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A SHARP ESTIMATE ON THE NORM OF THE MARTINGALE TRANSFORM

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1. Introduction

The boundedness of singular integral operators in $L^2(w)$ for $w \in A_2$ has been known for a long time, by the Hunt-Muckenhaupt-Wheeden Theorem. What is as of yet unknown is the sharp bound of these operators in terms of the A_2 norm of w. In his thesis, S. Buckley proved that the Hardy-Littlewood maximal operator is bounded linearly in $\|w\|_{A_2}$, the square function operator bound is no more than $\|w\|_{A_2}^{3/2}$ and that the Hilbert transform norm is no worse than quadratic. It is easily seen that each of these operators can not have a better bound than linear in $\|w\|_{A_2}$ (just look at power weights). In this paper, following the methods of [6], we show that the bound for the martingale transform, which is a dyadic analog of singular integral operators, is linear. We also give a simple proof that the dyadic square function is bounded linearly.

2. Notation

In what follows, h_I will denote the normalized Haar function for the dyadic interval I, i.e., $h_I = \frac{\chi_{I_l} - \chi_{I_r}}{\sqrt{|I|}}$, where I_l, I_r denote the left and right children of I respectively. The weight w and its inverse w^{-1} will be dyadic A_2 weights on $[0,1]$. w will be normalized to have $\int_{[0,1]} w(x) = 1$. Let $(f)_I$ denote $\frac{1}{|I|}\int_I f(x)dx$ for I a dyadic interval. Sometimes the parentheses are omitted when it is clear which function we are averaging. w_I^{-1} will denote $\frac{1}{|I|}\int_I \frac{1}{w(x)}dx$. Let $\mu(I) = w_I w_I^{-1}$ and $\|w\|_{A_2} = \sup_{I\in D} \mu(I)$.

We will write w in the form $w(x) = \prod(1 + c_I h_I)$ where $c_I = \frac{w_{I_l} - w_{I_r}}{2w_I}\sqrt{|I|}$. Similarly, we define $d_I = \frac{w_{I_l}^{-1} - w_{I_r}^{-1}}{2w_I^{-1}}\sqrt{|I|}$.

$\langle\, ,\rangle_\mu$ will denote the inner product in $L^2(d\mu)$. If the subscript is omitted, the measure is dx.

The family of operators which we are concerned with are the martingale transforms

$$T_r f = \sum_{I\in D[0,1]} r(I)\langle f, h_I\rangle h_I$$

where $r(I)$ assumes the values $+1$ and -1 only.

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3. The main theorem

Theorem 3.1.

$$\|T_r f\|_{L^2(w)} \leq c\,\|w\|_{A_2}\,\|f\|_{L^2(w)}\,,$$

for $w \in A_2$ and $f \in L^2(w)$.

We will prove this Theorem in section 5.

4. Useful lemmas and theorems

The following theorem can be found in [6], Section 2.

Theorem 4.1. *Let $\alpha_I \geq 0$. Then*

$$\sum_{I\in D[0,1]} (fw^{1/2})_I^2 \alpha_I \leq 4c\,\|f\|_2^2$$

for all $f \in L^2(dx)$ iff

$$\frac{1}{|J|}\sum_{I\subset J}(w_I)^2\alpha_I \leq c w_J$$

for all $J \in D[0,1]$.

Theorem 4.2. *Let $\alpha_I \geq 0$. If*

$$\frac{1}{|J|}\int_J (\sum_{I\subset J} \alpha_I w_I \chi_I(x))^2 w^{-1} dx \leq c_1 w_J$$

for all $J \in D$ and

$$\frac{1}{|J|}\int_J (\sum_{I\subset J} \alpha_I w_I^{-1} \chi_I(x))^2 w dx \leq c_1 w_J^{-1}$$

for all $J \in D$, then

$$\sum_{I\in D} \alpha_I (fw^{1/2})_I (gw^{-1/2})_I |I| \leq c_2\,\|f\|_{L^2}\,\|g\|_{L^2}\,,$$

for all $f, g \in L^2(dx)$, where $c_2 \leq c\sqrt{c_1}$ and c is an absolute constant.

Proof. From [6], section 4, we know that if

$$\frac{1}{|J|}\int_J (\sum_{I\subset J} \alpha_I w_I \chi_I(x))^2 v(x) dx \leq w_J$$

for all $J \in D$, then

$$\sum_{I\in D_1} \alpha_I (fw^{1/2})_I (gv^{1/2})_I |I| \leq c(\|f\|_{L^2}^2 + \|g\|_{L^2}^2),$$

where $D_1 = \{I \in D : \frac{(fw^{1/2})_I^2}{w_I} \geq \frac{(gv^{1/2})_I^2}{v_I}\}$.

To prove Theorem 4.2, we use this theorem with $v = \frac{w^{-1}}{c_1}$. Then, since in the conclusion v appears with a square root only, the bound for the conclusion would be of the order $\sqrt{c_1}$. Furthermore, imposing that the condition be fulfilled for both w and w^{-1} , we can sum over all dyadic intervals, since for all of them either

$$\frac{(fw^{1/2})_I^2}{w_I} \geq \frac{(g(w^{-1})^{1/2})_I^2}{w_I^{-1}},$$

or

$$\frac{(fw^{1/2})_I^2}{w_I} \leq \frac{(g(w^{-1})^{1/2})_I^2}{w_I^{-1}}$$

is true. Finally, letting $g^* = (\frac{\|g\|_{L^2}}{\|f\|_{L^2}})^{1/2} g$ and $f^* = (\frac{\|g\|_{L^2}}{\|f\|_{L^2}})^{-1/2} f$, we get

$$\sum_{I \in D_1} \alpha_I (f^* w^{1/2})_I (g^*(w^{-1})^{1/2})_I |I| \leq 2c(\|f^*\|_{L^2} \|g^*\|_{L^2}),$$

since this normalization does not change the right hand side at all. □

The following estimate is not used in the proof of the main theorem, but may be interesting in its own right. It gives the (unweighted) carleson norm of the sequence $\{c_I\}$. See [7], theorem 4.1 .

Lemma 4.3. *For $w \in A_2$ where $w(x) = C \prod (1 + c_I h_I)$, the following is true:*

$$\sum_{I \subset J} c_I^2 \leq 2 \log(\|w\|_{A_2}) |J|,$$

and this estimate is sharp.

Proof. By the definition of w,

$$(1 - c_I^2/|I|) = 1 - (\frac{w_{I_l} - w_{I_r}}{2w_I})^2 = \frac{4(1/2(w_{I_l} + w_{I_r}))^2 - (w_{I_l} - w_{I_r})^2}{(2w_I)^2} = \frac{w_{I_l} w_{I_r}}{(w_I)^2},$$

and therefore

$$\frac{w_{I_l} w_{I_r}}{(w_I)^2} \leq e^{-c_I^2/|I|}.$$

Taking square roots, this becomes $w_I \geq e^{c_I^2/2|I|}(w_{I_l} w_{I_r})^{1/2}$.

Since $(w_I^{-1})^2 \geq (w_{I_l}^{-1} w_{I_r}^{-1})$, we can multiply these two equations to get

$$\mu(I) \geq e^{c_I^2/2|I|}(\mu(I_l)\mu(I_r))^{1/2}.$$

Now use the same inequality to replace $\mu(I_l)$ and $\mu(I_r)$, and repeat this process. After n steps we get

$$\mu(I) \geq \exp\left(\sum_{K \subset I, |K| \geq 2^{-n}|I|} \frac{c_K^2 |K|}{2|K||I|} \right) \left(\prod_{|K| = 2^{-n}|I|} \mu(K) \right)^{2^{-n}}.$$

(The $|K|/|I|$ in the exp comes from the repeated square roots that we took.) Realizing that $\mu(K) \geq 1$ always, we get

$$\mu(I) \geq \exp\left(\sum_{K\subset I, |K|\geq 2^{-n}|I|} \frac{c_K^2}{2|I|}\right).$$

Taking the limit as $n \to \infty$, and taking logarithms, we get the lemma.

To see that the bound is indeed sharp, consider $w = \exp\left(\sum_{i\in\mathbb{N}} b_i\Phi_i\right)$ where Φ_i are Rademacher functions. Then $\sum_{I\subset J} c_I^2 = \sum_{i\geq j} \tanh^2(b_i)|J|$, while $\|w\|_{A_2} = \prod_{i\in\mathbb{N}}(1+\sinh^2(b_i))$. □

Theorem 4.4. *Let Sf be the dyadic square function of f. Then*

$$\|Sf\|_{L^2(w)} \leq c\,\|w\|_{A_2}\,\|f\|_{L^2(w)},$$

and this estimate is sharp.

Note. This estimate has also been proven, using a different method, in [5].

Proof. Let

$$f_I = \frac{h_I}{(w_I)^{1/2}},$$

and

$$g_I = h_I (w_I)^{1/2} w^{-1}.$$

Then $\langle f_I, g_J\rangle_w = 1$ if $I = J$ and 0 otherwise. Let $f \in L^2(w)$.

$$\|Sf\|_{L^2(w)} = \left(\sum_{I\in D} \langle f, h_I\rangle^2 w_I\right)^{1/2} = \left(\sum_{I\in D} \langle f, g_I\rangle_w^2\right)^{1/2}.$$

In that notation, the theorem to prove becomes

$$\sum_{I\in D} \langle f, g_I\rangle_w^2 \leq \|w\|_{A_2}^2 \|f\|_{L^2(w)}^2.$$

We will bound this by first showing that

$$\sum_{I\in D} \langle f, f_I\rangle_w^2 \leq \|w\|_{A_2} \|f\|_{L^2(w)}^2,$$

from which the desired inequality can be extracted by algebraic manipulations.

By the theorem on the lower bound of the square function in [3] we know the following:

$$\|f\|_{L^2(w)} \leq c\|w\|_{A_2}{}^{1/2} \left(\sum_{I\in D} \langle f, g_I\rangle_w^2\right)^{1/2}.$$

This allows us to compute the norm of the sequence valued operator $(Jg)_I = \langle g, f_I\rangle_w$ from $L^2(w)$ to l^2:

$$\sup_{\|g\|_{L^2(w)}=1} \sup_{\|\{k_I\}\|_{l^2}=1} \sum_{I\in D} k_I \int g f_I w =$$

$$\sup_{\|g\|_{L^2(w)}=1} \sup_{\|\{k_I\}\|_{l^2}=1} \int g \sum_{I\in D} k_I f_I w \leq$$

by Cauchy-Schwarz, and the above inequality,

$$\sup_{\|\{k_I\}\|_{l^2}=1} \sup_{\|g\|_{L^2(w)}=1} \left\| \sum_{I\in D} k_I f_I(x) \right\|_{L^2(w)} \|g\|_{L^2(w)} \leq$$

$$\|w\|_{A_2}{}^{1/2} \sup_{\|\{k_I\}\|_{l^2}=1} \left[\sum_{J\in D} \langle \left(\sum_{I\in D} k_I f_I \right), g_J\rangle_w^2 \right]^{1/2} =$$

$$\|w\|_{A_2}{}^{1/2} \sup_{\|\{k_I\}\|_{l^2}=1} \left(\sum_{I\in D} {k_I}^2 \right)^{1/2} = \|w\|_{A_2}{}^{1/2}.$$

This means that

$$\sum_{I\in D} \langle g, f_I\rangle_w^2 \leq \|w\|_{A_2} \|g\|_{L^2(w)}^2 ,$$

for every weight in A_2. In particular, this is true for the weight w^{-1}:

$$\sum_{I\in D} \langle g, \frac{h_I}{(w_I^{-1})^{1/2}}\rangle_{w^{-1}}^2 \leq \|w\|_{A_2} \|g\|_{L^2(w^{-1})}^2 .$$

Let $g^* = gw^{-1}$. Then this becomes

$$\sum_{I\in D} \langle g^*, \frac{h_I}{(w_I^{-1})^{1/2}}\rangle_{dx}^2 \leq \|w\|_{A_2} \|g^*\|_{L^2(w)}^2 .$$

The left hand side can be rewritten as

$$\sum_{I\in D} \langle g^*, h_I (w_I)^{1/2} w w^{-1}\rangle_{dx}^2 \frac{1}{\mu(I)}.$$

So finally, we have the inequality

$$\sum_{I\in D} \langle g^*, g_I\rangle_w^2 \leq \|w\|_{A_2}^2 \|g^*\|_{L^2(w)}^2 ,$$

which is what we needed to prove. □

Remark 4.5. We have the following inequality:

$$\frac{1}{|J|} \sum_{I\subset J} (w_I^{-1})^2 d_I^2 w_I \leq c \|w\|_{A_2}^2 w_J^{-1}.$$

This inequality follows by realizing that the left hand side is bounded by $\left\|S(w^{-1}\chi_J)\right\|_{L^2(w)}^2$, which by Theorem 4.4 is bounded by $c\|w\|_{A_2}^2 \left\|w^{-1}\chi_J\right\|_{L^2(w)}^2$.

Lemma 4.6.
$$\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2 w_I|I| \leq c\,\|w\|_{A_2}\, w_J|J|,$$
and this estimate is sharp.

Proof. By [1], 2.2 we have the following two estimates:
$$\sum_{I\subset J}(w_I)^p\left(\frac{w_{I_l}-w_{I_r}}{w_I}\right)^2|I| \leq \frac{c}{p(p-1)}\,\|w\|_{B_p}^p\,(w_J)^p|J|,$$
for $w\in B_p$, $p>1$, and
$$\sum_{I\subset J}(w_I)^q\left(\frac{w_{I_l}-w_{I_r}}{w_I}\right)^2|I| \leq \frac{c}{q(1-q)}(w_J)^q|J|,$$
for $0<q<1$. (c in both estimates is an absolute constant.)

Using Hölder's inequality on $(w_I)^{(1+\epsilon)/2}$ and $(w_I)^{(1-\epsilon)/2}$ we get
$$\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2 w_I|I| \leq \left(\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2(w_I)^{1+\epsilon}|I|\right)^{1/2}\left(\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2(w_I)^{1-\epsilon}|I|\right)^{1/2},$$
where we choose ϵ to be $c/\|w\|_{A_2}$ for c a small constant.

If c is chosen smaller than a dimensional constant, this ϵ is one for which the reverse Hölder inequality holds for w, i.e., $(\frac{1}{|I|}\int_I w^{1+\epsilon})^{\frac{1}{1+\epsilon}} \leq 2\frac{1}{|I|}\int_I w$ as can be seen by carefully reading the proof of the reverse Hölder inequality in [2]. Therefore w is in $B_{1+\epsilon}$ with norm 2.

Letting $p=1+\epsilon$ and $q=1-\epsilon$ in the above estimates, and writing $\frac{w_{I_l}-w_{I_r}}{2w_I}\sqrt{|I|}$ as c_I we get
$$\left(\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2(w_I)^{1+\epsilon}|I|\right)^{1/2} \leq (\frac{c}{\epsilon(1+\epsilon)})^{1/2}2^{\frac{1+\epsilon}{2}}(w_J)^{\frac{1+\epsilon}{2}}|J|^{1/2}$$
and
$$\left(\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2(w_I)^{1-\epsilon}|I|\right)^{1/2} \leq (\frac{c}{\epsilon(1-\epsilon)})^{1/2}(w_J)^{\frac{1-\epsilon}{2}}|J|^{1/2},$$
which, when combined, yields the required estimate. (Remember that $\epsilon = c/\|w\|_{A_2}$.)

To see that this estimate is sharp, let $w=\prod_{i\in\mathbb{N}}(1+a2^{-i/2}h_{I_i})$ where h_{I_i} is the Haar function whose support is of the form $[0,2^{-i}]$. Then $\|w\|_{A_2}$ is comparable to $\frac{1}{1-a}$, and $\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2 w_I|I| = \sum_{i\geq j} a^2(\frac{1+a}{2})^i$. □

Lemma 4.7. $\sum_{K\subset I}\frac{|c_K|}{\sqrt{|K|}}\frac{|d_K|}{\sqrt{|K|}}\mu(K)|K| \leq C\,\|w\|_{A_2}\,|I|$.

Proof. By Lemma 5.3 from [6], we have

$$\frac{1}{|J|}\sum_{I\subset J} v_I w_I \left|\frac{v_{I_l}-v_{I_r}}{v_I}\frac{w_{I_l}-w_{I_r}}{w_I}\right||I| \le C\sqrt{v_J w_J},$$

provided $v_I w_I \le 1$ for all $I\subset J$.

Letting $v_I = \frac{w_I^{-1}}{\|w\|_{A_2}}$, and clearing the denominators, we get the above lemma. □

Lemma 4.8. $\frac{1}{|J|}\sum_{I\subset J}\frac{|c_I|}{\sqrt{|I|}}\frac{|d_I|}{\sqrt{|I|}}w_I|I| \le C\|w\|_{A_2} w_J$.

Proof. This lemma will be proven by the method of Bellman functions.

Let $B(x,y) = x(\frac{-4A}{xy} - \frac{xy}{4A} + 4A + 1)$. Then $B(x,y)$ has the following properties, which can be checked by elementary calculus:

1. $0 \le B(x,y) \le 5Ax$ on $\{x,y>0; 1\le xy \le A\}$
2. $\begin{pmatrix} -B_{xx} & -B_{xy}-\frac{1}{y} \\ -B_{xy}-\frac{1}{y} & -B_{yy}\end{pmatrix}$ is positive semidefinite on $\{x,y>0; 1\le xy \le 2A\}$.
3. $\begin{pmatrix} -B_{xx} & -B_{xy}+\frac{1}{y} \\ -B_{xy}+\frac{1}{y} & -B_{yy}\end{pmatrix}$ is positive semidefinite on $\{x,y>0; 1\le xy \le 2A\}$.

Let us show that these conditions imply the following discrete condition:

$$(*) \qquad B(x,y) - \frac{B(x_-,y_-)+B(x_+,y_+)}{2} \ge C\left|(x_- - x_+)(y_- - y_+)\frac{1}{y}\right|,$$

where $x = \frac{(x_-+x_+)}{2}$ and $y = \frac{(y_-+y_+)}{2}$, and $(x,y),(x_-,y_-),(x_+,y_+)$ are in the domain $\{x,y>0; 1\le xy\le A\}$. Let $x(t) = \frac{x_-(1+t)+x_+(1-t)}{2}$ and $y(t) = \frac{y_-(1+t)+y_+(1-t)}{2}$ for $t\in[-1,1]$. Note that $y(t) \le \frac{2y_-+2y_+}{2} = 2y$. Denote by $b(t)$ the function $B(x(t),y(t))$. Note that $(x(t),y(t))$ are in the domain $\{x,y>0; 1\le xy\le 2A\}$. Then

$$\begin{aligned} -b''(t) &= (x'(t),y'(t))(-d^2B)(x'(t),y'(t))^t \\ &\ge (x'(t),y'(t))\begin{pmatrix} 0 & 1/y(t) \\ 1/y(t) & 0\end{pmatrix}(x'(t),y'(t))^t, \end{aligned}$$

and

$$\begin{aligned} -b''(t) &= (x'(t),y'(t))(-d^2B)(x'(t),y'(t))^t \\ &\ge (x'(t),y'(t))\begin{pmatrix} 0 & -1/y(t) \\ -1/y(t) & 0\end{pmatrix}(x'(t),y'(t))^t. \end{aligned}$$

Therefore, evaluating $x'(t)$ and $y'(t)$ we have

$$-b''(t) \ge \frac{2}{y(t)}\left|\frac{(x_- - x_+)}{2}\frac{(y_- - y_+)}{2}\right| \ge \frac{1}{4y}\left|(x_- - x_+)(y_- - y_+)\right|.$$

By the definition of b,

$$B(x,y)-\frac{B(x_-,y_-)+B(x_+,y_+)}{2}=b(0)-\frac{b(1)+b(-1)}{2}$$
$$=-1/2\int_{-1}^{1}b''(t)(1-|t|)dt\geq C\left|(x_--x_+)(y_--y_+)\frac{1}{y}\right|.$$

Now we are ready to run the usual Bellman function argument to prove the lemma.

Let $x=w_J, y=w_J^{-1}, x_-=w_{J_l}, x_+=w_{J_r}, y_-=w_{J_l}^{-1}, y_+=w_{J_r}^{-1}$ and $A=\|w\|_{A_2}$. These x, y, x_+, etc. satisfy the conditions for $(*)$. Therefore, we have

$$c|w_{J_l}-w_{J_r}||w_{J_l}^{-1}-w_{J_r}^{-1}|\frac{1}{w_J^{-1}}+\frac{B(w_{J_l},w_{J_l}^{-1})+B(w_{J_r},w_{J_r}^{-1})}{2}\leq B(w_J,w_J^{-1}).$$

Now x_-y_- and $x_+y_+\leq A=\|w\|_{A_2}$ again, so we can use estimate $(*)$ on the Bellman functions on the left side, too. This process can be repeated as often as we want. After n iterations, we have the following formula:

$$c\sum_{I\subset J,|I|\geq 2^{-n}|J|}\frac{|c_I|}{\sqrt{|I|}}\frac{|d_I|}{\sqrt{|I|}}w_I\frac{|I|}{|J|}+\text{ positive terms}\leq B(w_J,w_J^{-1})\leq 5\,\|w\|_{A_2}\,w_J.$$

So letting n go to ∞, we get the desired estimate. □

5. Dividing the estimate up into 4 sums

To begin with, let us estimate the norm of the martingale transform by duality,

$$\|T_rf\|_{L^2(w)}=\sup_{\|g\|_{L^2(w^{-1})}=1}\int T_rfg\,dx$$
$$=\sup_{\|g\|_{L^2(w^{-1})}=1}\int\sum_{I,J}r(I)\langle f,h_I\rangle h_I\langle g,h_J\rangle h_J\,dx.$$

Since the h_I's are orthonormal in $L^2(dx)$ this becomes

$$\sup_{\|g\|_{L^2(w^{-1})}=1}\sum_I r(I)\langle f,h_I\rangle\langle g,h_I\rangle.$$

Replacing f by $fw^{1/2}$, and g by $gw^{-1/2}$ this becomes:

$$\|T_r\|_{L^2(w)\to L^2(w)}=\sup_{\|g\|_{L^2(dx)}=1}\sup_{\|f\|_{L^2(dx)}=1}\sum_I r(I)\langle fw^{-1/2},h_I\rangle\langle gw^{1/2},h_I\rangle.$$

In order to estimate this, it is convenient to express the Haar functions in terms of a different family, which is more suited to working with weights.

The following "Haar functions" for $L^2(w)$ are normalized and orthogonal in $L^2(w)$:

$$h_I^w(x)=\frac{h_I(x)+\gamma_w^I\chi_I}{\delta_w^I},$$

where $\gamma_w^I = \frac{-c_I}{|I|}$ and ${\delta_w^I}^2 = w_I(1 - c_I^2/|I|) = \frac{w_{I_l} w_{I_r}}{w_I}$ are chosen to make these functions an orthonormal family.

Of course, there is the equivalent family for w^{-1}, with $\gamma_{w^{-1}}^I = \frac{-d_I}{|I|}$ and ${\delta_{w^{-1}}^I}^2 = w_I^{-1}(1 - d_I^2/|I|) = \frac{w_{I_l}^{-1} w_{I_r}^{-1}}{w_I^{-1}}$.

Substituting $h_I(x) = \delta_w^I h_I^w(x) - \gamma_w^I \chi_I$ (or the equivalent for w^{-1}) into our equation, and sorting the different types of terms, we can write what we are trying to estimate as four sums $I + II + III + IV$

$$
\begin{aligned}
I :& \sum_{I \in D[0,1]} r(I) \langle f w^{-1/2}, h_I^{w^{-1}} \rangle \delta_{w^{-1}}^I \langle g w^{1/2}, h_I^w \rangle \delta_w^I, \\
II :& - \sum_{I \in D[0,1]} r(I) \langle f w^{-1/2}, \chi_I \rangle \gamma_{w^{-1}}^I \langle g w^{1/2}, h_I^w \rangle \delta_w^I, \\
III :& - \sum_{I \in D[0,1]} r(I) \langle f w^{-1/2}, h_I^{w^{-1}} \rangle \delta_{w^{-1}}^I \langle g w^{1/2}, \chi_I \rangle \gamma_w^I, \\
IV :& \sum_{I \in D[0,1]} r(I) \langle f w^{-1/2}, \chi_I \rangle \gamma_{w^{-1}}^I \langle g w^{1/2}, \chi_I \rangle \gamma_w^I.
\end{aligned}
$$

We will estimate each sum separately in absolute value.

5.1. Sum I. Since $r(I)$ could be any combination of signs, we will sum in absolute value.

$$
I : \sum \left| \langle f w^{-1/2}, h_I^{w^{-1}} \rangle \langle g w^{1/2}, h_I^w \rangle \right| \sqrt{\frac{w_{I_l} w_{I_r} w_{I_l}^{-1} w_{I_r}^{-1}}{w_I w_I^{-1}}}.
$$

Since $w_{I_r} w_{I_l} \leq (w_I)^2$and $w_{I_r}^{-1} w_{I_l}^{-1} \leq (w_I^{-1})^2$, this is bounded by

$$
\begin{aligned}
&\sum \left| \langle f w^{-1/2}, h_I^{w^{-1}} \rangle \langle g w^{1/2}, h_I^w \rangle \right| \sqrt{\mu(I)} \\
&\qquad\qquad \leq \|w\|^{1/2} \sum \left| \langle f w^{-1/2}, h_I^{w^{-1}} \rangle \langle g w^{1/2}, h_I^w \rangle \right|.
\end{aligned}
$$

Taking the inner product in $L^2(w)$ and $L^2(w^{-1})$ instead of in $L^2(dx)$, this becomes

$$
\|w\|^{1/2} \sum \left| \langle f w^{1/2}, h_I^{w^{-1}} \rangle_{w^{-1}} \langle g w^{-1/2}, h_I^w \rangle_w \right|.
$$

Since h^w and $h^{w^{-1}}$ are orthonormal in $L^2(w)$ and $L^2(w^{-1})$, Cauchy Schwarz and the Bessel inequality allow us to estimate this by

$$
\|w\|^{1/2} \left\| f w^{1/2} \right\|_{L^2(w^{-1})} \left\| g w^{-1/2} \right\|_{L^2(w)} = \|w\|^{1/2} \|f\|_{L^2(dx)} \|g\|_{L^2(dx)}.
$$

5.2. Sum II and Sum III. Sums II and III are equivalent, so we will show Sum II only.

$$\sum_{I\in D[0,1]} \left|\langle fw^{-1/2},\chi_I\rangle \gamma^I_{w^{-1}}\langle gw^{1/2},h^w_I\rangle \delta^I_w\right|$$
$$= \sum_{I\in D[0,1]} \left|(fw^{-1/2})_I|I|\frac{|d_I|}{|I|}\langle gw^{1/2},h^w_I\rangle \sqrt{w_I(1-c_I^2/|I|)}\right|.$$

Using the fact that $(1-c_I^2/|I|)\leq 1$ and applying the Cauchy-Schwarz inequality, this becomes

$$\left(\sum_{I\in D[0,1]} |(fw^{-1/2})_I^2|\,|d_I|^2 w_I\right)^{1/2}\left(\sum_{I\in D[0,1]}\langle gw^{1/2},h^w_I\rangle^2\right)^{1/2}.$$

The second of these terms can again be estimated as in Sum I, and is bounded by $\|g\|_{L^2(dx)}$. To estimate the first term, we will use Theorem 4.1 with w^{-1} for w, and $\alpha_I = d_I^2 w_I$, and the inequality from Remark 4.5:

$$\frac{1}{|J|}\sum_{I\subset J}(w_I^{-1})^2 d_I^2 w_I \leq c\,\|w\|^2_{A_2}\, w_J^{-1}.$$

Theorem 4.1 gives us

$$\left(\sum_{I\in D[0,1]} |(fw^{-1/2})_I^2|\,|d_I|^2 w_I\right)^{1/2} \leq c\,\|w\|_{A_2}\,\|f\|_{L^2(dx)}\,.$$

Therefore Sum II is bounded by $c\,\|w\|_{A_2}\,\|f\|_{L^2(dx)}\,\|g\|_{L^2(dx)}$.

5.3. Sum IV.

$$IV:\sum_{I\in D[0,1]}\left|(fw^{-1/2})_I|I|(gw^{1/2})_I|I|\frac{|c_I d_I|}{|I|^2}\right|$$

In order to estimate this sum, we will make use of Theorem 4.2. with $\alpha_I = \frac{|c_I d_I|}{|I|}$. Since w and w^{-1} have the same A_2 norm, it is sufficient to calculate one of the conditions for the theorem.

$$\frac{1}{|J|}\int_J\left(\sum_{I\subset J}\alpha_I w_I\chi_I(x)\right)^2 w^{-1}\,dx$$
$$= \frac{1}{|J|}\sum_{I,K\subset J}\alpha_I w_I\alpha_K w_K\int_J \chi_I(x)\chi_K(x)w^{-1}(x)\,dx.$$

We can break this sum into the case where $I = K$ or $I \subset K$, since $K \subset I$ is equivalent to the latter. So we get

$$\frac{1}{|J|}\left(\sum_{I\subset J}\alpha_I^2(w_I)^2 w_I^{-1}|I| + 2\sum_{I,K\, K\subset I\subset J}\alpha_I\alpha_K w_K w_I w_K^{-1}|K|\right).$$

The first sum is

$$\frac{1}{|J|}\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2(\frac{d_I}{\sqrt{|I|}})^2 w_I|I|\mu(I).$$

Because w is an A_∞ weight, $|\frac{d_I}{\sqrt{|I|}}| \leq 1$ (see [4]), and so we can estimate the above by

$$\frac{1}{|J|}\sum_{I\subset J}(\frac{c_I}{\sqrt{|I|}})^2 w_I|I|\mu(I)$$

which is $\leq c\,\|w\|_{A_2}^2\, w_J$ by Remark 4.5.

This leaves us to estimate the second sum

$$\frac{2}{|J|}\sum_{I,K\, K\subset I\subset J}\frac{|c_I|}{\sqrt{|I|}}\frac{|d_I|}{\sqrt{|I|}}\frac{|c_K|}{\sqrt{|K|}}\frac{|d_K|}{\sqrt{|K|}}w_I\,\mu(K)|K|$$

$$= \frac{2}{|J|}\sum_{I\subset J}\frac{|c_I|}{\sqrt{|I|}}\frac{|d_I|}{\sqrt{|I|}}w_I\sum_{K\subset I}\frac{|c_K|}{\sqrt{|K|}}\frac{|d_K|}{\sqrt{|K|}}\mu(K)|K|.$$

The inside sum, by lemma 4.7 is bounded by $c\,\|w\|_{A_2}|I|$. That leaves us to estimate

$$\|w\|_{A_2}\frac{2}{|J|}\sum_{I\subset J}\frac{|c_I|}{\sqrt{|I|}}\frac{|d_I|}{\sqrt{|I|}}w_I|I|$$

which, by lemma 4.8 is bounded by $c\,\|w\|_{A_2}^2\, w_J$. Remembering that the final estimate for sum IV will be the square root of the estimate for the above, we have that sums I to IV are each individually bounded by $c\,\|w\|_{A_2}\|f\|_{L^2(dx)}\|g\|_{L^2(dx)}$, which concludes the proof.

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