A SIMPLE PROOF OF A THEOREM
OF LAPTEV AND WEIDL

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ABSTRACT. A new, elementary proof of a recent result of Laptev and Weidl [LW] is given. It is a sharp Lieb–Thirring inequality for one dimensional Schrödinger operators with matrix valued potentials.

I. Introduction

In this note we give a new and, we believe, simpler proof of a recent result of Laptev and Weidl. It is concerned with Lieb–Thirring inequalities for matrix valued Schrödinger operators of the type

$$
(1) \qquad H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I + V(x)
$$

acting on $L^2(\mathbb{R}; \mathbb{C}^N)$. The potential $V(x)$ is a negative definite hermitean $N \times N$ matrix. We assume that its matrix elements are smooth functions of compact support, say in the interval $[-a, a]$. The operator H has finitely many negative eigenvalues which, counting multiplicities, we denote by $-\lambda_j$, $j = 1, \ldots, L$.

The following theorem was proved in [LW].

Theorem 1. *With the above assumptions on V the following inequality holds*

$$
(2) \qquad \sum_{j=1}^{L} \lambda_j^{3/2} \le \frac{3}{16} \int_{\mathbb{R}} \mathrm{Tr}(V(x)^2)\, dx \ .
$$

From Weyl's law on the distribution of eigenvalues it is seen that this inequality is best possible. For the case where the potential is a scalar function this result was already proved in [LT] where it was realized that (2) follows from trace identities.

The matrix case, however, is important, since inequality (2) of Laptev and Weidl is the starting point for deriving sharp Lieb–Thirring inequalities in higher

dimensions. In particular, the argument of [AL] applies also in this case and yields sharp Lieb-Thirring inequalities for the sum of powers of eigenvalues where the power is larger than $3/2$. For the details we refer the reader to the original paper [LW] where a collection of beautiful results is presented. Their proof of Theorem 1 which corresponds to formula (2.1) in their paper is patterned after the proof of [BF] (see also [FZ]) and is fairly involved. Inequality (2) is derived from a trace identity, which in turn is a special case of a whole family of identities that express conservation laws of the Korteweg–de Vries equation. The derivation of these trace identities uses nontrivial results about scattering theory on the line and Laptev and Weidl prove these afresh for the matrix case. Since (2) is the central result in [LW] and of independent interest, it is of value to have a different, more elementary and more direct proof. It relies on the 'commutation method' and some elementary facts from the calculus of variations.

The 'commutation method' has a fairly long history, some versions of it were already known to Darboux [DG] and Jacobi[J]. Its modern appearance seems to be due to Crum [C]. For a rigorous discussion of these issues we refer to the papers of [G] and [DP]. In the latter more examples of the usefulness of this method are presented. Another work, closer to the spirit of ours, is the one of Schmincke [S] who uses the commutation method to prove that

$$\sum \lambda_j^{1/2} \geq -\frac{1}{4} \int V(x) \mathrm{d}x$$

for scalar potentials. This result was extended in [LW] to the matrix case which can also be obtained using the methods of the present work. This inequality should be contrasted with

$$\sum \lambda_j^{1/2} \leq -\frac{1}{2} \int V(x) \mathrm{d}x,$$

obtained in [HLT] for the scalar case and in [HLW] for the matrix case. Both inequalities are sharp in the sense that the constants cannot be improved.

To illustrate the ideas we give a short proof of Theorem 1 for the case where V is a scalar potential, thereby recovering the result in [LT]. This sets the stage for the proof of the matrix case in the following section. While it is certainly possible to prove Theorem 1 under fairly general conditions on the potential, we refrain from doing so. It would clutter the simple argument with technical details.

II. The scalar case

Let $-\lambda_1$ be the lowest eigenvalue of the Schrödinger operator (1) of Section I with a *scalar* potential. It is well known that this eigenvalue is not degenerate and the corresponding eigenfunction ϕ_1 can be chosen to be strictly positive. Moreover, outside the range of the potential we have

(1)
$$\phi_1(x) = \begin{cases} \text{const}.e^{-\sqrt{\lambda_1}x}, & \text{if } x > a, \\ \text{const}.e^{\sqrt{\lambda_1}x}, & \text{if } x < -a. \end{cases}$$

Thus the function

$$(2) \qquad F(x) = \frac{\phi_1'(x)}{\phi_1(x)},$$

is defined and satisfies the Riccati equation

$$(3) \qquad F' + F^2 = V + \lambda_1,$$

together with the conditions

$$(4) \qquad F(x) = \begin{cases} -\sqrt{\lambda_1}, & \text{if } x > a, \\ \sqrt{\lambda_1}, & \text{if } x < -a. \end{cases}$$

A simple computation shows that the Hamiltonian H can be written as

$$(5) \qquad H = D^*D - \lambda_1,$$

where

$$(6) \qquad D = \frac{\mathrm{d}}{\mathrm{d}x} - F,$$

and

$$(7) \qquad D^* = -\frac{\mathrm{d}}{\mathrm{d}x} - F.$$

It is a general fact [DP][G] that the operators D^*D and DD^* on $L^2(\mathbb{R})$ have the same spectrum with the possible exception of the zero eigenvalue. Note that D^*D has a zero eigenvalue which corresponds to the ground state of H. The operator DD^* does not have a zero eigenvalue. This follows from the fact that the corresponding eigenfunction ψ satisfies

$$(8) \qquad \psi' = -F\psi,$$

and hence $\psi(x) = \mathrm{const}./\phi_1(x)$ which grows exponentially and is not normalizable. Thus the new Schrödinger operator

$$(9) \qquad \tilde{H} = DD^* - \lambda_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - F' + F^2 - \lambda_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V - 2F',$$

has, except for the eigenvalue $-\lambda_1$, precisely the same eigenvalues as H. Also note that the potential $V - 2F'$ is smooth and has support in the same interval as the potential V.

Next, we compute using the Riccati equation (3)

$$\int (V - 2F')^2 \mathrm{d}x = \int V^2 \mathrm{d}x + 4 \int (\lambda_1 - F^2) F' \mathrm{d}x.$$

The last term can be computed explicitly using (4) and we obtain

$$(10) \qquad \int (V - 2F')^2 \mathrm{d}x = \int V^2 \mathrm{d}x - \frac{16}{3}\lambda_1^{3/2}.$$

Thus,

$$(11) \qquad \sum_{k=1}^{L} \lambda_k^{3/2} - \frac{3}{16}\int V^2 \mathrm{d}x = \sum_{k=2}^{L} \lambda_k^{3/2} - \frac{3}{16}\int (V - 2F')^2 \mathrm{d}x,$$

and the Schrödinger operator with the potential $V - 2F'$ has precisely the eigenvalues $-\lambda_2, \ldots, -\lambda_L$. Continuing this process we remove one eigenvalue after another. After the last one is removed a manifestly negative quantity is left over, and this proves Theorem 1 in the scalar case. $\qquad\square$

III. The matrix case

The proof of Theorem 1 is patterned after the scalar case. In addition to the usual eigenvalue equation for H in (1) of Section I

$$(1) \qquad -\phi''(x) + V(x)\phi(x) = -\lambda\phi(x),$$

we consider the following matrix version for an $N \times N$ matrix $M(x)$,

$$(2) \qquad -M''(x) + V(x)M(x) = -\lambda M(x).$$

The following Lemma is central.

Lemma 2. *Assume that $-\lambda$ is the ground state energy of H and let ϕ be any solution of the differential equation* (1) *with*

$$\phi(x) = e^{\sqrt{\lambda}x}u, \qquad \text{for } x < -a,$$

where $0 \neq u \in \mathbb{C}^n$ is constant. In particular, we do not require that ϕ is normalizable. Then $\phi(x)$ never vanishes. Moreover, the ground state energy is at most N–fold degenerate.

Proof. Suppose there exists a point x_0 with $\phi(x_0) = 0$. Consider the continuous function

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & \text{if } x < x_0 \\ 0, & \text{if } x \geq x_0. \end{cases}$$

Clearly, this function does not vanish identically and is square integrable. A simple integration by parts calculation shows that

$$(\tilde{\phi}, H\tilde{\phi}) = -\lambda(\tilde{\phi}, \tilde{\phi}).$$

Here $(\ ,\ )$ denotes the inner product on $L^2(\mathbb{R}, \mathbb{C}^N)$. Thus $\tilde{\phi}$ is a ground state and must be a solution of the Schrödinger equation (1) which is an ordinary differential equation of second order. Since $\tilde{\phi}$ vanishes to the right of x_0 the solution must vanish everywhere, which is a contradiction.

The last statement of the lemma is an immediate consequence of this.   □

Remark. The above Lemma clearly generalizes to potentials that do not have compact support but decay, e.g., exponentially, at infinity.

Consider any matrix solution $M(x)$ of the differential equation (2) subject to the condition

$$(3) \qquad M(x) = e^{\sqrt{\lambda}x} A, \qquad \text{for } x < -a,$$

where A is a *nonsingular* matrix. By the previous Lemma 1, any solution of (1) that decays exponentially must be a linear combination of the column vectors of $M(x)$. In particular, the ground states themselves must be linear combinations of the column vectors of $M(x)$. Also by Lemma 2 we know that the matrix $M(x)$ must be invertible for every $x \in \mathbb{R}$. Hence it makes sense to define

$$(4) \qquad F(x) = M'(x)M^{-1}(x).$$

The following Lemma 3 states all we need to know about $F(x)$. The number K below denotes the degeneracy of the ground state energy. We have that $K \leq N$ by Lemma 2.

Lemma 3. *The matrix $F(x)$ is hermitean for every $x \in \mathbb{R}$, independent of the choice of A and satisfies the matrix Riccati equation*

$$(5) \qquad F' + F^2 = V + \lambda I.$$

Moreover, for $x < -a$,

$$(6) \qquad F(x) = \sqrt{\lambda}I,$$

and for $x > a$, the eigenvectors of $F(x)$ are independent of x and its eigenvalues decay exponentially fast to the K fold eigenvalue $-\sqrt{\lambda}$ and the $N - K$ fold eigenvalue $\sqrt{\lambda}$ respectively.

Proof. Consider any two matrix solutions of (2), $M_1(x)$ and $M_2(x)$. From the Wronskian identity (with $*$ denoting adjoint)

$$(7) \qquad \frac{d}{dx}\left(M_1(x)^* M_2'(x) - M_1^{*\prime}(x) M_2(x)\right) = 0,$$

one obtains

$$(8) \qquad \left(M_1(x)^* M_2'(x) - M_1^{*\prime}(x) M_2(x)\right) = \text{const}.$$

First we set $M_1 = M_2 = M$ and assuming the initial condition (3) we get that

$$(9) \qquad\qquad M(x)^* M'(x) = M^{*\prime}(x) M(x),$$

which yields the hermicity of F. If we set $F_1 = M_1' M_1^{-1}$ and $F_2 = M_2' M_2^{-1}$ where M_1 and M_2 satisfy (2) and (3) for possibly two different, nonsingular matrices A_1 and A_2 we get from (8) that $F_1 \equiv F_2$. An elementary computation yields (5) and (6).

Fix $x_0 > a$. For $x > x_0$, the potential vanishes and the matrix $M(x)$ is given by

$$(10) \qquad M(x) = \cosh(\sqrt{\lambda}(x - x_0)) M(x_0) + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x - x_0)) M'(x_0),$$

and hence $F(x)$ is given by

$$(11) \quad F(x) =$$
$$\sqrt{\lambda} \left(\sqrt{\lambda} \tanh(\sqrt{\lambda}(x - x_0)) I + F(x_0) \right) \left(\sqrt{\lambda} I + \tanh(\sqrt{\lambda}(x - x_0)) F(x_0) \right)^{-1}.$$

From this it follows that the eigenvectors of $F(x)$ do not depend on x and since $F(x)$ exists for all x we must have $-\sqrt{\lambda} I \leq F(x_0)$. It follows from (10) that since the bound states are precisely those solutions $\phi(x)$ of the differential equation (1) that decay exponentially in both directions, they must be precisely of the form $\phi(x) = M(x) M^{-1}(x_0) u$ where u is an eigenvector of $F(x_0)$ with eigenvalue $-\sqrt{\lambda}$. Thus $F(x_0)$ has the K fold eigenvalue $-\sqrt{\lambda}$ and all the other eigenvalues ν_j satisfy the inequality $-\sqrt{\lambda} < \nu_j$. From (11) we see $-\sqrt{\lambda}$ is a K fold degenerate eigenvalue of $F(x)$ for all $x \geq x_0$ and that all the other eigenvalues converge exponentially fast to $\sqrt{\lambda}$. $\qquad\square$

Proof of Theorem 1 in the matrix case. From the Riccati equation (5) we get that

$$(13) \qquad\qquad H + \lambda_1 I = D^* D,$$

where

$$(14) \qquad\qquad D^* = \left(-\frac{d}{dx} \otimes I - F \right),$$

and

$$(15) \qquad\qquad D = \left(\frac{d}{dx} \otimes I - F \right).$$

Clearly

$$(16) \qquad\qquad D\phi = 0$$

for any ground state ϕ. Moreover,

$$(17) \qquad\qquad\qquad\qquad D^*\psi = 0$$

has no nontrivial normalizable solution on $\mathbb{R}$ since $F = \sqrt{\lambda_1}I$ for $x < -a$.

    Thus the operator

$$(18) \qquad\qquad\qquad\qquad H' := DD^* - \lambda_1 I,$$

has precisely the eigenvalues $-\lambda_{K+1}, \ldots, -\lambda_L$. A calculation shows that

$$(19) \qquad\qquad\qquad H' = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I + V(x) - 2F'(x),$$

where the potential

$$(20) \qquad\qquad\qquad\qquad V(x) - 2F'(x)$$

is smooth and decays exponentially fast at infinity by Lemma 3. One easily computes using (5) that

$$(21) \quad \int_{\mathbb{R}} \mathrm{Tr}\left((V - 2F')^2\right) \mathrm{d}x \;=\; \int_{\mathbb{R}} \mathrm{Tr}\left(V^2\right) \mathrm{d}x - 4 \int_{\mathbb{R}} \mathrm{Tr}\left(\left(F^2 - \lambda_1 I\right) F'\right) \mathrm{d}x,$$

which can be integrated to yield

$$(22) \qquad\qquad -\frac{4}{3}\mathrm{Tr}\left(F^3(x)\right)\big|_{-\infty}^{+\infty} + 4\lambda_1 \mathrm{Tr}\left(F(x)\right)\big|_{-\infty}^{+\infty}.$$

By Lemma 2 this equals

$$(23) \qquad\qquad\qquad\qquad -\frac{16}{3}K\lambda_1^{3/2}.$$

Again, we have shown that

$$(24) \quad \sum_{j=1}^{L} \lambda_j^{3/2} - \frac{3}{16}\int_{\mathbb{R}} \mathrm{Tr}(V^2)\mathrm{d}x = \sum_{j=K+1}^{L} \lambda_j^{3/2} - \frac{3}{16}\int_{\mathbb{R}} \mathrm{Tr}\left((V - 2F')^2\right)\mathrm{d}x,$$

and the Schrödinger operator with the potential $V - 2F'$ has precisely the eigenvalues $-\lambda_{K+1}, \ldots, -\lambda_L$.

    The potential $V - 2F'$ decays exponentially but, unfortunately, does not have compact support and hence the second step in the scalar case, i.e., the removal of the next eigenvalue, cannot be taken directly. However, the following approximation argument can be used to circumvent this difficulty. Cutting off the potential $V - 2F'$ sufficiently far out we are left with a new potential $V - 2F_c'$ which has compact support and whose eigenvalues are numbers $-\mu_{K+1}, \cdots -\mu_L$

which can be made to be as close to the old ones $-\lambda_{K+1}, \cdots - \lambda_L$ as we please. The cutoff might cause some new eigenvalues to appear, but all of those can be made to be as close to the continuum, i.e., as close to 0 as we please. Removing the ground state eigenvalue of this new potential $V - 2F'_c$ yields

$$\sum_{j=K+1}^{L} \mu_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}\left((V - 2F'_c)^2\right) \mathrm{d}x =$$

$$\sum_{j=K+P+1}^{L} \mu_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}\left((V - 2F'_c - 2G')^2\right) \mathrm{d}x \ .$$

Here P denotes the degeneracy of μ_{K+1} and G plays the same role for $V - 2F'_c$ as F does for V. Although tempting, one cannot remove the cutoff in this formula since the two terms on the right side are not separately continuous. E.g., the degeneracy of the eigenvalue μ_{K+1} is not necessarily the same as the degeneracy of the eigenvalue λ_{K+1}. Nevertheless we have the following

$$\sum_{j=1}^{L} \lambda_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}(V^2) \mathrm{d}x$$

$$= \sum_{j=K+1}^{L} \lambda_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}\left((V - 2F')^2\right) \mathrm{d}x$$

$$= \sum_{j=K+P+1}^{L} \mu_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}\left((V - 2F'_c - 2G')^2\right) \mathrm{d}x \ + e_1,$$

where e_1 is the error in the eigenvalues and the potential integral due to the cutoff in the potential. Again, the new potential has exponential decay.

By repeating the cutting and removing procedure finitely many, say $s \leq L$ times, we end up with

$$(25) \quad \sum_{j=1}^{L} \lambda_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}(V^2) \mathrm{d}x = -\frac{3}{16} \int \mathrm{Tr}(W^2) \mathrm{d}x + e_1 + \cdots + e_s,$$

where the e_i denotes the error stemming from the cutoff at the i-th step and W is the resulting potential. In particular (25) implies that

$$\sum_{j=1}^{L} \lambda_j^{3/2} - \frac{3}{16} \int \mathrm{Tr}(V^2) \mathrm{d}x \leq e_1 + \cdots e_s,$$

which we can make as small as we please. $\square$

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