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EXPONENTIAL BOUNDS OF THE RESOLVENT FOR A CLASS OF NONCOMPACTLY SUPPORTED PERTURBATIONS OF THE LAPLACIAN

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1. Introduction and statement of results

Let $\mathcal{O} \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with C^∞ boundary Γ and connected complement $\Omega = \mathbb{R}^n \setminus \overline{\mathcal{O}}$. Consider in Ω the operator

$$\Delta_g := c(x)^2 \sum_{i,j=1}^n \partial_{x_i}(g_{ij}(x)\partial_{x_j}),$$

where $c(x), g_{ij}(x) \in C^\infty(\overline{\Omega})$, $c(x) \geq c_0 > 0$ and

$$\sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \qquad \forall (x,\xi) \in T^*\Omega, \quad C > 0.$$

We suppose that for all multi-indices α such that $|\alpha| \leq 1$, we have

$$|\partial_x^\alpha(c(x)-1)| + \sum_{i,j=1}^n |\partial_x^\alpha(g_{ij}(x)-\delta_{ij})| \leq C_1 e^{-C_2\langle x\rangle^p}, \quad \forall x \in \Omega, \tag{1.1}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$, $C_1, C_2 > 0$ and $p > 2$. Here δ_{ij} denotes Kronecker's symbol. We also suppose that for all multi-indices α,

$$|\partial_x^\alpha c(x)| + \sum_{i,j=1}^n |\partial_x^\alpha g_{ij}(x)| \leq C_\alpha < \infty, \quad \forall x \in \Omega. \tag{1.2}$$

Denote by G the selfadjoint realization of Δ_g in the Hilbert space

$$H = L^2(\Omega; c(x)^{-2}dx)$$

with a domain of definition $D(G) = \{u \in H^2(\Omega), Bu|_\Gamma = 0\}$, where either $B = Id$ (Dirichlet boundary conditions) or $B = \partial_\nu$ (Neumann boundary conditions). Consider the resolvent $R(\lambda) := (G+\lambda^2)^{-1} : H \to H$ defined for $\operatorname{Im}\lambda < 0$. It is well known that for any $b > 0$ the modified resolvent $e^{-b\langle x\rangle}R(\lambda)e^{-b\langle x\rangle} : H \to H$ extends meromorphically to $\operatorname{Im}\lambda < Cb$ with poles (which do not depend on b) called resonances. In what follows $\|\cdot\|$ will denote the norm in $\mathcal{L}(H,H)$. The purpose of this work is to prove the following

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Theorem 1.1. *Under the assumptions* (1.1) *and* (1.2), *there exist positive constants* C_1, C_2 *and* λ_0 *so that*

$$\|e^{-\langle x\rangle}R(\lambda)e^{-\langle x\rangle}\| \le C_1 e^{C_2|\lambda|} \quad \text{for} \quad \lambda\in\mathbb{R},\ |\lambda|\ge\lambda_0. \tag{1.3}$$

Note that for compactly supported perturbations, Burq [1] has proved a similar result using the Carleman estimates proved by Lebeau-Robbiano [2], [3]. Another proof (much simpler and shorter) of Burq's result is presented in [5]. In the same way as in the case of compactly supported perturbations one can derive from (1.3) the following

Corollary 1.2. *Under the assumptions* (1.1) *and* (1.2), *there exist positive constants* $\widetilde{C}_1, \widetilde{C}_2$ *and* $\widetilde{C}_3$ *so that there are no resonances of* G *in the region*

$$\{\lambda\in\mathbf{C} : \operatorname{Im}\lambda \le \widetilde{C}_1 e^{-\widetilde{C}_2|\lambda|},\ |\operatorname{Re}\lambda|\ge\widetilde{C}_3\}.$$

To prove (1.3) we follow the approach developed in [5] and based on some estimates due to Lebeau-Robbiano [2], [3] (see the appendix in the present paper). We first show that (1.3) is equivalent to a similar bound of the resolvent of another operator, depending on λ, which is a compactly supported (in a ball of radius $a = O(\lambda^q)$ with $0<q<\frac{1}{p}<\frac{1}{2}$) perturbation of the free Laplacian. Then we paste the estimates of Lebeau-Robbiano [2], [3] mentioned above (applied in $\Omega_{a_0} := \{x\in\Omega : |x|\le a_0\}$, $a_0\gg 1$ being independent of λ (see (4.1))) with Carleman estimates in $a_0\le|x|\le a$. When a does not depend on λ these latter estimates follow from [2]. Here we modify the original proof in order to have estimates uniform in both λ and a. Finally, we combine these estimates with some properties of the Neumann operator on the sphere $S_a := \{x\in\mathbb{R}^n : |x| = a\}$ (see Lemma 3.1) to get the desired result.

2. Reduction of the problem

Clearly, it suffices to prove (1.3) for real $\lambda\gg 1$. Let $\widetilde{\chi}\in C_0^\infty(\mathbb{R}^n)$, $0\le\widetilde{\chi}\le 1$, $\widetilde{\chi}=1$ for $|x|\le 1/3$, $\widetilde{\chi}=0$ for $|x|\ge 1/2$. Given any $a>0$, set $\chi_a(x)=\widetilde{\chi}(x/a)$.

Lemma 2.1. *For any* $0<\delta\ll 1$, *we have*

$$\sum_{|\alpha|\le 2}\|e^{-\langle x\rangle}\partial_x^\alpha R(\lambda)e^{-\langle x\rangle}\| \le C + C\sum_{|\alpha|\le 2}\|\chi_{\lambda^\delta}\partial_x^\alpha R(\lambda)\chi_{\lambda^\delta}\|, \tag{2.1}$$

with a constant $C>0$ *independent of* λ.

Proof. Denote by G_0 the selfadjoint realization of the Laplacian Δ in $\mathbb{R}^n$ on the Hilbert space $L^2(\mathbb{R}^n)$, and set $R_0(\lambda)=(G_0+\lambda^2)^{-1}$. Denote by $\|\cdot\|_0$ the norm in $\mathcal{L}(L^2(\mathbb{R}^n))$. Take a $\rho_0\gg 1$, independent of λ, so that $\chi_{\rho_0}=1$ in a neighbourhood of $\mathcal{O}$. We have

$$(1-\chi_{\rho_0})R(z) = R_0(z)(1-\chi_{\rho_0}) + R_0(z)\left([\chi_{\rho_0},\Delta] + (1-\chi_{\rho_0})(\Delta_g-\Delta)\right)R(z)$$

for $\operatorname{Im} z < 0$, and hence

$$e^{-\langle x\rangle}(1-\chi_{\lambda^\delta})\partial_x^\alpha R(\lambda)e^{-\langle x\rangle} = e^{-\langle x\rangle}(1-\chi_{\lambda^\delta})\partial_x^\alpha R_0(\lambda)(1-\chi_{\rho_0})e^{-\langle x\rangle} + e^{-\langle x\rangle}(1-\chi_{\lambda^\delta})\partial_x^\alpha R_0(\lambda)\left([\chi_{\rho_0},\Delta]+(1-\chi_{\rho_0})(\Delta_g-\Delta)\right)R(\lambda)e^{-\langle x\rangle}.$$

This implies

$$\begin{aligned}\|e^{-\langle x\rangle}(1-\chi_{\lambda^\delta})\partial_x^\alpha R(\lambda)e^{-\langle x\rangle}\| &\le O(\lambda^{-\infty})\|e^{-\frac{1}{2}\langle x\rangle}\partial_x^\alpha R_0(\lambda)e^{-\langle x\rangle}\|_0 \\ &\quad + O(\lambda^{-\infty})\|e^{-\frac{1}{2}\langle x\rangle}\partial_x^\alpha R_0(\lambda)e^{-\langle x\rangle}\|_0 \sum_{|\beta|\le 2}\|e^{-\langle x\rangle}\partial_x^\beta R(\lambda)e^{-\langle x\rangle}\| \\ &\le O(\lambda^{-\infty})\left(1+\sum_{|\beta|\le 2}\|e^{-\langle x\rangle}\partial_x^\beta R(\lambda)e^{-\langle x\rangle}\|\right).\end{aligned}$$

Clearly, similar estimate holds for $\|e^{-\langle x\rangle}\partial_x^\alpha R(\lambda)(1-\chi_{\lambda^\delta})e^{-\langle x\rangle}\|$, and (2.1) follows easily from the above estimates. □

Given $a \gg 1$, denote

$$\Delta_g^a := c(x)^2\sum_{i,j=1}^n \partial_{x_i}(\chi_a g_{ij}(x)\partial_{x_j}) + \sum_{i,j=1}^n \partial_{x_i}((1-\chi_a)\delta_{ij}\partial_{x_j}).$$

Denote by G_a the selfadjoint realization of Δ_g^a on H, and set $R_a(z) := (G_a+z^2)^{-1}$.

Lemma 2.2. *For $a \gg \lambda^\delta$, $0<\delta\ll 1$, we have*

$$\left(1-O(e^{-Ca^p})\sum_{|\alpha|\le 2}\|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)\chi_{\lambda^\delta}\|\right)\sum_{|\alpha|\le 2}\|\chi_{\lambda^\delta}\partial_x^\alpha R(\lambda)\chi_{\lambda^\delta}\| \le \sum_{|\alpha|\le 2}\|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)\chi_{\lambda^\delta}\|, \tag{2.2}$$

with a constant $C>0$ independent of a and λ.

Proof. We have

$$\chi_{\lambda^\delta}\partial_x^\alpha R(\lambda)\chi_{\lambda^\delta} = \chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)\chi_{\lambda^\delta} + \chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)(G_a-G)R(\lambda)\chi_{\lambda^\delta},$$

and hence, in view of (1.1),

$$\|\chi_{\lambda^\delta}\partial_x^\alpha R(\lambda)\chi_{\lambda^\delta}\| \le \|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)\chi_{\lambda^\delta}\| + O(e^{-Ca^p})\|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)e^{-\langle x\rangle}\|\sum_{|\beta|\le 2}\|e^{-\langle x\rangle}(1-\chi_{\lambda^\delta})\partial_x^\beta R(\lambda)\chi_{\lambda^\delta}\|.$$

On the other hand, in the same way as in the proof of Lemma 2.1, we have

$$\sum_{|\alpha|\le 2}\|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)e^{-\langle x\rangle}\|\le C+C\sum_{|\alpha|\le 2}\|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)\chi_{\lambda^\delta}\|,$$

$$\sum_{|\beta|\le 2}\|e^{-\langle x\rangle}(1-\chi_{\lambda^\delta})\partial_x^\beta R(\lambda)\chi_{\lambda^\delta}\|\le C+C\sum_{|\beta|\le 2}\|\chi_{\lambda^\delta}\partial_x^\beta R(\lambda)\chi_{\lambda^\delta}\|.$$

Clearly, (2.2) follows from the above esimates. □

In the next sections we will prove the following

Proposition 2.3. *If $a=\lambda^q$ with $\delta<q<1/2$, we have*

$$\|\chi_{\lambda^\delta}\partial_x^\alpha R_a(\lambda)\chi_{\lambda^\delta}\|\le e^{C\lambda},\qquad \forall\alpha, 0\le|\alpha|\le 2, \tag{2.3}$$

with a constant $C>0$ independent of λ.

Clearly, taking $\frac{1}{p}<q<\frac{1}{2}$, (1.3) follows from (2.3) combinned with Lemmas 2.1 and 2.2.

3. Proof of Proposition 2.3

In what follows $a=\lambda^q$, $\lambda\gg 1$, with $0<q<1/2$. Consider the problem

$$\begin{cases}(\Delta_g^a+\lambda^2)u=v \text{ in } \Omega,\\ Bu=0 \text{ on } \Gamma,\\ u-\lambda-\text{outgoing},\end{cases}$$

where $v\in C^\infty(\Omega)$, $\operatorname{supp} v\subset\Omega_{\lambda^\delta}:=\{x\in\Omega:|x|<\lambda^\delta\}$. Clearly, (2.3) is equivalent to the estimate

$$\|u\|_{H^2(\Omega_{\lambda^\delta})}\le e^{C\lambda}\|v\|_{L^2(\Omega)}, \tag{3.1}$$

with a constant $C>0$ independent of λ. Denote $S_a=\{x\in\mathbb{R}^n:|x|=a\}$. Define the Neumann operator $N(\lambda):H^1(S_a)\to L^2(S_a)$ by $N(\lambda)g:=\lambda^{-1}\partial_{\nu'}w|_{S_a}$, where w solves the equation

$$\begin{cases}(\Delta+\lambda^2)w=0 \text{ in } |x|>a,\\ w=g \text{ on } S,\\ w-\lambda-\text{outgoing}.\end{cases}$$

Here ν' denotes the outer unit normal to S_a. Throughout this paper, given a domain K, $H^s(K)$ will denote the Sobolev space equipped with the semiclassical norm $\|f\|_{H^s(K)}:=\|\Lambda_s f\|_{L^2(K)}$, where Λ_s is a $\lambda-\Psi DO$ on K with principal symbol $(|\xi|^2+1)^{s/2}$.

Clearly, u and v satisfy the equation

$$\begin{cases}(\Delta_g^a+\lambda^2)u=v \text{ in } \Omega_a,\\ Bu=0 \text{ on } \Gamma,\\ \lambda^{-1}\partial_\nu u|_{S_a}+N(\lambda)f=0,\end{cases}$$

where $f = u|_{S_a}$ and $\nu = -\nu'$ denotes the inner unit normal to S_a. By Green's formula we have

$$-\text{Im}\,\langle N(\lambda)f, f\rangle_{L^2(S_a)} = -\text{Im}\,\langle u, c^{-2}v\rangle_{L^2(\Omega_{\lambda\delta})} \le e^{-\beta\lambda}\|u\|^2_{L^2(\Omega_{\lambda\delta})} + e^{\beta\lambda}\|v\|^2_{L^2(\Omega)}, \tag{3.2}$$

$\forall\beta$. Choose a real-valued function $\widetilde{\rho}(t) \in C_0^\infty(\mathbb{R})$, $0 \le \widetilde{\rho}(t) \le 1$, $\widetilde{\rho}(t) = 1$ for $t \le 1$, $\widetilde{\rho}(t) = 0$ for $t \ge 2$, and given a $\gamma > 0$, set $\rho_\gamma(t) = \widetilde{\rho}(\lambda^\gamma(t-1))$. Denote by Δ_{S_a} the Laplace-Beltrami operator on S_a, and set $L = \lambda^{-1}\sqrt{-\Delta_{S_a}}$.

Lemma 3.1. *There exist positive constants c_0 and C, independent of λ, so that we have*

$$[N(\lambda), L] = 0, \tag{3.3}$$

$$-\text{Im}\,\langle N(\lambda)f, f\rangle_{L^2(S_a)} \ge e^{-c_0\lambda}\|\rho_q(L)f\|^2_{L^2(S_a)}, \tag{3.4}$$

$$\|N(\lambda)f\|_{L^2(S_a)} \le C\|(L+1)f\|_{L^2(S_a)}, \tag{3.5}$$

$$a\text{Re}\,\langle N(\lambda)f, f\rangle_{L^2(S_a)} \le C\lambda^{-\frac{1}{3}(1-2q)}\|f\|^2_{L^2(S_a)}. \tag{3.6}$$

Proof. Let $\{\mu_j\}$ be the eigenvalues of $\sqrt{-\Delta_{S_1}}$ repeated according to multiplicity. Then $(a\lambda)^{-1}\mu_j$ are the eigenvalues of L, and let $\{e_j\}$ be the corresponding eigenfunctions, i.e. $Le_j = (a\lambda)^{-1}\mu_j e_j$. If $f \in L^2(S_a)$, we write $f = \sum \alpha_j e_j$, where $\{\alpha_j\}$ are such that

$$\|f\|^2_{L^2(S_a)} = \sum \alpha_j^2.$$

It is well known that $N(\lambda)f$ is given by the formula

$$N(\lambda)f = -\frac{1}{2a}(n-2+\lambda^{-1})f + \sum \frac{h'_\nu(a\lambda)}{h_\nu(a\lambda)}\alpha_j e_j, \tag{3.7}$$

where $h_\nu(z) = z^{1/2}H_\nu^{(2)}(z)$, $\nu = \sqrt{\mu_j^2 + (\frac{n}{2}-1)^2}$, $H_\nu^{(2)}(z)$ being the Hankel function of second type. For real $z > 0$, set $\psi_\nu(z) = -\text{Im}\,\frac{h'_\nu(z)}{h_\nu(z)}$, $\eta_\nu(z) = -\text{Re}\,\frac{h'_\nu(z)}{h_\nu(z)}$. Clearly, (3.7) implies (3.3). Moreover, we have

$$-\text{Im}\,\langle N(\lambda)f, f\rangle_{L^2(S_a)} = \sum \psi_\nu(a\lambda)\alpha_j^2, \tag{3.8}$$

$$\text{Re}\,\langle N(\lambda)f, f\rangle_{L^2(S_a)} \le -\sum \eta_\nu(a\lambda)\alpha_j^2, \tag{3.9}$$

$$\|N(\lambda)f\|^2_{L^2(S_a)} \le C'\|f\|^2_{L^2(S_a)} + \sum \left|\frac{h'_\nu(a\lambda)}{h_\nu(a\lambda)}\right|^2 \alpha_j^2, \tag{3.10}$$

where $C' > 0$ is independent of λ.

Since $h_\nu(z)$ satisfies the equation

$$h''_\nu(z) = \left(\frac{\nu^2 - 1/4}{z^2} - 1\right)h_\nu(z),$$

we have

$$\psi'_\nu(z) = \text{Im}\left(\left(\frac{h'_\nu(z)}{h_\nu(z)}\right)^2 - \frac{h''_\nu(z)}{h_\nu(z)}\right) = 2\eta_\nu\psi_\nu.$$

This implies

$$\psi_\nu(\nu z) = \psi_\nu(\nu z_0) \exp\left(2\nu \int_{z_0}^{z} \eta_\nu(\nu y) dy\right). \tag{3.11}$$

Set $z = a\lambda/\nu$. On $\operatorname{supp} \rho_q$ we have $z \geq 1 - O(\lambda^{-q})$. Take $z_0 = 1 + \nu^{-\gamma}$ with $\gamma = \frac{q}{q+1} < \frac{1}{3}$. By Olver's expansions (see [4])

$$\psi_\nu(\nu z_0) = \frac{\sqrt{z_0^2 - 1}}{z_0} + O(\nu^{-1/3}) = \nu^{-\gamma/2}(\sqrt{2} + o(1)),$$

and $\eta_\nu(\nu y) = O(1)$ for $1 - O(\lambda^{-q}) \leq y \leq 1 + \nu^{-\gamma}$. By (3.11),

$$\psi_\nu(\nu z) \geq \nu^{-\gamma/2} \exp\left(-\nu O(\lambda^{-q} + \nu^{-\gamma})\right).$$

On the other hand, we have $\nu \leq 2a\lambda$ on $\operatorname{supp}\rho_q$, and hence $\exists \nu_0 \gg 1$ so that for $\nu \geq \nu_0$ we have

$$\psi_\nu(\nu z) \geq (a\lambda)^{-\gamma/2} \exp\left(-\nu O(\lambda) - O(\lambda^{(1-\gamma)(1+q)})\right) \geq \exp(-c_0\lambda), \tag{3.12}$$

with $c_0 > 0$ independent of λ. Moreover, it is clear from (3.11) that

$$\psi_\nu(z) > 0, \qquad \forall z > 0, \forall \nu \geq \nu_0. \tag{3.13}$$

Let now $1/2 < \nu \leq \nu_0$. Using the well known asymptotics of the Hankel functions as $z \to +\infty$, $\nu > 1/2$ fixed, we get

$$\psi_\nu(z) = 1 + O(z^{-1}), \qquad 1/2 < \nu \leq \nu_0. \tag{3.14}$$

Clearly, (3.4) follows from (3.8) combined with (3.12)-(3.14).

It is clear from (3.10) that (3.5) would follow from the bound

$$\left|\frac{h_\nu'(\nu z)}{h_\nu(\nu z)}\right| \leq C(1 + z^{-1}), \qquad \forall z > 0, \tag{3.15}$$

while (3.6) would follow from (3.9) and the inequality

$$-\eta_\nu(z) \leq C z^{-1/3}, \qquad \forall z \geq 1, \tag{3.16}$$

where $C > 0$ is independent of z and ν. By Olver's expansions ([4]), uniformly for $0 < z \leq 1/2$, $\nu \geq \nu_0 \gg 1$, we have

$$z\frac{h_\nu'(\nu z)}{h_\nu(\nu z)} = -\sqrt{1 - z^2} + O(\nu^{-1}),$$

while for $z \geq 2$,

$$\frac{h_\nu'(\nu z)}{h_\nu(\nu z)} = -i\frac{\sqrt{z^2 - 1}}{z} + O(\nu^{-1}).$$

On the other hand,

$$\frac{h_\nu'(\nu z)}{h_\nu(\nu z)} = O(1) \qquad \text{for} \quad 1/2 \leq z \leq 2,$$

and (3.15) follows when $\nu \geq \nu_0 \gg 1$. If $1/2 < \nu \leq \nu_0$, (3.15) follows from the well known asymptotics:

$$\lim_{z\to 0^+} z\left|\frac{h'_\nu(z)}{h_\nu(z)}\right| = 1, \quad \lim_{z\to +\infty}\left|\frac{h'_\nu(z)}{h_\nu(z)}\right| = 1.$$

Moreover, $\eta_\nu(z) = O(z^{-1})$ as $z \to +\infty$, which proves (3.16) when $1/2 < \nu \leq \nu_0$. If $\nu \geq \nu_0 \gg 1$, (3.16) would follow from

$$-z^{1/3}\eta_\nu(\nu z) \leq C\nu^{-1/3}, \qquad \forall z \geq \nu^{-1}. \tag{3.17}$$

We have $-\eta_\nu(\nu z) = O(\nu^{-1})$ uniformly in $z \geq 2$. If $\nu^{-1} \leq z \leq 1/2$,

$$-z\eta_\nu(\nu z) = -\sqrt{1-z^2} + O(\nu^{-1}) \leq O(\nu^{-1}),$$

which clearly implies (3.17) in this case. Let now $1/2 \leq z \leq 2$. By Olver's expansions ([4]), we have $-\eta_\nu(\nu z) = k(z) + O(\nu^{-1/3})$ with a function $k(z)$ satisfying $\operatorname{Re} k(z) \leq 0$, which implies (3.17) in this case, too. □

It is easy to see that (3.1) would follow from (3.2), (3.4) and the following

Proposition 3.2. *There exist positive constants C and λ_0 so that for $\lambda \geq \lambda_0$ we have*

$$e^{-C\lambda}\|u\|_{H^2(\Omega_{\lambda\delta})} \leq \|v\|_{L^2(\Omega)} + \|\rho_q(L)f\|_{L^2(S_a)}. \tag{3.18}$$

Note that, in view of the coercivity of the boundary value problem, we have

$$\begin{aligned}\|u\|_{H^2(\Omega_{\lambda\delta})} &\leq O(1)\|u\|_{L^2(\Omega_{\lambda\delta})} + O(1)\|\Delta_g^a u\|_{L^2(\Omega_{\lambda\delta})}\\ &\leq O(1)\|u\|_{L^2(\Omega_{\lambda\delta})} + O(1)\|v\|_{L^2(\Omega)},\end{aligned}$$

so it suffices to prove (3.18) for $\|u\|_{L^2(\Omega_{\lambda\delta})}$ only.

4. Proof of Proposition 3.2

Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ for $|x| \leq a_0+2$, $\chi = 0$ for $|x| \geq a_0+3$, where $a_0 \gg 1$ does not depend on λ and will be fixed later on. Applying Theorem A.2 to the function χu (with $M = \Omega_{a_0+4}$) leads to the estimate

$$\begin{aligned}&\int_{\Omega_{a_0+2}} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) dx \\ &\qquad \leq e^{2\gamma_1\lambda}\int_{a_0+2\leq|x|\leq a_0+3}\left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) dx + e^{2\gamma_1\lambda}\|v\|^2_{L^2(\Omega)},\end{aligned} \tag{4.1}$$

with some $\gamma_1 > 0$.

Set $P = -\lambda^{-2}\Delta_g^a - 1$. If $\varphi \in C^\infty(\Omega_a)$, then $P_\varphi := e^{\lambda\varphi}Pe^{-\lambda\varphi}$ is a $\lambda - \Psi DO$ with principal symbol $p_\varphi(x,\xi) = p(x, \xi + i\nabla_g^a\varphi)$, p being the principal symbol of P considered as a $\lambda - \Psi DO$, and $\nabla_g^a\varphi$ is a vector-valued function defined by

$$(\nabla_g^a\varphi)_j = \sum_{i=1}^n (c^2\chi_a g_{ij} + (1-\chi_a)\delta_{ij})\partial_{x_i}\varphi, \quad j = 1, ..., n.$$

Fix an ε such that $0<\varepsilon<(2q)^{-1}-1$. Denote $r=|x|$ and for $a_0\le r\le a$, set

$$\varphi(r)=-1-\widetilde{C}(r^{-\varepsilon}-a_0^{-\varepsilon}),$$

where $\widetilde{C}>0$ is independent of r. Clearly, taking $\widetilde{C}=\widetilde{C}(a_0,\gamma_1)$ large enough we can arrange $\varphi(a_0+2)\ge\gamma_1+1$ and hence $\varphi(r)\ge\gamma_1+1$ for $a_0+2\le r\le a$. Since $\varphi(a_0)=-1$, there exist $a_0<a_1<a_2<a_0+1$ so that $\varphi(r)<0$ for $a_1\le r\le a_2$. Choose a function $\eta\in C^\infty(\mathbb{R}^n)$, $\eta=0$ for $|x|\le a_1$, $\eta=1$ for $|x|\ge a_2$. Set $w=e^{\lambda\varphi}\eta u$. In what follows $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will denote the norm and the scalar product in $L^2(a_0\le|x|\le a)$, $\|\cdot\|_1$ will denote the norm in $H^1(a_0\le|x|\le a)$, while $\|\cdot\|_0$ and $\langle\cdot,\cdot\rangle_0$ will denote the norm and the scalar product in $L^2(S_a)$. It is easy to see that Proposition 3.2 would follow from (4.1) and the following

Proposition 4.1. *There exist positive constants C and λ_0 so that for $\lambda\ge\lambda_0$ we have*

$$\|w\|_1+\lambda^{q/2}\|(L+1)w|_{S_a}\|_0 \le C\lambda^{1/2+q(1+\varepsilon)}\|P_\varphi w\|+C\lambda^{q/2}\|\rho_q(L)w|_{S_a}\|_0. \tag{4.2}$$

Proof. We will first prove the following

Lemma 4.2. *There exist positive constants C and λ_0 so that for $\lambda\ge\lambda_0$ we have*

$$\|w\|_1\le C\lambda^{1/2+q(1+\varepsilon)}\|P_\varphi w\|+C\lambda^{q/2}\|(L+1)w|_{S_a}\|_0. \tag{4.3}$$

Proof. When $q=0$ the lemma follows from the Carleman estimates of Lebeau-Robbiano [2]. We will modify their proof in a way allowing to get estimates uniform in both λ and a. Set $f_1:=w|_{S_a}$ and denote $\varphi'(r)=d\varphi(r)/dr$, $\varphi''(r)=d^2\varphi(r)/dr^2$. Then the boundary conditions on S_a become

$$\lambda^{-1}\partial_\nu w|_{S_a}=-(N(\lambda)+\varphi'(a))f_1.$$

Let P_φ^* be the formal adjoint to P_φ and denote $Q_1=\frac{P_\varphi+P_\varphi^*}{2}$, $Q_2=\frac{P_\varphi-P_\varphi^*}{2i}$ and $Q=i\lambda[Q_1,Q_2]$ with principal symbols $\operatorname{Re}p_\varphi$, $\operatorname{Im}p_\varphi$ and $\{\operatorname{Re}p_\varphi,\operatorname{Im}p_\varphi\}$, respectively. We are going to take advantage of the identity (see (16)–(18) of [2])

$$\lambda\|P_\varphi w\|^2=\lambda\|Q_1w\|^2+\lambda\|Q_2w\|^2+\langle Qw,w\rangle+\mathcal{B}(w), \tag{4.4}$$

where

$$\mathcal{B}(w)=\langle Q_2w|_{S_a},(i\lambda)^{-1}\partial_\nu w|_{S_a}\rangle_0+ \langle(i\lambda)^{-1}\partial_\nu Q_2w|_{S_a}+2\varphi'(a)Q_1w|_{S_a},f_1\rangle_0. \tag{4.5}$$

Using that $\Delta_g^a=\Delta$ near S_a, one can rewrite (4.5) in the form

$$\mathcal{B}(w)=-2\varphi'(a)\left(\|\lambda^{-1}\partial_\nu w|_{S_a}\|_0^2-\langle(L^2+O(1))f_1,f_1\rangle_0\right) +O((a\lambda)^{-1})\left|\langle\lambda^{-1}\partial_\nu w|_{S_a},f_1\rangle_0\right|,$$

and hence, in view of (3.5),

$$|\mathcal{B}(w)|\le C_1\varphi'(a)\left(\|N(\lambda)f_1\|_0^2+\|(L+1)f_1\|_0^2\right)\le C_2a^{-1-\varepsilon}\|(L+1)f_1\|_0^2, \tag{4.6}$$

with $C_2 > 0$ independent of λ.

Introduce the polar coordinates $r = |x|$, $a_0 \le r \le a$, $\theta = \frac{x}{|x|} \in S_1$, and denote by (ρ, σ) the dual variables of (r, θ). Set $\mathcal{D}_r = (i\lambda)^{-1}\partial_r$, $\mathcal{D}_\theta = (i\lambda)^{-1}\partial_\theta$. In view of (1.1), it is easy to see that the principal symbols of P, Q_1 and Q_2 can be written in these coordinates as follows:

$$\begin{aligned} p(r,\theta,\rho,\sigma) &= (1 + b_1(r,\theta))\rho^2 + \frac{|\sigma|^2}{r^2}(1 + b_2(r,\theta)) - 1, \\ \operatorname{Re} p_\varphi(r,\theta,\rho,\sigma) &= (1 + b_1(r,\theta))\rho^2 + \frac{|\sigma|^2}{r^2}(1 + b_2(r,\theta)) - 1 - \varphi'(r)^2, \\ \operatorname{Im} p_\varphi(r,\theta,\rho,\sigma) &= 2(1 + b_1(r,\theta))\varphi'(r)\rho, \end{aligned}$$

where $b_j \in C^\infty$, $b_j = 0$ for $\frac{2}{3}a \le r \le a$, $b_j = O(r^{-\infty})$ with all its first derivatives. Furthermore, an easy computation gives

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = 4(1 + O(r^{-\infty}))\left(\varphi''\rho^2 + \frac{\varphi'|\sigma|^2}{r^3} + \varphi'^2\varphi''\right).$$

More precisely, the operator Q can be written in the form

$$Q = \widetilde{b}_1 Q_1 + \widetilde{b}_2 Q_2 + \mathcal{D}_r(\widetilde{b}_3 \mathcal{D}_r) + Q_0, \tag{4.7}$$

where $\widetilde{b}_j \in C^\infty$, $\widetilde{b}_j = O(\varphi'') = O(r^{-2-\varepsilon})$, $\widetilde{b}_3$ real-valued. Q_0 is a second order differential operator with coefficients, $c_\beta(r,\theta;\lambda)$, which in view of (1.2) satisfy

$$|\partial_r^k c_\beta(r,\theta;\lambda)| + |(r^{-1}\partial_\theta)^\alpha c_\beta(r,\theta;\lambda)| \le C_{k,\alpha} < \infty,$$

for all multi-indeces (k,α) with constants $C_{k,\alpha}$ independent of λ, whose principal symbol is of the form $q_0(r,\theta,\sigma)$ and satisfies, for $a_0 \le r \le a$,

$$q_0(r,\theta,\sigma) \ge Cr^{-2-\varepsilon}\left(\frac{|\sigma|^2}{r^2} + 1\right) \ge Ca^{-2-\varepsilon}\left(\frac{|\sigma|^2}{r^2} + 1\right), \quad C > 0.$$

Hence, by Gärding's inequality

$$\operatorname{Re}\langle Q_0 w, w\rangle \ge Ca^{-2-\varepsilon}\int_{a_0}^{a} \|(1 - \lambda^{-2}\Delta_{S_r})^{1/2} w\|^2_{L^2(S_r)} dr - O(\lambda^{-1})\|w\|_1^2. \tag{4.8}$$

Integrating by parts we get

$$\begin{aligned} \langle Qw, w\rangle &= \operatorname{Re}\langle Q_0 w, w\rangle + \operatorname{Re}\langle \widetilde{b}_1 Q_1 w, w\rangle + \operatorname{Re}\langle \widetilde{b}_2 Q_2 w, w\rangle \\ &\quad + \operatorname{Re}\langle \widetilde{b}_3 \mathcal{D}_r w, \mathcal{D}_r w\rangle + \widetilde{b}_3(a)\operatorname{Re}\langle \lambda^{-1}\partial_\nu w|_{S_a}, f_1\rangle_0 \\ &\ge \operatorname{Re}\langle Q_0 w, w\rangle - O(a_0^{-2-\varepsilon})\left(\lambda\|Q_1 w\|^2 + \lambda\|Q_2 w\|^2 + \|\mathcal{D}_r w\|^2\right) \\ &\quad - O(\lambda^{-1})\|w\|^2 - O(a^{-2-\varepsilon})\|(L+1)f_1\|_0^2. \end{aligned} \tag{4.9}$$

Combining (4.4), (4.6), (4.8) and (4.9) and taking $a_0 \gg 1$ large enough, independent of λ, lead to

$$\begin{aligned}\lambda\|P_\varphi w\|^2 &\geq C_1\lambda\varphi'(a)^2\|\mathcal{D}_r w\|^2 + C_1 a^{-2-\varepsilon}\int_{a_0}^a \|(1-\lambda^{-2}\Delta_{S_r})^{1/2}w\|^2_{L^2(S_r)}dr\\ &\qquad - O(\lambda^{-1})\|w\|_1^2 - O(a^{-1-\varepsilon})\|(L+1)f_1\|_0^2\\ &\geq \left(C_2 a^{-2-\varepsilon} - O(\lambda^{-1})\right)\|w\|_1^2 - O(a^{-1-\varepsilon})\|(L+1)f_1\|_0^2\\ &\geq C_3 a^{-2-\varepsilon}\|w\|_1^2 - O(a^{-1-\varepsilon})\|(L+1)f_1\|_0^2,\end{aligned}$$

with a constant $C_3 > 0$, independent of λ, which clearly implies (4.3). □

Set $\zeta_q(t) = (1-\rho_q(t))(1+t)$. Clearly, Proposition 4.1 would follow from Lemma 4.2 and the following

Lemma 4.3. *There exist positive constants C and λ_0 so that for $\lambda \geq \lambda_0$ we have*

$$\|\zeta_q(L)f_1\|_0 \leq C\lambda^{1/2+q}\|P_\varphi w\| + C\lambda^{-1/2}\|w\|_1. \tag{4.10}$$

Proof. Let $\phi(t) \in C_0^\infty(\mathbb{R})$, $\phi(t) = 1$ for $|t| \leq 1/5$, $\phi(t) = 0$ for $|t| \geq 1/4$, and set $\phi_a(r) = \phi(1-r/a)$, $w_1 = \phi_a(r)\zeta_q(L)w$. Clearly, $\Delta_g^a = \Delta$ on $\operatorname{supp} w_1$. Integrating by parts one gets

$$\operatorname{Im}\langle Q_2 w_1, w_1\rangle = O(\lambda^{-1}\varphi'(a))\|\zeta_q(L)f_1\|_0^2, \tag{4.11}$$

$$\begin{aligned}\operatorname{Re}\langle (Q_1-\mathcal{D}_r^2)w_1, w_1\rangle + \|\mathcal{D}_r w_1\|^2 &= \operatorname{Re}\langle P_\varphi w_1, w_1\rangle\\ &\quad + \lambda^{-1}\operatorname{Re}\langle N(\lambda)\zeta_q(L)f_1, \zeta_q(L)f_1\rangle_0\\ &\quad + O(\lambda^{-1}\varphi'(a))\|\zeta_q(L)f_1\|_0^2.\end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{aligned}\operatorname{Re}\langle (Q_1-\mathcal{D}_r^2)w_1, w_1\rangle &= \left\langle \left(\left(\frac{a}{r}\right)^2 L^2 - 1 - \varphi'(r)^2\right) w_1, w_1\right\rangle\\ &\quad + O(\lambda^{-1})\langle \mathcal{D}_r w_1, w_1\rangle\\ &\geq \left\langle \left(L^2 - 1 - (\varphi'(a/2))^2\right) w_1, w_1\right\rangle\\ &\quad - O(\lambda^{-1})\left(\|\mathcal{D}_r w_1\|^2 + \|w_1\|^2\right).\end{aligned} \tag{4.13}$$

Since

$$\left(t - 1 - (\varphi'(a/2))^2\right)\zeta_q(t) \geq C\lambda^{-q}(t+1)\zeta_q(t), \qquad C > 0,$$

we have that the scalar product in the RHS of (4.13) is estimated from below by

$$C\lambda^{-q}\|(L+1)w_1\|^2.$$

Thus, by (4.12) and (4.13) together with (3.6), $\forall \varepsilon_0 > 0$,

$$\begin{aligned}\lambda^{-q}\|(L+1)w_1\|^2 + \|\mathcal{D}_r w_1\|^2 &\leq O_{\varepsilon_0}(\lambda^q)\|(L+1)^{-1}P_\varphi w_1\|^2\\ &\quad + \varepsilon_0\lambda^{-q}\|(L+1)w_1\|^2 + \lambda^{-1-q}O\left(\lambda^{-\frac{1}{3}(1-2q)}\right)\|\zeta_q(L)f_1\|_0^2,\end{aligned}$$

and hence

$$\begin{aligned}(4.14)\quad & \lambda\|(L+1)w_1\|^2+\lambda\|\mathcal{D}_r w_1\|^2 \\ & \le O(\lambda^{1+2q})\|(L+1)^{-1}P_\varphi w_1\|^2+O\left(\lambda^{-\frac{1}{3}(1-2q)}\right)\|\zeta_q(L)f_1\|_0^2.\end{aligned}$$

On the other hand,

$$\|\zeta_q(L)f_1\|_0^2\le\lambda\|w_1\|^2+\lambda\|\mathcal{D}_r w_1\|^2. \tag{4.15}$$

By (4.14) and (4.15),

$$\begin{aligned}\|\zeta_q(L)f_1\|_0 &\le O(\lambda^{1/2+q})\|(L+1)^{-1}P_\varphi w_1\| \\ &\le O(\lambda^{1/2+q})\|P_\varphi(\phi_a w)\| \\ &\le O(\lambda^{1/2+q})\|P_\varphi w\|+O(\lambda^{1/2+q})\|[P_\varphi,\phi_a]w\| \\ &\le O(\lambda^{1/2+q})\|P_\varphi w\|+O(\lambda^{-1/2})\|w\|_1,\end{aligned}$$

since $\phi_a'=O(a^{-1})$, $\phi_a''=O(a^{-2})$. □

Appendix

Let (M,g) be a compact, connected Riemannien manifold with a C^∞-smooth boundary ∂M, and denote by Δ the Laplace-Beltrami operator on (M,g). Let $U\subset M$ be an arbitrary open domain such that $\partial M\cap\partial U=\emptyset$. Set $X=(-1,1)\times M$, $Y=(-\frac{1}{2},\frac{1}{2})\times M$, $Z=(-1,1)\times U$, and denote $Q=\partial_t^2+\Delta$. The following estimates are due to Lebeau-Robbiano [2], [3]:

Theorem A.1. *Let $v(t,x)\in C^\infty(X)$ satisfy either the Dirichlet or Neumann boundary conditions on ∂M for every $t\in(-1,1)$. Then there exist positive constants C and μ, $0<\mu<1$, such that*

$$\|v\|_{H^1(Y)}\le C\left(\|v\|_{H^1(X)}\right)^{1-\mu}\left(\|Qv\|_{L^2(X)}+\|v\|_{H^1(Z)}\right)^{\mu}. \tag{A.1}$$

In the case of Dirichlet boundary conditions this theorem is proved in Section 3 of [2]. The main idea is to prove (A.1) locally (which in turn is done by obtaining local Carleman estimates) and then to propagate this estimate up to an arbitrary open domain in M. In the case of Neumann boundary conditions the proof goes in the same way except that in this case the Carleman estimates are harder to prove. Such Carleman estiamtes are established in Section 5 of [3].

Let us apply the above theorem to the function $v(t,x)=e^{t\lambda}u(x)$, where $\lambda\in\mathbb{R}$ and $u\in C^\infty(M)$. Denote $P=\Delta+\lambda^2$ and observe that $Qv=e^{t\lambda}Pu$. We have the following

Theorem A.2. *Let $u(x)\in C^\infty(M)$ satisfy either Dirichlet or Neumann boundary conditions on ∂M. Then there exist positive constants C and γ, independent of λ, such that*

$$\|u\|_{H^1(M)}\le Ce^{\gamma|\lambda|}\left(\|Pu\|_{L^2(M)}+\|u\|_{H^1(U)}\right). \tag{A.2}$$

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