the Todd-Bott-Chern secondary form attached to the sequence $\mathcal{E}^*_{\mathbf{C}}$ (see [GS3], p. 503) and Td is the usual Todd form. We are just interested in the terms of degree 0 and 1 of $\widehat{\mathrm{Td}}(X)$. If $\bar{E}$ is a hermitian vector bundle, we have

$$\widehat{\mathrm{Td}}(\bar{E}^*) = 1 + \frac{\hat{c}_1(\bar{E}^*)}{2} + \cdots, \quad \widehat{\mathrm{Td}}^{-1}(\bar{E}^*) = 1 - \frac{\hat{c}_1(\bar{E}^*)}{2} + \cdots.$$

The $(0,0)$ component of $\mathrm{Td}(\bar{N}^*_{\mathbf{C}})^{-1}$ is 1. We can also see that the $(0,0)$ component of the secondary form $\widetilde{\mathrm{Td}}(\mathcal{E}^*_{\mathbf{C}})$ is given by

$$\widetilde{\mathrm{Td}}(\mathcal{E}^*_{\mathbf{C}})^{(0,0)} = \frac{\tilde{c}_1(\mathcal{E}^*_{\mathbf{C}})}{2},$$

where $\tilde{c}_1(\mathcal{E}^*_{\mathbf{C}})$ the "secondary" first Bott-Chern form associated to $\mathcal{E}^*_{\mathbf{C}}$. This gives

$$\widehat{\mathrm{Td}}(X) = 1 + \frac{\hat{c}_1(\bar{P}^*) - \hat{c}_1(\bar{N}^*)}{2} + a(\frac{\tilde{c}_1(\mathcal{E}^*_{\mathbf{C}})}{2}) + \cdots = 1 + \frac{\hat{c}_1(\Omega^*)}{2} + \cdots,$$

and therefore

$$\widehat{\mathrm{Td}}(X) = \widehat{\mathrm{Td}}(\Omega^*) \mod \widehat{\mathrm{CH}}^{\geq 2}(X)_{\mathbb{Q}}. \tag{2.6}$$

Let us now consider the γ operations on the λ-ring $\widehat{\mathrm{K}}_0(X)$ with augmentation $\epsilon : \widehat{\mathrm{K}}_0(X) \to \mathbb{Z}$ given by $\epsilon((\bar{E}, \eta)) = \mathrm{rk}(E)$ (see [R, §4]). If $\epsilon(x) = d$, then (as in [CPT1] §1) we have:

$$(-1)^d \gamma^d (x - \epsilon(x)) = \sum_{i=0}^{d} (-1)^i \lambda^i (x). \tag{2.7}$$

Therefore $\widehat{\mathrm{ch}}(\sum_{i=0}^{d}(-1)^i\lambda^i(x))$ is concentrated in degrees d and $d+1$ only and so by 2.6

$$\widehat{\mathrm{ch}}\Big(\sum_{i=0}^{d}(-1)^i\lambda^i(\Omega)\Big) \cdot \widehat{\mathrm{Td}}(X) = \widehat{\mathrm{ch}}\Big(\sum_{i=0}^{d}(-1)^i\lambda^i(\Omega)\Big) \cdot \widehat{\mathrm{Td}}(\Omega^*).$$

By the above and 2.7 it is enough to show that for $x \in \widehat{\mathrm{K}}_0(X)$ we have

$$(\widehat{\mathrm{ch}}(\gamma^d(x - \epsilon(x)) \cdot \widehat{\mathrm{Td}}(x^*))^{(d+1)} = -\frac{d+1}{2}\hat{c}_{d+1}(x).$$

Let $a_1, \ldots, a_{d+1}$ be the "arithmetic Chern roots" of x. By definition, these are formal symbols such that the arithmetic Chern classes of x are the elementary symmetric functions of a_i; we can perform our calculation using these symbols. The Chern roots of the dual x^* are $-a_1, \ldots, -a_{d+1}$. A standard argument using [GS1, Theorem 4.1] shows that we have

$$\widehat{\mathrm{ch}}(\gamma^d(x - \epsilon(x))) = \sum_{i=0}^{d+1} \prod_{j \neq i} (e^{a_j} - 1),$$

while by definition

$$\widehat{\mathrm{Td}}(x^*) = \prod_{i=0}^{d+1} \frac{-a_i}{1-e^{-(-a_i)}} = \prod_{i=0}^{d+1} \frac{a_i}{e^{a_i}-1}.$$

The product is equal to

$$\sum_{j=1}^{d+1} \frac{a_1 a_2 \cdots a_{d+1}}{e^{a_j}-1} = \sum_{j=1}^{d+1} (a_1 \cdots \widehat{a}_j \cdots a_{d+1} - \frac{a_1 \cdots a_{d+1}}{2}) + \cdots$$

$$= \hat{c}_d(x) - \frac{d+1}{2} \hat{c}_{d+1}(x) + \cdots$$

which gives the desired result. □

3. Localized Chern classes.

We continue with the same assumptions and notations. Recall the homomorphism

$$z_S : \mathrm{CH}_0(X_S)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(X)_{\mathbb{Q}} = \widehat{\mathrm{CH}}^{d+1}(X)_{\mathbb{Q}}.$$

Proposition 3.1. $z_S(c_{d+1}^{X_S}(\Omega^1_{X/\mathbb{Z}})) = \hat{c}_{d+1}(\Omega)$.

Theorem 2.2 follows from Propositions 3.1, 2.4, 2.5 and equation 2.3.

Proof of Proposition 3.1. We review the construction of the localized Chern class via the Grassmannian graph construction (as described in [B] §1, or in [GS3] §1) applied to the complex $0 \to N \overset{\delta}{\to} P$ with cokernel $\Omega^1_{X/\mathbb{Z}}$. Set $U = X - X_S$. Let p be the projection $X \times \mathbf{P}^1 \to X$. Set $M := p^*N(1) \oplus p^*P$ where (1) denotes the Serre twist (which we view as tensoring with the pull-back of $\mathcal{O}_{\mathbf{P}^1}(\infty)$ under $X \times \mathbf{P}^1 \to \mathbf{P}^1$). Let us consider the Grassmannian $\mathrm{Gr}(r, M)$ over $X \times \mathbf{P}^1$ of rank $r = \mathrm{rk}(N)$ local direct summands of M. Denote by $\pi_0 : \mathrm{Gr}(r, M) \to X \times \mathbf{P}^1$ the natural projection morphism. The diagonal embedding $p^*N \subset p^*N(1) \oplus p^*P$ gives a section s of π_0 over the subscheme $(X \times \mathbf{A}^1) \cup (U \times \mathbf{P}^1)$. In fact, over $X \times \mathbf{A}^1$ the image of p^*N can be identified with the graph of δ. Denote by W the Zariski closure of the image

$$s((X \times \mathbf{A}^1) \cup (U \times \mathbf{P}^1)) \subset \mathrm{Gr}(r, M);$$

this is an integral subscheme of $\mathrm{Gr}(r, M)$ which is called the Grassmannian graph of $N \to P$. The morphism $\pi := \pi_{0|W}$ is projective and gives an isomorphism on the generic fibers. Let W_∞ be the effective Cartier divisor on W given by the inverse image of $X \times \{\infty\}$ under π. Also let $\tilde{X}$ be the Zariski closure in W_∞ of the restriction of the section s to $U \times \{\infty\}$. Then $\pi_{|\tilde{X}} : \tilde{X} \to X$ is birational (an isomorphism over U). As in [GS3], we see that the cycle

$$Z = [W_\infty] - [\tilde{X}]$$

is supported in the inverse image of X_S. Looking at supports, we have $|W_\infty| = |\tilde{X}| \cup |Z|$.

Denote by ξ_1 the universal subbundle of rank r on $\mathrm{Gr}(r, M)$ and by ξ_0 the "constant" bundle which is the base change of P under the (smooth) morphism $\mathrm{Gr}(r, M) \to X$. The section s gives

$$s_{\mathbf{C}} : X_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}} = W_{\mathbf{C}} \to \mathrm{Gr}(r, M)_{\mathbf{C}}.$$

The pull-back of ξ_0 under $s_{\mathbf{C}}$ is $p^*P_{\mathbf{C}}$; the pull-back of ξ_1 under $s_{\mathbf{C}}$ is $p^*N_{\mathbf{C}}$. Denote the restrictions $\xi_{0|W}$, $\xi_{1|W}$ by ζ_0, ζ_1. Equip $\zeta_{0\mathbf{C}}$, $\zeta_{1\mathbf{C}}$ with the hermitian metrics which correspond to the hermitian metrics on $p^*P_{\mathbf{C}}$, $p^*N_{\mathbf{C}}$ obtained via base change from the metrics on $P_{\mathbf{C}}$, $N_{\mathbf{C}}$. We will denote by $\bar{\zeta}_1$, $\bar{\zeta}_0$ the vector bundles ζ_1, ζ_0 on W endowed with the above hermitian metrics on $W_{\mathbf{C}}$. Set $\bar{\zeta} = (\bar{\zeta}_0, 0) - (\bar{\zeta}_1, 0) \in \widehat{\mathrm{K}}_0(W)$.

There is a natural morphism $\xi_1 \to \xi_0$ obtained by the natural inclusion $\xi_1 \subset \pi_0^*M$ followed by the projection $\pi_0^*M \to \xi_0 = \pi_0^*p^*P$. After restricting to $W_{\mathbf{C}}$ this corresponds to the composition $p^*N_{\mathbf{C}} \to p^*P_{\mathbf{C}}$.

Over $X_{\mathbf{C}}$ we have the exact sequence

$$\mathcal{E}_{\mathbf{C}} \ : \ 0 \to N_{\mathbf{C}} \to P_{\mathbf{C}} \to \Omega^1_{X_{\mathbf{C}}} \to 0.$$

This gives an exact sequence over $X_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}} = W_{\mathbf{C}}$:

$$p^*\mathcal{E}_{\mathbf{C}} : 0 \to p^*N_{\mathbf{C}} \to p^*P_{\mathbf{C}} \to p^*\Omega^1_{X_{\mathbf{C}}} \to 0.$$

Consider $A = (0, \widetilde{\mathrm{ch}}(p^*\mathcal{E}_{\mathbf{C}}))$ in $\widehat{\mathrm{K}}_0(W)$. Let us now define the elements

$$b = \hat{c}_{d+1}(\bar{\zeta} + A) \in \widehat{\mathrm{CH}}^{d+1}(W)_{\mathbb{Q}},$$
$$\mu = \pi_*(b) \in \widehat{\mathrm{CH}}^{d+1}(X \times \mathbf{P}^1)_{\mathbb{Q}}.$$

Lemma 3.2. *The restrictions of μ to $X \times \{0\}$ and $X \times \{\infty\}$ are equal.*

Proof. By [GS2, Theorem 4.4.6] the restrictions are well defined and their difference is given by

$$a\Big(\int_{\mathbf{P}^1(\mathbf{C})} \omega(\mu) \log |z|^2 \Big)$$

where ω and a are defined in [GS2, 3.3.4]; $\omega(\mu)$ is a $(d+1, d+1)$-form on $(X \times \mathbf{P}^1)(\mathbf{C})$ and the integral in the parenthesis gives a (d, d)-form on $X(\mathbf{C})$. Since π is an isomorphism on the generic fibers, by the definition of $\bar{\zeta}$ and A, we can see that the form $\omega(\mu)$ is obtained by pulling back via the projection $p_{\mathbf{C}} : X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C}) \to X(\mathbf{C})$ a $(d+1, d+1)$-form on $X(\mathbf{C})$. It follows that

$$\int_{\mathbf{P}^1(\mathbf{C})} \omega(\mu) \log |z|^2 = 0$$

(the integral changes sign when z is replaced by $1/z$). □

Recall that the morphism $\pi : W \to X \times \mathbf{P}^1$ restricts to give a projective morphism $\pi^{|Z|} : |Z| \to X_S \times \infty = X_S$. Here $|Z|$ is the (reduced) support of Z. Set $\xi = \xi_0 - \xi_1 \in K_0(\mathrm{Gr}(r, M))$ and denote by $[Z]$ the fundamental cycle of Z in $\mathrm{CH}_{d+1}(|Z|)$.

Lemma 3.3. (a) *The restriction of* μ *to* $X\times\{0\}$ *is equal to* $\hat{c}_{d+1}(\Omega)$*;*
(b) *The restriction of the class* μ *to* $X\times\{\infty\}$ *is equal to the image of* $\pi_*^{|Z|}(c_{d+1}(\xi_{||Z|})\cap[Z])\in \mathrm{CH}_0(X_S)_{\mathbb{Q}}$ *under* z_S*.*

Before we continue with the proof, let us point out that since by definition $c_{d+1}^{X_S}(\Omega^1_{X/\mathbb{Z}})=c^X_{d+1,X_S}(\Omega^1_{X/\mathbb{Z}})\cap[X]=\pi_*^{|Z|}(c_{d+1}(\xi_{||Z|})\cap[Z])$, Lemmas 3.2 and 3.3 together imply the proof of Proposition 3.1.

Proof. Part (a) is straightforward; indeed $\bar{\xi}_0$ restricts to give $\bar{P}$, $\bar{\xi}_1$ gives $\bar{N}$ and A gives $(0,\widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}}))$.

Let us show part (b). Recall W is integral of dimension $d+2$, W_∞ is an effective Cartier divisor in W and we have $[W_\infty]=Z+[\tilde{X}]$. Denote by $|W_\infty|$ the reduced support of W_∞ in W. Since $\pi^{|W_\infty|}:|W_\infty|\to X\times\{\infty\}=X$ is a projective morphism which is an isomorphism on the generic fiber,

$$\pi_*^{|W_\infty|}:\widehat{\mathrm{CH}}^{d+1}(|W_\infty|)_{\mathbb{Q}}\to\widehat{\mathrm{CH}}^{d+1}(X)_{\mathbb{Q}}$$

is well-defined. Also, since $i:W_\infty\to W$ is the inclusion of an effective Cartier divisor with smooth generic fiber, the pull-back $i^*(b)$ makes sense in

$$\widehat{\mathrm{CH}}^{d+1}(W_\infty)_{\mathbb{Q}}=\widehat{\mathrm{CH}}^{d+1}(|W_\infty|)_{\mathbb{Q}}=\widehat{\mathrm{CH}}_0(|W_\infty|)_{\mathbb{Q}},$$

and we have

$$\mu_{|X\times\{\infty\}}=\pi_*(b)_{|X\times\{\infty\}}=\pi_*^{|W_\infty|}(i^*(b)).$$

(see for example [GS3, 2.2.7]).

In what follows, we will calculate $i^*(b)$. For simplicity set $G=\mathrm{Gr}(r,M)$. Equip the bundles ξ_1, ξ_0 on G with hermitian metrics and set

$$\bar{\xi}=(\bar{\xi}_0,0)-(\bar{\xi}_1,0)\in\widehat{\mathrm{K}}_0(G).$$

Consider $B=\hat{c}_{d+1}(\bar{\xi})$ in $\widehat{\mathrm{CH}}^{d+1}(G)$ and $B_{|W}=\hat{c}_{d+1}(\bar{\xi}_{|W})$ in $\widehat{\mathrm{CH}}^{d+1}(W)$. Note that $\bar{\xi}_{|W}\in\widehat{\mathrm{K}}_0(W)$ need not agree with $\bar{\zeta}$ because the metrics might not agree. In any case, we can write

$$B_{|W}-\hat{c}_{d+1}(\bar{\zeta}+A)=a(\eta) \tag{3.4}$$

with η a (d,d)-form on $W(\mathbf{C})$. The pull-back $i^*(B_{|W})$ is the $d+1$-st arithmetic Chern class of the restriction of the bundle $\bar{\xi}_0-\bar{\xi}_1$ to W_∞. We have

$$i^*(b)=i^*(B_{|W})-a(i^*_{\mathbf{C}}(\eta)). \tag{3.5}$$

By [GS3, Theorem 4 (1)], $i^*(B_{|W})=B\cdot_j[W_\infty]$ in

$$\widehat{\mathrm{CH}}^{d+1}(|W_\infty|)_{\mathbb{Q}}=\widehat{\mathrm{CH}}_0(|W_\infty|)_{\mathbb{Q}};$$

here $j:|W_\infty|\to G$ is the natural embedding and $[W_\infty]\in\widehat{\mathrm{CH}}_{d+1}(|W_\infty|)_{\mathbb{Q}}$ is the fundamental cycle of W_∞ (the notations are as in loc.cit.). We may also consider

$[\tilde{X}] \in \widehat{\mathrm{CH}}_{d+1}(|W_\infty|)_{\mathbb{Q}}$ so that we have $[W_\infty] = Z + [\tilde{X}]$ in $\widehat{\mathrm{CH}}_{d+1}(|W_\infty|)_{\mathbb{Q}}$. We obtain

$$i^*(B_{|W}) = B \cdot_j [W_\infty] = B \cdot_j Z + B \cdot_j [\tilde{X}]. \tag{3.6}$$

Denote by $\phi : |Z| \to G$ and $\psi : \tilde{X} \to G$ the natural immersions. By [GS3, Theorem 3 (4)] the elements $B \cdot_j Z$ and $B \cdot_j [\tilde{X}]$ are the images of the elements $B \cdot_\phi Z$ and $B \cdot_\psi [\tilde{X}]$ of $\mathrm{CH}_0(|Z|)_{\mathbb{Q}}$ and $\widehat{\mathrm{CH}}_0(\tilde{X})_{\mathbb{Q}}$ under the maps

$$\mathrm{CH}_0(|Z|)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(|W_\infty|)_{\mathbb{Q}}$$

and

$$\widehat{\mathrm{CH}}_0(\tilde{X})_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(|W_\infty|)_{\mathbb{Q}}$$

respectively. We have

$$B \cdot_\phi Z = c_{d+1}(\xi_{||Z|}) \cap [Z]$$

and by [GS1, Theorem 4 (1)],

$$B \cdot_\psi [\tilde{X}] = \hat{c}_{d+1}(\bar{\xi}_{|\tilde{X}}) \cap [\tilde{X}] = \hat{c}_{d+1}(\bar{\xi}_{|\tilde{X}})$$

in $\widehat{\mathrm{CH}}_0(\tilde{X})_{\mathbb{Q}} = \widehat{\mathrm{CH}}^{d+1}(\tilde{X})_{\mathbb{Q}}$ (recall $\tilde{X}$ is integral of dimension $d+1$).

Now subtract $a(i^*_{\mathbf{C}}(\eta))$ from both sides of 3.6. Using 3.5 and the above, we obtain that $i^*(b)$ can be written as a sum of the image of the class $\hat{c}_{d+1}(\bar{\xi}_{|\tilde{X}}) - a(i^*_{\mathbf{C}}(\eta))$ under the map $\widehat{\mathrm{CH}}_0(\tilde{X})_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(|W_\infty|)_{\mathbb{Q}}$ plus the image of $c_{d+1}(\xi_{||Z|}) \cap [Z]$ under $\mathrm{CH}_0(|Z|)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(|W_\infty|)_{\mathbb{Q}}$. Since W_∞ and $\tilde{X}$ have the same generic fiber we can see from 3.4 that

$$\hat{c}_{d+1}(\bar{\xi}_{|\tilde{X}}) - a(i^*_{\mathbf{C}}(\eta)) = \hat{c}_{d+1}((\bar{\zeta} + A)_{|\tilde{X}}).$$

Hence, part (b) will follow if we show that $\hat{c}_{d+1}((\bar{\zeta} + A)_{|\tilde{X}}) = 0$.

Over $\tilde{X}$, there is an exact sequence of vector bundles

$$0 \to \zeta_{1|\tilde{X}} \to \zeta_{0|\tilde{X}} \to Q \to 0$$

with Q of rank d. We have $\tilde{X}_{\mathbf{C}} = X_{\mathbf{C}}$ and, as we have seen before, there is an isomorphism $Q_{\mathbf{C}} \simeq \Omega^1_{X_{\mathbf{C}}}$ which can be used to identify the above exact sequence with $\mathcal{E}_{\mathbf{C}}$. This implies that

$$((\bar{\zeta}_0)_{|\tilde{X}}, 0) - ((\bar{\zeta}_1)_{|\tilde{X}}, 0) + (0, \widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}})) = (\bar{Q}, 0)$$

in $\widehat{\mathrm{K}}_0(\tilde{X})$. Since $A_{|\tilde{X}} = A_{|X\times\{\infty\}} = (0, \widetilde{\mathrm{ch}}(\mathcal{E}_{\mathbf{C}}))$, this translates to $(\bar{\zeta} + A)_{|\tilde{X}} = (\bar{Q}, 0)$ in $\widehat{\mathrm{K}}_0(\tilde{X})$. Since $d+1 > \mathrm{rk}(Q) = d$, by [GS1, 4.9, p. 198], $\hat{c}_{d+1}((\bar{Q}, 0)) = 0$. Therefore, we obtain

$$\hat{c}_{d+1}((\bar{\zeta} + A)_{|\tilde{X}}) = 0.$$

This completes the proof of Lemma 3.3 and therefore also of Proposition 3.1. $\square$

Remark. Let $\bar{\mathcal{F}}$ be a hermitian coherent sheaf on X. Suppose that $Y \subset X$ is a *fibral* closed subscheme and assume that $\mathcal{F}$ is locally free of rank m on the complement $X - Y$. Let $z_{i,Y} : \mathrm{CH}_{d+1-i}(Y) \to \widehat{\mathrm{CH}}_{d+1-i}(X)$ be the natural homomorphism. The same argument as in the proof above can be used to show that for $i > m$,

$$z_{i,Y}(c^X_{i,Y}(\mathcal{F}) \cap [X]) = \hat{c}_i(\bar{\mathcal{F}}),$$

where $c^X_{i,Y}(\mathcal{F})$ is the localized Chern class of [B, §1].

4. Tame reduction

Here we show Theorem 1.3. Write I for an index set for the irreducible components of the singular fibers of $X \to \mathrm{Spec}(\mathbb{Z})$. If $i \in I$, we denote by T_i the corresponding irreducible component and by m_i its multiplicity in the divisor of the corresponding special fiber. For a non-empty subset J of I, set

$$T_J = \cap_{i \in J} T_i$$

(scheme-theoretic intersection). Under our assumptions, T_J is either empty or a smooth projective scheme of dimension $d + 1 - |J|$ over a finite field. The union $\cup_{J \neq \subset J'} T_{J'}$ is a divisor with strict normal crossings on T_J. We start with the following proposition:

Proposition 4.1. *With the assumptions of Theorem* 1.3*, we can consider the sheaf of relative logarithmic differentials* $\Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_S / \log S)$ *(see below); it is locally free of rank* d *on* X*. There is a morphism*

$$\omega : \Omega^1_{X/\mathbb{Z}} \to \Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_S / \log S),$$

whose kernel and cokernel are isomorphic to the kernel and cokernel of the morphism

$$a : \oplus_{p \in S} \mathcal{O}_X / p\mathcal{O}_X \;\to\; \oplus_{i \in S}\, \mathcal{O}_{T_i}.$$

Proof. The statement is local on the base, and so to simplify notation we will assume there is only one prime in S. We will use the logarithmic differentials $\Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_p)$ defined in [K] §2. By definition,

$$\Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_p) := (\Omega^1_{X/\mathbb{Z}} \oplus (\mathcal{O}_X \otimes j_* \mathcal{O}^*_{X[\frac{1}{p}]}))/\mathcal{F},$$

where j is the open immersion $j : X[\frac{1}{p}] \to X$ and $\mathcal{F}$ is the $\mathcal{O}_X$-subsheaf generated by elements of the form $(da, 0) - (0, a \otimes a)$ for $a \in \mathcal{O}_X \cap j_* \mathcal{O}^*_{X[\frac{1}{p}]}$. We will write the element $a \otimes b$ as $a \cdot d\log(b)$. Notice that $j_* \mathcal{O}^*_{X[\frac{1}{p}]}$ is the sheaf of elements of the function field of X whose divisor has support contained in the special fiber. By definition, $\Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_S / \log S)$ is the quotient of $\Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_p)$ by the $\mathcal{O}_X$-subsheaf generated by $d\log(p)$. There is an exact sequence

$$\mathcal{O}_X / p\mathcal{O}_X \;\xrightarrow{\phi}\; \Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_p) \;\xrightarrow{\omega_1}\; \Omega^1_{X/\mathbb{Z}}(\log X^{\mathrm{red}}_S / \log S) \to 0,$$

where the homomorphism ϕ maps f to $f \cdot d\log(p)$. There is also a natural exact sequence

$$0 \to \Omega^1_{X/\mathbb{Z}} \xrightarrow{\omega_2} \Omega^1_{X/\mathbb{Z}}(\log X_p^{\mathrm{red}}) \xrightarrow{\oplus_i \mathrm{Res}_i} \oplus_i \mathcal{O}_{T_i} \to 0. \tag{4.2}$$

Here the right hand homomorphism is given by taking residues along T_i. The homomorphism ω is equal to the composition $\omega_1 \cdot \omega_2$.

Under our assumptions, the scheme X is locally étale isomorphic to

$$Y = \mathrm{Spec}(\mathbb{Z}[t_1, \dots, t_d]/(t_1^{m_1} \cdots t_d^{m_d} - p))$$

with all m_i prime to p. The above constructions of logarithmic differentials etc. make sense for the scheme Y; we can see by an explicit calculation that ϕ_Y is injective and that the analogue of the sequence 4.2 for Y is exact. It follows from the fact that taking (logarithmic) differentials commutes with étale base change that ϕ is injective and that the sequence 4.2 is exact. On Y we have $t_1^{m_1} \cdots t_d^{m_d} = p$ and so

$$d\log(p) = m_1 \frac{dt_1}{t_1} + \cdots + m_d \frac{dt_d}{t_d}.$$

This shows that for $f \in \mathcal{O}_Y/p\mathcal{O}_Y$, $\phi_Y(f)$ gives an element in the kernel of ω if and only if $f \in (t_1 \cdots t_d)$; this translates to $a(f) = 0$. Furthermore, $f \cdot d\log(p) = 0$ if and only if $f = 0$ in $\mathcal{O}_Y/p\mathcal{O}_Y$. This shows the statement about the kernels for X. Let us now discuss the cokernels: Let $\beta : \oplus_i \mathcal{O}_{T_i} \to \oplus_i \mathcal{O}_{T_i}$ be the automorphism defined by $\beta((f_i)_i) = (m_i f_i)_i$ (recall that all the m_i are prime to p). The above calculation on Y implies that the composition

$$\mathcal{O}_X/p\mathcal{O}_X \xrightarrow{\phi} \Omega^1_{X/\mathbb{Z}}(\log X_p^{\mathrm{red}}) \xrightarrow{\oplus_i \mathrm{Res}_i} \oplus_i \mathcal{O}_{T_i}$$

coincides with $f \mapsto (m_1 f, \cdots, m_d f)$. The residue homomorphism $\mathrm{Res} = \oplus_i \mathrm{Res}_i$ now gives a surjection:

$$\Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S) \xrightarrow{\beta^{-1}\cdot \mathrm{Res}} \mathrm{coker}(a) \to 0$$

and we have $\ker(\beta^{-1} \cdot \mathrm{Res}) = \ker(\mathrm{Res}) = \omega(\Omega^1_{X/\mathbb{Z}})$. This implies $\mathrm{coker}(\omega) \simeq \mathrm{coker}(a)$. $\square$

Let $K_0^{X_S}(X)$ be the Grothendieck group of complexes of locally free $\mathcal{O}_X$-sheaves which are exact off X_S; since X is regular, $K_0^{X_S}(X)$ can be identified with $K_0'(X_S)$. Set $q = \prod_{p \in S} p$. Consider the following complexes of locally free $\mathcal{O}_X$-sheaves which are exact off X_S:

$$\mathcal{E}_1 : N \xrightarrow{\delta} P \to \Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S)$$

$$\mathcal{E}_2 : \mathcal{O}_X \xrightarrow{(q,-q)} \mathcal{O}_X \oplus (\oplus_i \mathcal{O}_X(-T_i)) \to \oplus_i \mathcal{O}_X$$

concentrated in degrees $-1, 0, 1$. The second homomorphism of $\mathcal{E}_1$ is the composition of $P \to \Omega^1_{X/\mathbb{Z}}$ with ω; the second homomorphism of $\mathcal{E}_2$ is given by

$(g,(h_i)_i) \mapsto (g+h_i)_i$. Proposition 4.1 implies that $[\mathcal{E}_1]=[\mathcal{E}_2]$ in $K_0^{X_S}(X)$. Consider also the complex

$$\mathcal{E}_3 : N \stackrel{(\delta,0)}{\to} P \oplus \Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S) \stackrel{(0,id)}{\to} \Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S)$$

concentrated in degrees $-1,0,1$. The complex $\mathcal{E}_3$ is quasi-isomorphic to the complex $N \stackrel{\delta}{\to} P$ (in degrees -1 and 0). There is an exact sequence of complexes

$$0 \to \mathcal{E}_1 \to \mathcal{E}_3 \stackrel{pr}{\to} \Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S) \to 0,$$

where on the right end, $\Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S)$ is considered as a complex supported on degree 0. Therefore, the main result of [A] (see loc. cit. Proposition 1.4 also [B] Prop. 1.1) implies that

$$c^{X_S}_{d+1}(\Omega^1_{X/\mathbb{Z}}) = \sum_{k+l=d+1} c_k(\Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S))\cdot c_l^{X_S}([\mathcal{E}_1]). \tag{4.3}$$

In fact, since $[\mathcal{E}_1]=[\mathcal{E}_2]$ we can replace $c_l^{X_S}([\mathcal{E}_1])$ by $c_l^{X_S}([\mathcal{E}_2])$ in this equality. We have

$$[\mathcal{E}_2] = [\mathcal{O}_X/q\mathcal{O}_X] - \sum_i [\mathcal{O}_{T_i}]$$

(here we identify $K_0^{X_S}(X)$ with $K'_0(X_S)$) and so

$$c_l^{X_S}([\mathcal{E}_2]) = c_l^{X_S}([\mathcal{O}_X/q\mathcal{O}_X] + \sum_i(-[\mathcal{O}_{T_i}])). \tag{4.4}$$

We have $c_1^{X_S}([\mathcal{O}_X/q\mathcal{O}_X]) = \sum_i m_i[T_i]$, $c_l^{X_S}([\mathcal{O}_X/q\mathcal{O}_X])=0$ for $l>1$. Similarly, $c_1^{X_S}(-[\mathcal{O}_{T_i}]) = -[T_i]$, $c_l^{X_S}(-[\mathcal{O}_{T_i}])=0$, for $l>1$. Combining these with 4.4 we obtain from the usual Chern class identities

(4.5)
$$c_l^{X_S}([\mathcal{E}_2]) = \sum_{J\subset I, |J|=l} (-1)^{|J|}[T_J] + \Big(\sum_{i\in I} m_i[T_i]\Big)\Big(\sum_{J'\subset I,|J'|=l-1}(-1)^{|J'|}[T_{J'}]\Big).$$

Now since $\sum_i m_i[T_i]$ is a principal divisor in X we get for $l\geq 2$

$$\Big(\sum_{i\in I} m_i[T_i]\Big)\Big(\sum_{J'\subset I,|J'|=l-1}(-1)^{|J'|}[T_{J'}]\Big) = 0 \in \mathrm{CH}_*(X_S).$$

Combining this with 4.3 and 4.5 we get

$$\begin{aligned} c^{X_S}_{d+1}(\Omega^1_{X/\mathbb{Z}}) = {} & \sum_{i\in I}(m_i-1)c_d(\Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S))\cdot[T_i] + \\ & + \sum_{J\subset I,|J|\geq 2}(-1)^{|J|}c_{d+1-|J|}(\Omega^1_{X/\mathbb{Z}}(\log X_S^{\rm red}/\log S))\cdot[T_J]. \end{aligned} \tag{4.6}$$

Therefore

$$(4.7)\qquad c_{d+1}^{X_S}(\Omega^1_{X/\mathbb{Z}}) = \sum_{i\in I}(m_i-1)c_d(\Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S)_{|T_i}) + \\ + \sum_{J\subset I, |J|\geq 2}(-1)^{|J|}c_{d+1-|J|}(\Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S)_{|T_J}).$$

Proposition 4.8. *For a non-empty subset J of I, set $T_J^* = T_J - \cup_{J\neq\subset J'}T_{J'}$. We have*

$$\deg(c_{d+1-|J|}(\Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S)_{|T_J})) = (-1)^{d+1-|J|}\chi_c(T_J^*),$$

where $\chi_c(T_J^*)$ is the l-adic ($l \notin S$) Euler characteristic with compact supports of T_J^*.

Proof. Denote by $\log X_p^{\mathrm{red}}|T_J$ the logarithmic structure on T_J obtained by restricting the logarithmic structure given by (X, X_p^{red}) to T_J. This is isomorphic to the logarithmic structure defined on T_J by its divisor with strict normal crossings $\cup_{J\neq\subset J'}T_{J'}$. We will show that

$$(4.9)\qquad [\Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S)_{|T_J}] = [\Omega^1_{T_J/k}(\log X_p^{\mathrm{red}}|T_J)] + (|J|-1)[\mathcal{O}_{T_J}]$$

in $\mathrm{K}_0(T_J)$. The proposition will follow from 4.9 and the well-known fact (see for example [S], p. 402) that

$$\deg(c_{d+1-|J|}(\Omega^1_{T_J/k}(\log X_p^{\mathrm{red}}|T_J))) = (-1)^{d+1-|J|}\chi_c(T_J^*).$$

From the proof of Proposition 4.1 there is an exact sequence

$$(4.10)\qquad 0 \to \mathcal{O}_{T_i} \to \Omega^1_{X/\mathbb{Z}}(\log X_p^{\mathrm{red}})_{|T_i} \to \Omega^1_{X/\mathbb{Z}}(\log X_S^{\mathrm{red}}/\log S)_{|T_i} \to 0.$$

By [K] §2 (see also [S], p. 404) there are also exact sequences

$$(4.11)\qquad 0 \to \Omega^1_{T_i/\mathbf{F}_p}(\log X_p^{\mathrm{red}}|T_i) \to \Omega^1_{X/\mathbb{Z}}(\log X_p^{\mathrm{red}})_{|T_i} \to \mathcal{O}_{T_i} \to 0,$$

and for $|J'| = |J|+1$,

$$(4.12)\quad 0 \to \Omega^1_{T_{J'}/\mathbf{F}_p}(\log X_p^{\mathrm{red}}|T_{J'}) \to \Omega^1_{T_J/\mathbf{F}_p}(\log X_p^{\mathrm{red}}|T_J)_{|T_{J'}} \to \mathcal{O}_{T_{J'}} \to 0.$$

We can now see that 4.9 follows by induction on the cardinality of J. □

Proposition 4.8 and 4.7 give for $p \in S$:

$$(4.13)\quad \deg((-1)^{d+1}c_{d+1}^{X_S}(\Omega^1_{X/\mathbb{Z}})_{|X_p}) = -\sum_{i\in I_p}(m_i-1)\chi_c^*(T_i) + \sum_{J\subset I_p, |J|\geq 2}\chi_c^*(T_J) \\ = -\sum_{i\in I_p} m_i\chi_c^*(T_i) + \chi(X_p)$$

where I_p is the subset of I that corresponds to components over p.

Under our assumption, the ramification is tame (there is no Swan term in the conductor) and for each $p \in S$,

$$\chi(X_{\mathbb{Q}}) = \sum_{i\in I_p} m_i\chi_c^*(T_i)$$

(see for example [S], Cor. 2, p. 407). Therefore,

$$A(X) = \prod_{p \in S} p^{\chi(X_{\mathbb{Q}}) - \chi(X_p)} = \prod_{p \in S} p^{\sum_i m_i \chi_c^*(T_i) - \chi(X_p)}.$$

This together with 4.13 completes the proof of 1.3.

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