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ON MODIFIED MABUCHI FUNCTIONAL AND MABUCHI MODULI SPACE OF KÄHLER METRICS ON TORIC BUNDLES

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1. Introduction

Mabuchi introduced the Mabuchi functional in [Mb1], and it turns out that it is very useful for dealing with Kähler metrics with constant scalar curvatures on compact manifolds (see [BM], etc.). One can also expect that the existence of Kähler metric with constant scalar curvature is almost equivalent to the existence of a lower bound of the Mabuchi functional (see, e.g., [Ti]). But for the case of extremal metrics, the Mabuchi functional is not applicable. Therefore, we need a new (or a modified) functional for metrics which are invariant under a maximal compact connected subgroup K of $\mathrm{Aut}(M)$. We did not obtain this functional until the appearing of [FM] (while we were reviewing [FM] in 1995). Mabuchi also found this functional independently [Mb3] (see also [Sm]). A definition of this functional was given in [GC]. We shall give some results and applications in this paper.

It turns out that our modified Mabuchi functional $M(\omega_1, \omega_2)$ has the property that

$$M(\omega_1,\ g^*\omega_2) = M(\omega_1, \omega_2),$$

for any $g \in C_{K^{\mathbf{C}}}(K)$, where $C_{K^{\mathbf{C}}}(K)$ is the centralizer of K in the complexification $K^{\mathbf{C}}$ of K. Moreover, the extremal metrics are exactly the local minimal points of this functional. Therefore, we expect that the existence of an extremal metric is almost equivalent to the existence of a lower bound of this functional.

Surprising enough that the first application of this functional is not the existence but the uniqueness of extremal metrics on smooth toric varieties, i.e., smooth Kähler manifolds with an open $(\mathbf{C}^*)^n$-orbit. Therefore, there is for example at most one extremal metric in any Kähler class of the manifold obtained by blowing up two points or three points of a two dimensional complex projective space.

To have the uniqueness, we consider the Mabuchi moduli space of the Kähler metrics on the toric varieties (see [Mb2], which was rediscovered by Semmes [Se1] and Donaldson [Ch]). It turns out that the moduli space is flat in this situation (see also [Se1,2]). Moreover, for any two Kähler metrics there is a unique geodesic

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in the Kähler cone which connects them (I was told in [Mb3] that the original method which Bando and Mabuchi used for [BM] is trying to find the geodesics, but it was too difficult for them at that time. Therefore, they eventually apply a different method). By [Se1], the existence of this geodesic is reduced to a solution of a complex Monge-Ampère equation. But in our situation, this reduces to a solution of a real Monge-Ampère equation and its property near the infinities. And we can actually linearize the real Monge-Ampère equation by applying the Legendre transformation to the space direction (we keep the time direction no change). Hence, we can construct a C^∞ solution and check that the conditions near the infinities are available. Then by the convex property of our functional (one might call this functional the *height*), we obtain the uniqueness.

After a while, we find that the same argument gives the uniqueness for extremal metrics in any Kähler class on *toric bundles over compact homogeneous manifolds* considered in [Mb4 Appendixes B and C] and [Nk], which is a much larger class of compact complex manifolds. We can also regard the manifolds considered in [Gu4] as a special case.

A modified method here had been applied in [Gu2] to obtain the uniqueness therein. Mabuchi also applied our method to obtain the uniqueness of certain generalization of Kähler-Einstein metrics in Ricci classes.

This paper is an off shoot of [Gu2] in which we try to prove the uniqueness of Kähler metrics of constant scalar curvature and was motivated by a talk given by Calabi in Princeton university. Here I thank Professor Semmes for his kindly help in dealing with the real Monge-Ampère equations, he also showed me his papers in 1993. I thank Dr. X. Chen for showing me [Ar], [Ch] and helpping me about ODE in [GC]. I also thank Professors P. Yang and X. Xu for their discussions. And finally I thank Professor T. Mabuchi for his kindly discussion in [Mb3] about the history.

2. Modified Mabuchi functional

For the metrics which are invariant under a maximal connected compact subgroup K of the automorphism group, one can define a modified Mabuchi functional

$$M(\omega_1,\ \omega_2) = -\int_a^b \int_X \dot{\varphi}_t (R - HR - \phi_t)\omega_t^n dt,$$

where ϕ_t are the function corresponding to the extremal vector field E in [FM].

Lemma 1. *M does not depend on the choice of the path.*

Then we have:

Theorem 1. *A local minimal of this functional is achieved by an extremal metric and*

$$M(\omega_1,\ g^*\omega_2) = M(\omega_1,\ \omega_2),$$

for any $g \in C_{K^{\mathbf{C}}}(K)$, where $C_{K^{\mathbf{C}}}(K)$ is the centralizer of K in $K^{\mathbf{C}}$.

Proof of Lemma 1. We prove that this functional is well defined as in [Mb p.579–580] as following.

Let $\psi(s,\ t) = s\varphi_t$, then this problem is reduced to the proof of the closeness of $\Phi(s,\ t) = \int_X \frac{\partial\psi}{\partial s}\phi_{s,t}\omega_{s,t}^n ds + \int_X \frac{\partial\psi}{\partial t}\phi_{s,t}\omega_{s,t}^n dt$.

Arguing as in [FM p.208–209] we see that $\phi_{s,t} = \phi + \frac{1}{2}s(d\phi, d\varphi_t)$ (here we notice that ϕ is a real function). Then

$$\begin{aligned}
d\Phi &= \int_X \frac{\partial}{\partial t}\left(\varphi_t\Big(\phi + \frac{1}{2}s\,(d\phi,\ d\varphi_t)\Big)\omega_{s,t}^n\right) dt\wedge ds \\
&\qquad + \int_X \frac{\partial}{\partial s}\left(s\dot{\varphi}_t\Big(\phi + \frac{1}{2}s\,(d\phi,\ d\varphi_t)\Big)\omega_{s,t}^n\right) ds\wedge dt \\
&= \frac{1}{2}s\int_X \Big(\varphi_t\left((d\phi,\ d\dot{\varphi}_t) + \phi_{s,t}\Delta_{s,t}^d\dot{\varphi}_t\right) \\
&\qquad - \dot{\varphi}_t\big((d\phi,\ d\varphi_t) + \phi_{s,t}\Delta_{s,t}^d\varphi_t\big)\Big)\omega_{s,t}^n dt\wedge ds \\
&= \frac{1}{2}s\int_X \big(\varphi_t(E\dot{\varphi}_t + \phi_{s,t}\Delta_{s,t}^d) - \dot{\varphi}_t(E\varphi_t + \phi_{s,t}\Delta_{s,t}^d\varphi_t)\big)\omega_{s,t}^n dt\wedge ds \\
&= \frac{1}{2}s\int_X \Big(\varphi_t\big((d\phi_{s,t}, d\dot{\varphi}_t)_{s,t} + \phi_{s,t}\Delta_{s,t}^d\dot{\varphi}_t\big) \\
&\qquad - \dot{\varphi}_t\big((d\phi_{s,t}, d\varphi_t)_{s,t} + \phi_{s,t}\Delta_{s,t}^d\varphi_t\big)\Big)\omega_{s,t}^n dt\wedge ds \\
&= \frac{1}{2}s\int_X \phi_{s,t}\big(-(d\varphi,\ d\dot{\varphi}_t) + (d\dot{\varphi}_t,\ d\varphi_t)\big)\omega_{s,t}^n dt\wedge ds = 0.
\end{aligned}$$

Therefore, the functional

$$F(\omega_1,\ \omega_2,\ E) = \int_b^a \int_X \dot{\varphi}_t\phi_t\omega_t^n dt,$$

is independent of the choice of the path. □

Proof of Theorem 1. If ω is a local minimal point of M, then $R-\phi$ is a constant. Therefore, ω is extremal. Moreover, if we let

$$\omega_t = g_t^*\omega_1 = \omega_1 + i\partial\bar{\partial}\varphi_t,$$

be a path of Kähler metrics with g_t generated by the real part of a holomorphic vector field H, then

$$L_{H_{\mathbf{R}}}\omega_t = i\partial\bar{\partial}\dot{\varphi}_t.$$

Hence, $\dot{\varphi}_t$ is the real part of the function corresponding to H (see [Gu3]). If H commuts with K, then ω_t is invariant under K. Therefore, ϕ_t is real. We have

$$M(\omega_1,\ g^*\omega_1) = -\int_0^1 \int_X \dot{\varphi}_t(R - HR - \phi_t)\omega_t^n dt = 0.$$

as desired. □

One might expect that the existence of an extremal metric is almost equivalent to the existence of a lower bound of this functional as in [BM], [DT1,2], [Ti].

Remark 1. The gradient flow of this functional is exactly the Calabi flow. And the derivative of this functional along this flow is the negative of the modified Calabi functional

$$\int_X R^2\omega^n - (HR)^2 \int_X \omega^n - F(E),$$

which is always nonnegative if ω is invariant under the action of a maximal connected compact subgroup of the automorphism group, and only be zero when ω is an extremal metric (this can be proved easily with L^2 decomposition of the *real* functions as in [Hw], but in general it seems that the proof there does not work since one confronts a *complex* decomposition). Applying Calabi's calculation in [Cl1] to our flow we obtain that the second derivative of the functional is

$$2\int_X R(\dot{\varphi}_{t,\alpha\beta})^{\alpha\beta}\omega_t^n = 2(R_{,\alpha\beta},\ (\dot{\varphi}_t)_{,\alpha\beta}) = 2\|\dot{\varphi}_{t,\alpha\beta}\|^2 > 0.$$

In other words, the Calabi functional is also decreasing under this flow.

We shall consider more properties of this functional in the next section.

3. Mabuchi Moduli Space and the Functional

In this section, we shall first recall some results on the Mabuchi moduli space of Kähler metrics.

In [Mb2] Mabuchi defined a Riemannian metric of all the Kähler metrics in a Kähler class. Let ω_0 be a Káhler metric, then all other Kähler metrics in the same Kähler class have the form $\omega_0 + i\partial\bar{\partial}\varphi$ with unique φ up to a constant. Therefore, the tangent space at ω_0 can be described by $\mathcal{T}_0 = \{\varphi|_{\int_X \varphi\omega_0^n=0}\}$. One can defined an inner product

$$\langle\langle\varphi,\phi\rangle\rangle_0 = \int_X \varphi\phi\omega_0^n,$$

on $C^\infty(X)$ which induces an inner product on $\mathcal{T}_0$. In this way one has a Riemannian metric of all the Kähler metrics in a Kähler class. We call this infinite dimensional Riemannian manifold the Mabuchi moduli space of Kähler metrics.

The Riemannian connection is given by

$$\frac{D}{dt} = \frac{d}{dt} - \frac{1}{2}(d\dot{\varphi}_t,\ d\ )_t,$$

along a curve φ_t. Therefore, the equation for a geodesic curve is

$$\ddot{\varphi}_t - \frac{1}{2}\,|d\dot{\varphi}_t|_t^2 = 0.$$

It is an observation of Semmes [Se1] that this is a homogeneous Monge-Ampère equation of φ_t if we regard t as the real part of z_0 and φ_t is independent of the imaginary part of z_0.

Now we consider the second variation of the modified Mabuchi functional:

Theorem 2.
$$\frac{d^2}{dt^2}M=\int_X \left|\bar\partial Y_t\right|_t^2\omega_t^n-\int_X(\ddot\varphi_t-\frac{1}{2}\left|d\dot\varphi_t\right|_t^2)(R-HR-\phi_t)\omega_t,$$
where $Y_t=\omega_t^*(\ ,\bar\partial\dot\varphi_t)$ and ω_t^* is the dual of ω_t. In particular, the second derivative along any geodesic curve is nonnegative and is positive if and only if the geodesic curve does not come from any one parameter subgroup of the automorphism group.

Proof. To prove this formula we apply Mabuchi [Mb2 Lemma 5.1] and Calabi's [Cl1] calculation. We see that $\frac{D}{dt}\phi_t=0$. Therefore,

$$\begin{aligned}\frac{d^2}{dt^2}M &= -\frac{d}{dt}\int_X\dot\varphi_t(R-HR-\phi_t)\omega_t^n\\ &= -\left\langle\left\langle\frac{D}{dt}\dot\varphi_t,\ R-HR-\phi_t\right\rangle\right\rangle-\left\langle\left\langle\dot\varphi_t,\ \frac{D}{dt}(R-\phi_t)\right\rangle\right\rangle\\ &= -\int_X\left(\ddot\varphi_t-\frac{1}{2}\left|d\dot\varphi_t\right|_t^2\right)(R-HR-\phi_t)\,\omega_t^n+\int_X\left|\bar\partial Y_t\right|_t^2\omega_t^n,\end{aligned}$$

as desired. □

Remark 2. From the calculation of the curvatures of the Mabuchi moduli space in [Mb2], [Se1], [Ch] we observe that if the manifold is almost homogeneous with an action of a reductive group then the moduli space is flat. This comes from the properties of the Poission bracket.

4. Existence of geodesic curves of Kähler metrics on toric varieties

In this section we shall prove the existence of geodesic curve connecting two arbitary Kähler metrics in a Kähler class on any smooth toric variety. By the observation of [Se1,2,3] we only need to solve a homogeneous Monge-Ampère equation. In the case of smooth toric varieties, this is further reduced to a real Monge-Ampère equation. By applying the Legendre transformation to the space direction and keeping the time direction unchanged, we can linearized this equation (see [Se1], [Lp], etc.). Moreover, the Legendre transformation of a convex function is still a convex function. Therefore, by connecting the two boundary functions $G_0,\ G_1$ by a line $G_t=tG_1+(1-t)G_0$ we obtain a solution of the linearized equation. G_t is also convex. Apply Legendre transformation we obtain a "geodesic curve of convex functions" connecting the initial two metrics. By checking the positive conditions on the boundaries, we see that all the convex functions correspond some metrics (a similar treatment of toric varieties can be found in [Ar]). In this way, we obtain following Theorem.

Theorem 3. *For any two Kähler metrics in a Kähler class which are invariant under a maximal connected compact subgroup of the automorphism group on a smooth toric variety there is a geodesic curve connecting them. In particular, there is at most one extremal metric in a Kähler class on a smooth toric variety,* *and if there is an extremal metric, the modified Mabuchi functional is bounded from below.*

Proof. Let X be a toric variety. Then, X is a completion of $(C^*)^n$ with coordinate $(z_1,\dots,z_n)$. Let $w_k=x_k+iy_k$ such that $z_k=e^{w_k}$. If ω is a T^n-invariant metric, then $\omega=\frac{i}{2}\sum_{j,k}\frac{\partial^2 F}{\partial x_k\partial x_j}dz_j\wedge d\bar z_k>0$ with F defined on $\mathbf{R}^n$. We let the Legendre transformation of F to be $G(s)=x(\frac{\partial F}{\partial x})^T-F(x)$ with $s=\frac{\partial F}{\partial x}$ being the moment map. Then

$$\frac{\partial G}{\partial s}=x+s\left(\frac{\partial x}{\partial s}\right)^T-\frac{\partial F}{\partial s}=x+s\left(\frac{\partial x}{\partial s}\right)^T-\frac{\partial F}{\partial x}\frac{\partial x}{\partial s}=x$$

and $F(x)=s(\frac{\partial G}{\partial s})^T-G(s)$. Therefore, $\frac{\partial s}{\partial x}=\frac{\partial^2 F}{\partial x^2}=(\frac{\partial x}{\partial s})^{-1}=(\frac{\partial^2 G}{\partial s^2})^{-1}$. By F being convex, G is convex on its domain.

We have $\frac{\partial G}{\partial t}=\dot G(s)+\frac{\partial G}{\partial s}(\dot s)^T=\dot G+x(\dot s)^T$ and $\frac{\partial}{\partial t}(x(\frac{\partial F}{\partial x})^T-F)=x(\dot s)^T-\dot F(x)$. Therefore, $\dot G(s)=-\dot F(x)$.

For the second derivative of t we obtain

$$\ddot G(s)+\frac{\partial \dot G}{\partial s}(\dot s)^T=\ddot G(s)-\left(\frac{\partial \dot F}{\partial x}\frac{\partial x}{\partial s}\right)\left(\frac{\partial \dot F}{\partial x}\right)^T=\ddot G(s)-\frac{1}{2}\left|\frac{\partial \dot F}{\partial x}\right|_t^2=-\ddot F(x).$$

Therefore, the geodesic equation $\ddot F-\frac{1}{2}\left|\frac{\partial \dot F}{\partial x}\right|_t^2=0$ is the same as $-\ddot G(s)=0$.

Therefore, if $G_0,\ G_1$ are the two functions corresponding to the metrics $\omega_0,\ \omega_1$, then $G_t=tG_1+(1-t)G_0$ is a solution of $-\ddot G(s)=0$. We obtain a C^∞ solution $F_t=s(\frac{\partial G_t}{\partial s})^T-G_t$ of the geodesic curve equation.

Now we want to see that F_t define a curve of Kähler metrics on X. By the moment map having the same image, we observe that G_t have same domains. By G_t being convex we obtain that F_t are convex. Therefore, F_t define Kähler metrics on the open set of X which is the preimage O of the domain of G_t.

We only need to check the positivity of the metrics on the boundary of O, i.e., that the metrics do not degenerate[1] outside O. Here we apply a description of a germ of a neighborhood of a point on the boundary in [Dl Lemma 2.5]. Let m be a point such that the orbit through m has codimension $d>0$. Then there is a neighborhood of m with the form $((C^*)^{n-d}\times B(\epsilon)^d,m)$. Near m a metric can have a form $\sum_{1\le k\le n-d}d\alpha_k\wedge da_k+\sum_{1\le j\le d}db_j\wedge dc_j$, where $w_j=b_j+ic_j$ is the coordinate of the j-th copy of the complex ball $B(\epsilon)$, $\alpha_k,\ a_k$ are the real and imaginary parts of the C^* action. This gives a complex decomposition of the tangent space at m. We call the first space the parallel tangent space, which is parallel to the orbit. We call the second space the vertical tangent space. They only depend on the torus action. The positivity of the metrics at m is the same as the positivity of them on both the parallel and the vertical tangent space at m. The parallel part comes from the limit of $(\frac{\partial^2 F_t}{\partial x_j\partial x_k})_{1\le j,k\le n-d}$ which corresponds to the eigenvalues of $\partial^2 F_t$ which do not turn to zero — the inverse

[1]A warnning here is that the argument in [Ar p.645 line 4–7] does not work.

of the eigenvalues of $\partial^2 G_t$ which do not turn to infinity, these eigenvalues of $\partial^2 F_t$ have lower positive bound given by the upper bound of those of G_t. The vertical part comes from the limit of

$$A = \frac{1}{4}\left(w_j^{-1}\bar{w}_k^{-1}\frac{\partial^2 F_t}{\partial x_{n-d+k}\partial x_{n-d+j}}\right)_{1\le j,k\le d}$$

that corresponds to the eigenvalues of $\partial^2 F_t$ which turn to zero — the inverse of the eigenvalues of $\partial^2 G_t$ which turn to infinity, the eigenvalues of A for any point near m have lower bound coming from those of F_0 and F_1 as in the parallel case. Therefore, we have proved the nondegeneracy of the metrics outside O as desired. □

Remark 3. The proof shows that we can obtain a better formula of the modified Mabuchi functional on toric varieties. We shall deal with this in our future research. In [Ch] Dr. Chen is trying to apply an elliptic Monge-Ampère equation argument to handle the the existence of the geodesics which seemly can only produce a weakly $C^{1,1}$ solution even in our situation and does not achieve our goal (see also [CNS] etc.), while we apply a direct "homogeneous" Monge-Ampère equation argument here and obtain a C^∞ solution. I personally believe that the "homogeneous" Monge-Ampère equation argument can still go through for the general situation.

5. Toric Bundles

After we finished the proof for toric varieties, we realized that the same proof works for the toric bundles on compact complex homogeneous manifolds considered in [Mb4 sections 8 and 9] and [Nk].

The point is that on the open orbit we might think that the Kähler metric is

$$\partial\bar{\partial}F = \sum \frac{\partial^2 F}{\partial x_j \partial x_k} dw_k \wedge d\bar{w}_j + \omega_0 - \sum y_j C_1(L_j)$$

with ω_0 a Kähler metric on the base manifold Q and L_j the corresponding $\mathbf{C}^*$ bundle on Q as well as the function F being invariant under the maximal connected compact group (see [Mb4 (8.1.8) and p.727 lines 7–8]).

By choosing a good local coordinate at a point in Q we can assume that $\partial\dot{F}$ is a linear combination of dw_j's. Therefore the geodesic equation is reduced to that of the fiber.

Therefore, we have:

Theorem 4. *The same results as in Theorem* 3 *holds for any compact toric bundle over any compact homogeneous manifold.*

Remark 4. A similar reduction holds for the situation in [Gu2]. Therefore, we can reduce the uniqueness in [Gu2] to the case of $\mathbf{C}P^2$ with $SU(2)$ invariant metrics, which corresponds to a 3 dimension homogeneous Monge-Ampère equation. In [Gu2] we are able to reduce this situation further to apply our result here and finally obtain a partial solution of our original problem. This leads to a proof of uniqueness of the situation in [Gu2].

References

[Ar] M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. **9** (1998), 641–651.

[BM] S. Bando and T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic Geometry, Sendai 1985, 11–40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam-New York, 1987.

[Ch] X. Chen, *The space of Kähler metrics*, preprint, 1998.

[Cl1] E. Calabi, *Extremal Kähler metrics*, Seminars on Differential Geometry, pp. 259–290, Annals of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[Cl2] ———, *Extremal Kähler metrics II*, Differential Geometry and Complex Analysis, pp. 95–114, Springer-Verlag, Berlin-New York, 1985.

[CNS] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for the degenerate Monge-Ampère equation*, Rev. Mat. Iberoamericana **2** (1986), 19–27.

[Dl] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339.

[DT1] W. Ding and G. Tian, *Kähler-Einstein metrics and the generalized Futaki invariant*, Invent. Math. **110** (1992), 315–335.

[DT2] ———, *The generalized Moser-Trudinger inequality*, Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.

[FM] A. Futaki and T. Mabuchi, *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.

[GC] D. Guan and X. Chen, *Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one*, Asian J. Math., to appear.

[Gu1] D. Guan, *Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one — II*, preprint.

[Gu2] ———, *Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one — III*, in preparation.

[Gu3] Z. Guan, *Quasi-Einstein metrics*, Internat. J. Math. **6** (1995), 571–579.

[Gu4] ———, *Existence of extremal metrics on almost homogeneous spaces with two ends*, Trans. Amer. Math. Soc. **347** (1995), 2255–2262.

[Hw] A. Hwang, *On the Calabi energy of Eetremal Kähler metrics*, Internat. J. Math. **6** (1995), 825–830.

[Lp] L. Lempert, *Symmetries and other transformations of the complex Monge-Ampère equation*, Duke Math. J. **52** (1985), 869–885.

[Mb1] T. Mabuchi, *K-energy Maps Integrating Futaki Invariants*, Tōhuku Math. J. (2) **38** (1986), 575–593.

[Mb2] ———, *Some symplectic geometry on compact Kähler manifolds I*, Osaka J. Math. **24** (1987), 227–252.

[Mb3] ———, private communication, 1999.

[Mb4] ———, *Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties*, Osaka J. Math. **24** (1987), 705–737.

[Nk] Y. Nakagawa, *Classification of Einstein-Kähler toric Fano fourfolds*, Tōhoku Math. J. (2) **46** (1994), 125–133.

[Se1] S. Semmes, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. **114** (1991), 495–550.

[Se2] ———, *The homogeneous complex Monge-Ampère equation and the infinite dimensional versions of classic symmetric spaces*, The Gelfand Mathematical Seminars 1993–1995, pp. 225–242 Birkhäuser Boston, Boston, MA, 1996.

[Se3] ———, *Interpolation of Banach spaces, differential geometry, and differential equations*, Rev. Mat. Iberoamericana **4** (1988), 155–176.
[Sm] S. Simanca, *A K-energy characterization of extremal metrics*, preprint, 1999.
[Ti] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **137** (1997), 1–37.

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