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INTEGRAL GEOMETRY ON $SL(2;\mathbb{R})$

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Abstract. We define a complex horospherical transform on the group $SL(2;\mathbb{R})$ which corresponds to the Plancherel formula on it.

Gelfand and Graev found that the computation of the Plancherel measure for complex semisimple Lie groups or noncompact Riemannian symmetric spaces is equivalent to a problem of geometrical analysis - the inversion of the horospherical transform [GGr,GGrV]. Unfortunately, this result does not admit a direct generalization on real semisimple Lie groups or, more generally, on semisimple pseudo-Riemannian symmetric spaces: the horospherical transform for them has a kernel, corresponding to the discrete series of representations. Gelfand several times suggested the problem to find a version of the integral geometry, corresponding to the harmonic analysis on real groups, starting with $SL(2;\mathbb{R})$. In this paper we develop such integral geometry on $SL(2;\mathbb{R})$. The starting idea is very simple. The desired integral geometry uses complex horospheres in the $SL(2;\mathbb{C})\backslash SL(2;\mathbb{R})$ instead of real horospheres in $SL(2;\mathbb{R})$ and Cauchy kernels instead of δ-functions. This "complex" horospherical transform already has no kernel and gives the ability to write the Plancherel formula, including discrete series. Let us remark that a complex language is useful already for the real Radon transform [Gi1]. Connections of representations of $SL(2;\mathbb{R})$ with a complex geometry of $SL(2;\mathbb{C})\backslash SL(2;\mathbb{R})$ was considered in [GGi] and our considerations here in a sense develop the ideology of this paper (cf.also [Gi2]). I hope that these results give a good chance to develop the integral geometry on pseudo-Riemannian symmetric spaces, including real semisimple Lie groups, which corresponds to the harmonic analysis on them.

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Complex geometry. Essentially we do not use the group structure of $G_{\mathbb{C}} = SL(2;\mathbb{C})$ of 2×2-matrices:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det g = 1$$

and consider it as the hyperboloid in $\mathbb{C}^4$:

$$\Box g = \alpha\delta - \beta\gamma = 1. \tag{1}$$

It is a (pseudo-Riemannian) symmetric space $G_{\mathbb{C}} \times G_{\mathbb{C}}/G_{\mathbb{C}}$ (relative to left and right multiplications). The group $G_{\mathbb{C}} \times G_{\mathbb{C}}$ is locally isomorphic to $SSO(2,2)$.

Let the bilinear form $g_1 \cdot g_2$ in $\mathbb{C}^4$ be the polarization of the quadratic form $\Box g$ such that we can can rewrite (1) as

$$g \cdot g = 1.$$

If $\det g_2 = 1$ then

$$g_1 \cdot g_2 = \frac{1}{2}\,\mathrm{tr}(g_1 g_2^{-1}).$$

Often in our computations we will transform the hyperboloid (1) in the hyperboloid

$$\Box z = z_1^2 + z_2^2 - z_3^2 - z_4^2 = 1, \tag{1'}$$

replace correspondingly the form $z \cdot w$ and consider the action of the group $SO(2;2)$, locally isomorphic to $G_{\mathbb{C}} \times G_{\mathbb{C}}$.

Basic objects of integral geometry are *horospheres*. There are two alternative types of horospheres which can be put in the base of the integral geometry on $SL(2;\mathbb{C})$: *one-dimensional horocycles* and *two-dimensional horospheres*. Let us start from the description of horospheres. They are parameterized by the points of the cone

$$\Xi = \{\zeta; \Box\zeta = \zeta\cdot\zeta = \alpha\delta - \beta\gamma = 0, \zeta \neq 0\}. \tag{2}$$

We have

$$\Xi = G_{\mathbb{C}} \times G_{\mathbb{C}}/C, \qquad \zeta \mapsto g_1^{\top}\zeta g_2, \tag{3}$$

where C is the direct product of Borelian (triangle) subgroups B with the joint Cartan (diagonal) subgroups.

For $\zeta \in \Xi$ we define *the horosphere* $\Omega(\zeta)$ (of the dimension 2) *as the section of the hyperboloid* G (1) *by the hyperplane*

$$\zeta \cdot g = 1, \quad \Box\zeta = 0. \tag{4}$$

These sections are 2-dimensional hyperbolic paraboloids (of course in the complex picture all quadrics are equivalent) .

Usually we will work with an extended family of horospheres $\tilde{\Xi}$. It includes also "limit" horospheres $\Omega(\zeta, 0)$ - sections of G by hyperplanes

$$\zeta \cdot g = 0, \quad \Box\zeta = 0.$$

These sections are cylinders. We can represent hyperplanes from the extended family $\tilde{\Xi}$ as $\Omega(\zeta, p)$:

$$\zeta \cdot g = p, \quad \Box\zeta = 0; \tag{5}$$

(ζ, p) are homogeneous coordinates in $\tilde{\Xi}$. Of course, the extended space of horospheres $\tilde{\Xi}$ is not homogeneous relative to the $SO(2, 2)$-action.

Let us define another (projective) set of coordinates on the manifold Ξ of horospheres. Let $U, V \in \mathbb{C}^2$ be realized as row-vectors. Then elements of Ξ can be present as

$$\zeta = U^\top V$$

and we can parameterize horospheres by triplets (U, V, p) such that

$$(\lambda U, \mu V, \lambda\mu p) \sim (U, V, p), \quad \lambda, \mu \in \mathbb{R}$$

and the action $\zeta \mapsto g_1^\top \zeta g_2$ corresponds to the action $U \mapsto U g_1, V \mapsto V g_2$. We will use affine coordinates for ζ in the coordinate chart

$$\begin{gathered} V_1 = v, V_2 = 1, U_1 = \lambda u, U_2 = \lambda, \\ \zeta(\lambda|u, v) = \begin{pmatrix} \lambda uv & \lambda u \\ \lambda v & \lambda \end{pmatrix}. \end{gathered} \tag{6}$$

Correspondingly we will write $\Omega(\zeta, p) = \Omega(\lambda|u, v; p)$. In these coordinates $G_{\mathbb{C}} \times G_{\mathbb{C}}$ acts in the following way;

$$\begin{gathered} g_1^\top (\lambda|u, v) g_2 = (\tilde{\lambda}|\tilde{u}, \tilde{v}) \\ \tilde{u} = g_1(u), \tilde{v} = g_2(v), \quad g(u) = \frac{\alpha u + \gamma}{\beta u + \delta}, \\ \tilde{\lambda} = \lambda(\beta_1 u + \delta_1)(\beta_2 v + \delta_2). \end{gathered} \tag{7}$$

Horocycles are exactly *the linear generators on the hyperboloid* G (3-parametric family). They have the form $\omega(\zeta, a)$:

$$g = a + t\zeta, \quad \Box\zeta = 0, \zeta \cdot a = 0. \tag{8}$$

The point a here is defined up to any multiple of ζ. Let Ξ' be a set of horocycles; as a homogeneous space $\Xi' = G_{\mathbb{C}} \times G_{\mathbb{C}}/C$, where C is the product of two copies of Borel subgroups B with the joint Cartan subgroups.

The family Ξ' of horocycles includes the unipotent subgroup $Z = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and coincides with all its translations by elements $G_{\mathbb{C}} \times G_{\mathbb{C}}$:

$$\omega[g_1, g_2] = g_1^{-1} Z g_2. \tag{9}$$

It is possible to define a projective parameterization of Ξ' similar to the parameterization of Ξ, but we give here only the affine parameterization, induced by $g_1, g_2 \in B^\top$ at (9): the horocycle $\omega[\lambda|u, v]$ consists of matrices

$$\begin{pmatrix} \lambda^{-1} + vt & t \\ -\lambda^{-1}u + \lambda v - uvt & \lambda - ut \end{pmatrix}. \tag{10}$$

The parameters $(\lambda|u, v)$ here transform on formulas (7). We have the connections between two parameterizations:

$$\omega[\lambda|u, v] = \omega(\zeta, a), \quad \text{where} \quad \zeta = \begin{pmatrix} v & 1 \\ -uv & -u \end{pmatrix}, \quad a = \begin{pmatrix} \lambda^{-1} & 0 \\ -\lambda^{-1}u + \lambda v & \lambda \end{pmatrix}.$$

Between horospheres and horocycles there is a simple connection of incidence. Firstly, a limit horosphere $\Omega(\zeta, 0)$ is the union of horocycles $\omega(\zeta, a)$ with the same ζ (let us remind that limit horospheres are cylinders). Limit horospheres for nonproportional ζ do not intersect.

The horosphere $\Omega(\zeta, p)$ contains a horocycle $\omega(\tilde{\zeta}, a)$ if and only if

$$\zeta \cdot \tilde{\zeta} = 0, \quad \zeta \cdot a = p. \tag{11}$$

For a fixed ζ the first equation gives the section of the cone Ξ which is a pair of 2-subspaces intersecting on the line $\{t\zeta\}$. If we are to consider $\tilde{\zeta}$ as homogeneous coordinates, we have 2 lines, intersecting at ζ. If $p \neq 0$ and $\tilde{\zeta}$, ζ are nonproportional, we can find an unique $a \in G$, up to a multiple of $\tilde{\zeta}$, from (11) and the condition $\tilde{\zeta} \cdot a = 0$. Therefore the family of horocycles $\omega(\tilde{\zeta}, a)$ on the horosphere $\Omega(\zeta, p), p \neq 0$, is parameterized by $\tilde{\zeta}$ on the pair of affine lines, which are obtained from the intersection of Ξ by the hyperplane in (11) and by the removal of the point ζ. Let us recall that horospheres $\Omega(\zeta, p), p \neq 0$, are hyperbolic paraboloids and they have 2 families of linear generators. Correspondingly, horospheres $\Omega(\zeta, p)$, containing a horocycle $\omega(\tilde{\zeta}, a)$, are parameterized by points of the pair of (projective) lines, intersecting in $\tilde{\zeta}$ and defined by (11); the point $\tilde{\zeta}$ corresponds to the limit horosphere $\Omega(\tilde{\zeta}, 0)$. In such a way we have the duality between horospheres and horocycles.

In the integral geometry on $SL(2; \mathbb{C})$ we consider the horospherical transform: the integration either along horospheres or (more often) along horocycles ([GGrV]). These two versions of the horospherical transform have a very simple connection: in a sense 2-dimensional transform is the composition of the 1-dimensional one and the usual 2-dimensional (complex) Radon transform. For both transforms there are explicit inversion formulas which can be transformed in the Plancherel formula for the group $SL(2; \mathbb{C})$, using the Mellin transform.

Real geometry. Let us consider the real group (hyperboloid) $G_{\mathbb{R}} = SL(2;\mathbb{R})$ and the action of $G_{\mathbb{R}} \times G_{\mathbb{R}}$ on $G_{\mathbb{C}}$. Let $g = x + iy$. We have

$$\Box x - \Box y = x \cdot x - y \cdot y = 1, x \cdot y = 0. \tag{12}$$

On $G_{\mathbb{C}} \backslash G_{\mathbb{R}}$ we have a different kind of orbits which we will characterize by some canonical representatives. There are five domains, corresponding to different types of orbits for this action: the domains $G_{\pm}$ are the connect components of the set $\{\Box y > 0\}$; the domain G_0 is defined by the condition $\{\Box y < 0, \Box x > 0\}$ and the domains G_1, G_2 are the connect components of the set $\{\Box y < 0, \Box x < 0\}$.We give below descriptions of these domains on the language of canonical representatives of orbits.

Elements of $G_{\pm}$ can be transform to the form:

$$\begin{pmatrix} \lambda & i\mu \\ -i\mu & \lambda \end{pmatrix}, \lambda^2 - \mu^2 = 1, \lambda > 0, \mu \gtrless 0 \tag{13}$$

correspondingly. Points of G_0 can be transform to the canonical form:

$$\begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}, \quad \lambda^2 + \mu^2 = 1 \tag{14}$$

and points of G_1, G_2 to the form:

$$\begin{pmatrix} \lambda & i\mu \\ i\mu & -\lambda \end{pmatrix}, \lambda^2 - \mu^2 = -1, \lambda > 0, \mu \gtrless 0. \tag{15}$$

The first three domains are tubes with $G_{\mathbb{R}}$ as the edge; $G_{\pm}$ are Stein manifolds (in [GGi] was shown that they are equivalent Zarisky open parts in Siegel half-planes); G_0 is 1-pseudoconcave. The boundaries of the domains G_1, G_2 do not intersect $G_{\mathbb{R}}$ and as a result these domains are not essential for our considerations.

In the coordinates $(1')$ we can take

$$z = (\lambda, i\mu, 0, 0), \lambda^2 - \mu^2 = 1, \lambda > 0, \mu \gtrless 0 \tag{13'}$$

as representatives of $G_{\pm}$,

$$z = (\lambda, 0, i\mu, 0), \lambda^2 + \mu^2 = 1 \tag{14'}$$

as representatives of G_0, and

$$z = (0, 0, \lambda, i\mu), \lambda^2 - \mu^2 = -1, \lambda > 0, \mu \gtrless 0 \tag{15'}$$

as representatives of G_1, G_2.

Complex horospheres in the real picture. Our principal geometrical problem is to find complex horospheres without real points (which do not intersect $G_{\mathbb{R}}$). We consider complex horospheres (including limit ones) $\Omega(\zeta, p)$ (5). It is sufficient to consider $p = 0, 1$. Let $\zeta = \xi + i\eta$. Then

$$\Box\xi = \Box\eta = 0, \quad \xi \cdot \eta = 0. \tag{16}$$

Proposition. *Complex horosphere $\Omega(\zeta, p)$ has no real points in three situations:*

(i) $\Box\xi - |p|^2 > 0$;
(ii) $\Box\xi = 0, \eta = 0, \Im p \neq 0$;
(iii) $\Box\xi = \Box\eta = 0, p = 0, \xi \neq r\eta, r \neq 0$.

Proof. We will use the representation (1′) and we are interesting when (5) has no solutions $z = x, \Box x = 1, \Im z = 0$.

Let $\Box\xi > 0, p = 1$. Then we can transform ξ to the form $\xi = (\lambda, 0, 0, 0), \lambda > 0$, η to the form $\eta = (0, \pm\lambda, 0, 0)$. It must be $\xi \cdot x = 1, \eta \cdot x = 0$, therefore $x_1 = \lambda^{-1}, x_2 = 0, x_3^2 + x_4^2 = -1 + \lambda^{-2}$. So we have a real solution if and only if $\lambda > 1$ what corresponds in this case to (i).

If $\Box\xi > 0, p = 0$, then for similar reasons we take $\zeta = (\lambda, \pm i\lambda, 0, 0)$ and it must be that $x_1 = x_2 = 0$ and we have no solutions since we have the condition $x_3^2 + x_4^2 = -1$.

If $\Box\xi < 0$, then we can transform ζ to $\zeta = (0, 0, \lambda, i\lambda)$ and for both $p = 0, 1$ we can find real solutions x.

Let $\Box\xi = \Box\eta = 0$. If $\eta = 0, \Im p = 0$, we have a real horosphere on $G_{\mathbb{R}}$. If $\eta = 0, \Im p \neq 0$, the horospheres apparently have no real points. If $\eta \neq 0$, we can suppose, that $\eta = (a, 0, a, 0), a \neq 0$, and then on (16) we have $\xi = (b, c, b, \pm c)$. For a solution x with $p = 1$ we have $x_1 = x_3, c(x_2 \mp x_4) = 1$ and solutions exist if and only if $c \neq 0$. If $c = 0$ we can to change parameters such that $\eta = 0$. So we obtain a condition equivalent to (ii).

If $p = 0$ and $c \neq 0$, then again $x_1 = x_3$ and either $x_2 = x_4$ or $x_2 = -x_4$. In both cases we have $\Box x = 0$ and we obtain the contradiction with $\Box x = 1$. So these horospheres have no real points and we obtain the condition (iii). □

The set of parameters of horospheres satisfying condition (i) of Proposition has two connected components $\Xi_\pm$. Let us describe them in the coordinates (6). Let $p = 1$ and then $\Xi_\pm$ will be the sets of $\zeta \in \Xi$, equivalent to $(\lambda, \pm i\lambda, 0, 0), \lambda > 1$, correspondingly. In matrix coordinates (1) such points have representatives

$$\zeta(\lambda| \pm i, \mp i) = \begin{pmatrix} \lambda & \pm i\lambda \\ \mp i\lambda & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}, \lambda > 1 \tag{17}$$

and we need to describe the union of corresponding orbits of $G_{\mathbb{R}} \times G_{\mathbb{R}}$ (7). Let us start of Ξ_+. Points $u = i, v = -i$ have the joint isotropy subgroup of orthogonal matrices g. As the result of its action we have $\zeta(\tilde{\lambda}|i, -i), \tilde{\lambda} \in \mathbb{C}, |\tilde{\lambda}| > 1$. The translation which transforms $(i,-i)$ in $(u,v), \Im u>0, \Im v<0$, transforms $\tilde{\lambda}$ in λ with $|\lambda| = |\tilde{\lambda}|(-\Im u \Im v)^{-1/2}$. Then we obtain $\zeta(\lambda|u,v)$ with $(-\Im u\Im v)|\lambda|^2 > 1, \Im u>0, \Im v<0$. Correspondingly, for Ξ_- we have $\Im u<0, \Im v>0$. For $p\neq 1$ we obtain the description of $\Xi_\pm$:

$$(-\Im u \Im v)|\lambda|^2 > |p|^2, \quad \Im(\pm u)>0, \Im(\mp v)>0. \tag{18}$$

Apparently, this description is correct also for $p=0$.

The set of parameters of horospheres satisfying to (ii) we will denote through Ξ_0: we can take for them real $\zeta=\xi$ and nonreal p. Horospheres, satisfying to (iii), will not participate in our results and we do not introduce a notation for this set (it lies on the boundary of $\Xi_\pm$).

In the relation to horocycles in the real picture we will consider only horocycles $\omega(\xi,a)$, where ξ is real. We will distinguish three types of such horocycles. Let us remark that a in their parameterization are unique up to the addition of an multiple of ξ: $\tilde{a} = a + c\xi$. We will say that the horocycle has the real (imaginary) type if it is possible to choose real (imaginary) a (such that if $\Re a$ (correspondingly $\Im a$) is proportional to ξ). In other cases we will say that we have a generic horocycle.

Connection between horospheres, horocycles and domains in the real picture. Let $G^\mp$ be the supplements to the closure of $G_\pm$ correspondingly.

Proposition. *(i)The domains $G^\pm$ coincide with the unions of horospheres $\Omega(\zeta,p)$, $(\zeta,p)\in\Xi_\pm$.*

(ii) The domain G_0 coincide with the union of generic horocycles $\omega(\xi,a)$. The union of such horocycles of real (imaginary) type coincides with the part of the boundary of G_0 where $\Box y=0, \Box x=1$ (correspondingly $\Box x=0, \Box y=-1$).

Proof. (i)We will work in the representation (1′). The basic thing which we need to proof is that the horospheres $\Omega(\zeta,p), (\zeta,p)\in\Xi_-$, do not pass through points of G_+. We can reformulate it the following way. Let Π be the set of such $\zeta, \Box\zeta=0$, that $\Box\xi=1$. They can be transformed to one of 2 canonical forms $(1,\pm i,0,0)$. Correspondingly, the set Π has 2 components $\Pi_\pm$. We need to prove that

$$|\zeta\cdot z|>1 \quad \text{for} \quad \zeta\in\Pi_-, z\in G_+. \tag{19}$$

By virtue of invariance, it is enough to prove for $\zeta=(1,-i,0,0)$:

$$|z_1 - iz_2|>1 \quad \text{for} \quad z\in G_+. \tag{19′}$$

We have $|z_1-iz_2|^2 = (x_1+y_2)^2+(y_1-x_2)^2 = (x_1^2+x_2^2)+(y_1^2+y_2^2)+(x_1y_2-x_2y_1)$. As far as $z\in G_+$, then $\Box y>0$ and $(y_1^2+y_2^2)>0$; $\Box x>1$ and $(x_1^2+x_2^2)>1$. Therefore it is sufficient to check that

$$(x_1y_2-x_2y_1)>0 \qquad \text{on} \quad G_+. \tag{19″}$$

Let us verify that this expression can not be zero on G_+. If $\tilde{x} = (-x_2, x_1, 0, 0)$ (we have in (19″) $\tilde{x} \cdot y$), then $\Box\tilde{x} > 1$, since $\Box x > 1$; $\tilde{x} \cdot x = 0$. Suppose that $\tilde{x} \cdot y = 0$. Together with the condition $x \cdot y = 0$ (on the definition) it gives $y = 0$. Indeed, using the invariance, we can suppose $x = (a, 0, 0, 0), \tilde{x} = (0, b, 0, 0), a \neq 0, b \neq 0$, and it must be $y_1 = y_2 = 0$ and which is a contradiction to the condition $\Box y > 0$. Thus $\tilde{x} \cdot y$ conserves a sign on G_+ and therefore positive since it is positive for canonical elements (13′) in G_+.

It remains to check, that for almost every element of G^- there is a horosphere with parameters in Ξ_- , which passes through it. It is sufficient to consider the canonical representatives of G_-, G_0, G_1, G_2 of (13′) - (15′) and to find such $\zeta \in \Pi_-$ that $|\zeta \cdot z| < 1$. In all cases $\zeta = (1, -i, 0, 0)$ satisfies to this condition.

Remark. Horospheres $\Omega(\zeta, 1), \zeta \in \Pi$, are supporting ones for the real hyperboloid $G_{\mathbb{R}}$. Moreover, they intersect the hyperboloid only in one point $x = \xi$. It is sufficient to consider $\zeta = (1, \pm i, 0, 0)$. Then for the intersection points must be true that $x_1 = 1, x_2 = 0$ and on $G_{\mathbb{R}}$ there is only one point – (1,0,0,0) – with this condition. Supporting horospheres in each point of $G_{\mathbb{R}}$ are parameterized by points of a 2-dimensional hyperboloid of 2 sheets. They will play an important role in our considerations.

(ii) The basic fact is that there is no points z with $\Box(\Im z) > 0$ on limit horospheres $\Omega(\xi, 0)$ with a real ξ. Indeed, let $\xi \cdot z = 0, z = x + iy, \Box y > 0$. Using the invariance, we can consider only $\xi = (1, 0, 1, 0)$. Then $z_1 = z_3$ and $z_2^2 - z_4^2 = 1$. So

$$x_2^2 - x_4^2 - y_2^2 + y_4^2 = 1, \quad x_2 y_2 - x_4 y_4 = 0. \tag{20}$$

Let $\Box y > 0$. By the action of a transform, conserving ξ, we can obtain y with $y_4 = 0, y_2 \neq 0$. Then $x_2 = 0$ and we obtain the nonrealizable condition:$-x_4^2 - y_2^2 = 1$.

If $\Box y = 0$, then $x_2^2 - x_4^2 = 1$ and we can transform x to the case $x_4 = 0, x_2 = \pm 1$ in (20). Then $y_2 = 0, y_4 = 0, y = c\xi$ and such points lie on horocycles $\omega(\xi, a)$ with real a. On the other side, $\Im z = c\xi$ on horocycles of real type and hence $\Box y = 0$. Each point z with $\Box y = 0$ can be transform in the point $(1, i\lambda, i\lambda, 0)$, which lies on the horocycle $\omega(\xi, a)$ with $a = (1, 0, 0, 0), \xi = (0, 1, 1, 0)$ of real type. Such a way we connect a part of the boundary of G_0, where $\Box y = 0$, with horocycles of real type.

For other types of horocycles $\Box y < 0$. Then in the same notation as in (20) we can suppose that $y_4 \neq 0, y_2 = 0$ and as a result $x_4 = 0$ and $\Box x \geq 0$.

For our representatives the condition $\Box x = 0$ means that $x_2 = x_4 = 0$ and we have a horocycle of imaginary type since $\Re x = c\xi$. Again for points of any such a horocycle we have $\Re z = c\xi, \Box x = 0$. Points with conditions $\Box x = 0, \Box y = -1$ can be transform in points $(0, \lambda, \lambda, i)$ which lie on horocycles of imaginary type with $\xi = (0, 1, 1, 0), a = (0, 0, 0, i)$.

Finally, on generic horocycles we have $\Box x > 0, \Box y < 0$ hence they belong to the domain G_0. Since there is a generic horocycle $\omega(\xi, a)$ passing through the canonical point $z = (0, \lambda, i\mu, 0)$ $(\xi = (1, 0, 0, 1), a = z)$, for any point of G_0 there is a passing generic horocycle. The proof is finished. □

Complex horospherical transform. Let us start our analytic constructions from some notations. Let

$$dx = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

and

$$\mu(x, dx) = d(\Box x)\rfloor dx = \frac{dx_2 \wedge dx_3 \wedge dx_4}{2x_1} \tag{21}$$

be the invariant measure on the hyperboloid $G_{\mathbb{R}}$ (in the form (1′)). In our formulas we denote through $[a_1, a_2, \ldots, a_k]$ the determinant with the columns $a_1, \ldots, a_k$. Some of them can be columns of 1-forms and we use the cup-product for the computing of the determinant. As a result a determinant with identical columns of 1-forms can be different from zero. The important example of such a determinant is the form Leray

$$\mathcal{L}(\xi, d\xi) = [\xi, d\xi, \ldots, d\xi], \xi \in \mathbb{R}^k, \tag{22}$$

where the column $d\xi$ repeats $(k-1)$ times. This form plays an important role in the projective analysis.

Let $\mathcal{H}$ be the set of horospheres $\Omega(\zeta, p)$ without real points, $\mathcal{H}_{\pm}$ be components of the set horospheres satisfying to the condition (i) of Proposition and $\mathcal{H}_0$ corresponds to the condition (ii). For $p = 1$ we will identify these sets with $\Xi_{\pm}, \Xi_0$ correspondingly. For $f(x) \in C_0^{\infty}(G_{\mathbb{R}})$ we define the *(complex) horospherical transform* as

$$\hat{f}(\zeta, p) = \int_{G_{\mathbb{R}}} \frac{f(x)}{\zeta \cdot x - p} \mu(dx), \quad (\zeta, p) \in \mathcal{H}. \tag{23}$$

Since $\Omega(\zeta, p)$ has no real points, this transform is well defined. Let $\hat{f}_{\pm}, \hat{f}_0$ be its restrictions on $\mathcal{H}_{\pm}, \mathcal{H}_0$ correspondingly. The functions $\hat{f}_{\pm}$ are holomorphic and $\hat{f}_0$ is a CR-function.

Our principal result is the possibility to reconstruct f through the horospherical transform $\hat{f}$. We will give an explicit formula for the reconstruction of f through boundary values of $\hat{f}$. For each $x \in G_{\mathbb{R}}$ we consider the sets $\Gamma_{\pm}(x)$ of horospheres $\Omega(\zeta, 1)$, lying on the boundaries of $\mathcal{H}_{\pm}(\Xi_{\pm})$ and passing through x. As we saw for them

$$\zeta = x + i\eta, \quad x \cdot \eta = 0, \Box\eta = 1. \tag{24}$$

We use η as parameters and the conditions (24) defines a 2-dimensional hyperboloid of two sheets, giving $\Gamma_{\pm}(x)$. If $x = (1,0,0,0)$, then (24) means $\eta_1 = 0$ and $\Gamma_{\pm}(x)$ are the sheets

$$(\eta_2)^2 - (\eta_3)^2 - (\eta_4)^2 = 1, \quad \eta_2 \gtrless 0. \tag{25}$$

Let us denote $\mathcal{L}_x(\tilde{\eta}, d\tilde{\eta})$ the Leray form (22) on $\Gamma_{\pm}(x)$ on coordinates in which $\Box$ has the form (25) (for $x = (1,0,0,0)$ we take $\tilde{\eta} = (\eta_2, \eta_3, \eta_4)$).

Relative to the taking of boundary values, we remark that the horospheres $\Omega(x + i\eta, 1 + \varepsilon), \varepsilon > 0$, lies in $\mathcal{H}_{\pm}$ and we take the boundary values as $\varepsilon \to 0$ which exist as distributions since we have the singularity only in the point x:

$$\hat{f}_{\pm}(x + i\eta, 1) = \lim_{\varepsilon \to 0} \hat{f}_{\pm}(x + i\eta, 1 + \varepsilon). \tag{26}$$

Correspondingly, we take as $\Gamma_0(x)$ the set of (real) horospheres $\Omega(\xi, 1), x \cdot \xi = 1$. Let $\xi = x + \tilde{\xi}, x \cdot \tilde{\xi} = 0$. Then $\Box \tilde{\xi} = -1$ and we have a hyperboloid of one sheet. If $x = (1,0,0,0)$, then $\tilde{\xi}_1 = 0$ and we have the hyperboloid of one sheet

$$(\tilde{\xi}_2)^2 - (\tilde{\xi}_3)^2 - (\tilde{\xi}_4)^2 = -1. \tag{27}$$

We define $\mathcal{L}_x(\tilde{\xi}, d\tilde{\xi})$ as above. Let $\hat{f}_0(\xi, 1)$ be the boundary values in the sense of distributions:

$$\hat{f}_0(x + \tilde{\xi}, 1) = \lim_{\varepsilon \to 0} (\hat{f}_0(x + \tilde{\xi}, 1 + i\varepsilon), x \cdot \tilde{\xi} = 0, \Box(\tilde{\xi}) = -1. \tag{28}$$

We take here the limit of $\hat{f}$ for horospheres of $\mathcal{H}_0$.

The horospherical transform $\hat{f}_0(\xi, 1)$ has a natural connection with the real horospherical transform

$$\mathcal{R}f(\xi, p) = \int_{G_{\mathbb{R}}} f(x) \delta(\xi \cdot x - p) \mu(x, dx), \quad p \in \mathbb{R}. \tag{29}$$

Namely,

$$\hat{f}_0(\xi, 1 + i\varepsilon) = \int_{-\infty}^{\infty} \frac{\mathcal{R}f(\xi, q)}{q - 1 - i\varepsilon} dq.$$

Now we can formulate the principal result.

Main Theorem. *For $f(x) \in C_0^\infty(G_{\mathbb{R}})$ there exists an inversion formula*

$$\begin{gathered} f(x) = f_+(x) + f_-(x) + f_0(x), x \in G_{\mathbb{R}}, \\ f_{\pm}(x) = \frac{1}{4\pi^3} \int_{\Gamma_{\pm}(x)} L\hat{f}_{\pm}(x + i\eta, 1) \mathcal{L}(\tilde{\eta}, d\tilde{\eta}), \\ f_0(x) = \frac{1}{4(\pi i)^3} \int_{\Gamma_0(x)} L\hat{f}_0(x + \tilde{\xi}, 1) \mathcal{L}(\tilde{\xi}, d\tilde{\xi}), \\ L = \frac{\partial}{\partial p} + p \frac{\partial^2}{\partial p^2}. \end{gathered} \tag{30}$$

The proof of this theorem uses a version [Gi3] of the central construction of integral geometry - the operator κ of Gelfand-Graev-Shapiro [GGrSh].

Operator κ (the proof of Main Theorem). *The operator κ acts from functions on $G_{\mathbb{R}}$ to differential 2-forms on $G_{\mathbb{R}} \times \mathbb{C}^4_\zeta$:*

$$\kappa f[x] = \frac{f(u)}{(\zeta \cdot (u-x))^3} \mu(u, du) \wedge [u+x, \zeta, d\zeta, d\zeta], \qquad \Box x = \Box u = 1, \tag{31}$$

where x is a fixed point of $G_{\mathbb{R}}$ and we write the coordinates of $u+x$ as a column in the determinant. It is a general construction for arbitrary quadrics [Gi1, Gi3]. It plays a role of the decomposition of δ-function on plane waves for the case of quadrics. The basic result is

Proposition. *The form $\kappa f[x]$ is closed.*

The proof is a direct consequence of the differentiation formula

$$d[a(\xi), \xi, d\xi, \dots, d\xi] = (\sum \xi_j \partial a^j / \partial \xi_j) \mathcal{L}(\xi, d\xi), \tag{32}$$

where $a(\xi)$ is a column of homogeneous functions $a^j(\xi)$ of the degree $-k$. The proof of this formula is a direct calculation involving the Euler formula for homogeneous functions (cf. [Gi3]). Relative to u we have the form of the maximal degree, so we only need to care about the differentiation on ζ. We include the factor $(\xi \cdot (u-x))^{-3}$ in the first column and after the application of (32) we obtain the factor $\Box u - \Box x = 0$.

The form κ is the basic tool to obtain new inversion formulas from simplest ones. We integrate the form $\kappa f[x]$ on $G_{\mathbb{R}} \times \Gamma$, where Γ is a cycle in ζ-space. We show that for a cycle Γ this integral coincides with the Radon inversion formula and gives $cf(x)$. Then we deformate this cycle in the cycle of horospheres and through the virtue of the closeness of κ it gives us the inversion of the horospherical transform. It is essential that we realize this deformation in complex parameters.

To be more exact, we extend the set $\mathcal{H}$ of horospheres $\Omega(\zeta, p)$ without real points up to set $\tilde{H}$ of all sections $L(\zeta, p)$ of G by hyperplanes

$$\zeta \cdot z = p$$

without real points and also consider sections, corresponding to boundary parameters $(\zeta, p) \in \partial\tilde{\mathcal{H}}$. For $(\zeta, p) \in \tilde{\mathcal{H}}$ we define $\hat{f}(\zeta, p)$ by the formula (23) and then take the boundary values on holomorphic parameters. Thus all real pairs (ξ, p) correspond to "boundary" sections and we define $\hat{f}$ for them as in (28).

Let us put in (31) $\zeta \cdot x = p$ and integrate it in the beginning on $u \in G_{\mathbb{R}}$. This integral has a sense if $(\zeta, p) \in \tilde{\mathcal{H}}$ and gives a 2-form $F(p; \zeta, d\zeta)$, which can be expressed through $\hat{f}(\zeta, p)$ by a differential operator of 2-nd order, which can be written down explicitly [Gi1, Gi3](but we do not need this expression in the general case). It is important that this form is also closed.

Then we substitute $p = \zeta \cdot x$ and integrate the result on different cycles Γ in ζ-variables. In this situation we cannot find cycles with the condition $(\zeta, \zeta \cdot x) \in \tilde{\mathcal{H}}$, but we can find cycles with the condition $(\zeta, \zeta \cdot x) \in \partial\tilde{\mathcal{H}}$.

Firstly, we can take cycles with real $\zeta = \xi$ and $\hat{f}(\xi, p + i0)$. Let us take as the cycle $\Gamma^0(x)$ any cycle homological to the unit sphere in the hyperplane $\{\xi; x \cdot \xi = 0\}$ (the integrals for all such cycles will coincide in view of the closeness of κ). Then

$$\int_{G_{\mathbb{R}} \times \Gamma^0(x)} \kappa f = 4(\pi i)^3 f(x). \tag{33}$$

It is a simple consequence of the usual Radon inversion formula in the projective form [GGiGr,Gi1].Namely if $x = (1, 0, 0, 0)$, then on $\Gamma^0(x)$ we have $\xi_1 = 0$ and let $\xi = (0, \tilde{\xi}), u = (u_1, \tilde{u})$. Hence on $\Gamma^0(x)$ the form κf coincides with

$$\frac{(u_1 + 1) f(u)}{(\tilde{\xi} \cdot \tilde{u} - i0)^3} \mathcal{L}(\tilde{\xi}, d\tilde{\xi}).$$

Then (33) for even f coincides with the projective Radon inversion formula (restricted on the hyperboloid) and for odd f with the same formula for the $u_1 f(u)$.

Let us deformate this cycle $\Gamma^0(x)$ in the cycle of horospheres $\Gamma^1(x)$ inside $\tilde{\mathcal{H}} \bigcup \partial\tilde{\mathcal{H}}$ and through the virtue of closeness of κ we will obtain the inversion formula for the horospherical transform. We will describe this deformation for $x = (1, 0, 0, 0)$. Let $\xi = (0, \tilde{\xi} \in \Gamma^0(x)$ and

$$q(\tilde{\xi}) = \sqrt{(-\Box \tilde{\xi})},$$

where in the real case we take the positive evaluation of the root and in the imaginary case we take the evaluation of the form $ai, a > 0$.

Let us

$$\zeta(\varepsilon) = (\varepsilon q(\tilde{\xi}), \tilde{\xi}), \quad p(\varepsilon) = \varepsilon q(\tilde{\xi}), 0 \leq \varepsilon \leq 1, \tag{34}$$

and let $\Gamma^\varepsilon(x)$ be the cycle of the sections $L(\zeta(\varepsilon), p(\varepsilon))$.

All these sections lie in $\partial\tilde{\mathcal{H}}$. Indeed we need verify it only for the imaginary q. Then, for $\varepsilon \neq 0$, L has only one intersection point $x = (1, 0, 0, 0)$ with $G_{\mathbb{R}}$. The proof is the same as above, when we consider the support planes $\Pi_\pm$. If we replace $p(\varepsilon)$ on $p(\varepsilon) + i\delta, \delta > 0$, we obtain the section without real points. In this way the integral of κf has the same value on any cycle $\Gamma^\varepsilon(x)$. For $\varepsilon = 1$ we have the cycle $\Gamma^1(x)$ of horospheres: $\Box(\zeta(0)) = 0$. It decomposes on three parts: $\Gamma_0(x)$ corresponds to $\tilde{\xi}$ with $\Box(\xi) < 0$; $\Gamma_\pm$ correspond to 2 components of the set $\{\Box(\tilde{\xi}) > 0$. If $q(\tilde{\xi}) \neq 0$ we can transform pairs $(\zeta(1), p(1))$ to the form $(\zeta, 1)$.

The last step is a transformation of the determinant in κ. We have $\zeta_1 = 1$. Replace the first row by the combination of all rows with the coefficients ζ_j. Then in the first row only the first element will be different of zero and equal $\zeta \cdot u + 1$. The whole determinant will be equal to $(\zeta \cdot u + 1)\mathcal{L}(\zeta . d\zeta)$ and we can see directly that the factor in κf before $\mathcal{L}(\tilde{\zeta}, d\tilde{\zeta})$ is exactly the result of the application of the differential operator L to $\hat{f}$ and it gives (30).

Spherical Fourier transform (discrete series). The connection with representations goes through the spherical Fourier transform. As we already mentioned the domains $\Sigma_\pm$ are invariant (but not homogeneous) relative to the group $G_\mathbb{R}\times G_\mathbb{R}$. It is very important that on these domains there is also a commutating action of the circle $\mathbb{T}$. In the coordinates (λ,u,v) (7) it is the multiplication of λ on $\exp(i\rho)$.

Let us consider $\hat{f}_\pm(\zeta)$ as $\hat{f}_\pm(\lambda|u,v)$ which are holomorphic in the domains $\Xi_\pm$ (18):

$$|\lambda|^2 > \frac{1}{-\Im u\Im v}, \qquad \Im(\pm u)>0, \Im(\mp v)>0.$$

Let us decompose them in the Taylor series in these domains on λ:

$$\hat{f}_\pm(\lambda|u,v)=\sum_{k>0}\tilde{f}_\pm(k|u,v)\lambda^{-k-1}, \quad |\lambda|>(-\Im u\Im v)^{-1/2}. \tag{35}$$

Let us call functions $\tilde{f}_\pm(k|u,v)$ which are holomorphic in the products of two half-planes by *the discrete spherical Fourier transforms* of f. As follows from (7) the group $G_\mathbb{R}\times G_\mathbb{R}$ acts on them in the following way:

$$\begin{gathered} T^k_{(g_1,g_2)}\tilde{f}_\pm(k|u,v)=\tilde{f}_\pm(k|g_1(u),g_2(v))(\beta_1 u+\delta_1)^{-k-1}(\beta_2 v+\delta_2)^{-k-1}, \\ \Im(\pm u)>0, \Im(\mp v)>0. \end{gathered} \tag{36}$$

We see that the group $G_\mathbb{R}\times G_\mathbb{R}$ acts on the discrete spherical Fourier transform as the corresponding discrete (holomorphic and antiholomorphic) series of representations of this group.

Now we will rewrite the inversion formula (30) for $f_\pm$ through the discrete spherical Fourier transform. We need to start from the definition of the action of the operator L (30). Let us pass from $\hat{f}_\pm(\lambda|u,v)$ to $\hat{f}_\pm(\lambda|u,v;p)$. We need to replace in the series (35) λ^{-k-1} by $\lambda^{-k-1}p^k$, to apply the operator L (on p) and put $p=1$. We obtain

$$L\hat{f}_\pm(\lambda|u,v)=\sum_{k>0}k^2\tilde{f}_\pm(k|u,v)\lambda^{-k-1}, \quad |\lambda|>(-\Im u\Im v)^{-1/2}. \tag{37}$$

So the operator L corresponds in the spherical Fourier transform to the multiplication on k^2.

The next step is to investigate the cycles $\Gamma_\pm(x)$ in (30) in the parameters (λ,u,v). Let us work in the group coordinates (1) and let $x=e$ - the unit element. Then $\Gamma_\pm(e)$ consists of matrices

$$\begin{pmatrix} 1+i\eta_1 & i\eta_2 \\ i\eta_3 & 1-i\eta_1 \end{pmatrix}, \qquad -\eta_1^2-\eta_2\eta_3=1, \eta_2 \lesseqgtr 0.$$

We put here on the matrix η the conditions of the orthogonality to e (the trace is equal to 0) and we wrote down that the determinant is equal 1. So we have the sheets of the two-sheeted hyperboloid. We have on $\Gamma_{\pm}(e)$, according to (6),

$$\lambda = 1 - i\eta_1, u = \frac{i\eta_2}{1+(i\eta_1)^2}, v = \frac{i\eta_3}{1+(i\eta_1)^2}.$$

As the result

$$v = -\frac{1}{\overline{u}}, \qquad \lambda^{-1} = \frac{1}{2}(uv+1) = \frac{-u+\overline{u}}{2\overline{u}}. \tag{38}$$

It corresponds to the parameterization of one sheet of the hyperboloid by points of the upper (lower) half-plane. Thus we need to substitute these v and λ in (37) and for summation of the series, since we have a point on the boundary of the domain of convergence ($|\lambda| = (-\Im u \Im v)^{-1/2}$), we need for the regularization to include factors $\rho^{-k-1}, \rho > 1$ and take the limit as $\rho \to 1$. We need also to express $\mathcal{L}(\tilde{\eta}, d\tilde{\eta})$ through the parameters u (apparently it must be invariant relative to $SL(2;\mathbb{R})$). Finally,

$$f_{\pm}(x) = \frac{2}{\pi^3} \int_{\Im u \lessgtr 0} \left(\sum_{k>0} k^2 \tilde{f}_{\pm}\left(k|u, -\frac{1}{\overline{u}}\right) \left(\frac{-u+\overline{u}}{2\overline{u}}\right)^{-k-1} \right) \frac{du \wedge d\overline{u}}{|u|^2}. \tag{39}$$

Spherical Fourier transform (continuous series). Using $\hat{f}(\xi, 1), \xi \in \Xi_{\mathbb{R}}$, we will define the continuous part of the spherical Fourier transform. Let us express ξ in parameters (λ, u, v). Then the boundary values $(\xi, 1 + i0)$ will correspond to the $\lambda - \operatorname{sign}(\lambda) i0$. The set $\Xi_{\mathbb{R}}$ is invariant relative to the action of $\mathbb{R}^{\times}$ on λ and this action commutates with the action of $G_{\mathbb{R}} \times G_{\mathbb{R}}$ on Ξ_0. Let us decompose $\hat{f}(\xi)$ in the Mellin integral on λ:

$$\hat{f}_0(\lambda|u,v) = \int_0^{\infty} \tilde{f}_e(\rho|u,v)|\lambda|^{i\rho-1} d\rho + \int_0^{\infty} \tilde{f}_o(\rho|u,v)|\lambda|^{i\rho-1} \operatorname{sign}(\lambda) d\rho. \tag{40}$$

We will call $\tilde{f}_e, \tilde{f}_o$ by *the continuous parts of the spherical Fourier transform(its even and odd parts).* It is clear that they transform relative to the action of the group $G_{\mathbb{R}} \times G_{\mathbb{R}}$ by the corresponding representations of the continuous series of representations (cf.(36)).To reproduce the inversion formula (30) for f_0 using the spherical Fourier transform we need to remark that the operator L corresponds to the multiplication of $\tilde{f}$ on $-\rho^2$.

The computations are similar to the discrete part and we will not reproduce them in all details. We give formulas for the reconstruction in the point $e \in G_{\mathbb{R}}$. In this case(in the group parameters) the cycle $\Gamma_0(e)$ consists of matrices

$$\begin{pmatrix} 1+\xi_1 & \xi_2 \\ \xi_3 & 1-\xi_1 \end{pmatrix}, \qquad \xi_1^2+\xi_2\xi_3=1.$$

It is a hyperboloid of one sheet. Correspondingly, we have

$$u=\frac{\xi_2}{1-\xi_1}, \quad v=\frac{\xi_3}{1-\xi_1}, \quad \lambda=1-\xi_1.$$

The pair $u, v \in \mathbb{R}$ is the usual parameterization on the hyperboloid $\Gamma_0(e)$.Eventualy we transform the corresponding part of (30) in

$$\begin{aligned} f_0(e) = \frac{2}{(\pi i)^3}\Bigg\{ & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \rho^2 \tilde{f}_e(\rho|u,v) \left|\frac{uv+1}{2}\right|^{i\rho-1} \\ & \times \frac{(u-v)(uv-1)}{(uv+1)^3} d\rho \wedge du \wedge dv \\ & + \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \rho^2 \tilde{f}_o(\rho|u,v) \left|\frac{uv+1}{2}\right|^{i\rho-1} \operatorname{sign}\left(\frac{uv+1}{2}\right) \\ & \times \frac{(u-v)(uv-1)}{(uv+1)^3} d\rho \wedge du \wedge dv \Bigg\}. \end{aligned} \tag{41}$$

Remarks.

(1) If instead of horospheres we consider horocycles we will have essentially different formulas not only for the horospherical transform but also for the corresponding spherical Fourier transform (in the continuous part we will have the inversion formula which includes $\rho \tanh \rho$ instead ρ^2). We will give these formulas in the next publication.

(2) In the next paper we will give several applications: Plancherel formula, models of series, descriptions of projections on series $f \mapsto f_\pm, f_0$ on the language of complex analysis.

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