CHIRAL POINCARÉ DUALITY

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1. Let X be a smooth algebraic variety over $\mathbb{C}$. In the note [MSV] we introduced a sheaf of vertex superalgebras Ω_X^{ch} on X. (Below we will often omit the prefix "super"; we will live mainly in the $\mathbb{Z}/2$-graded world, the tilde over a letter will denote its parity.) This sheaf has a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$-grading

$$\Omega_X^{ch} = \oplus_{p \in \mathbb{Z}, \ i \in \mathbb{Z}_{\geq 0}} \ \Omega_i^{ch,p},$$

by *fermionic number p* and *conformal weight i*. The $\mathbb{Z}/2$-grading is $p \pmod 2$. The conformal weight zero part $\Omega_0^{ch} = \oplus_p \Omega_0^{ch,p}$ is identified with the usual de Rham algebra $\Omega_X = \oplus_p \Omega_X^p$ of differential forms. The sheaf Ω_X^{ch} will be called the *chiral de Rham algebra* of X.

Each component $\Omega_i^{ch,p}$ carries a canonical finite filtration whose factors are locally free $\mathcal{O}_X$-modules of finite rank. However, there is *no* natural $\mathcal{O}_X$-module structure on Ω_X^{ch} itself. Let us consider the cohomology

$$H^*(X, \Omega_X^{ch}) = \oplus_{p,q,i} \ H^q(X, \Omega_i^{ch,p}).$$

This is a conformal vertex superalgebra. The $\mathbb{Z}/2$-grading is $(p + q) \pmod 2$. When X is complete we call this algebra the *chiral Hodge cohomology* of X. The conformal weight zero part of it coincides with the usual Hodge cohomology algebra $H^*(X, \Omega_X) = \oplus_{p,q} H^q(X, \Omega_X^p)$. If X is Calabi-Yau then $H^*(X, \Omega_X^{ch})$ is a $N = 2$ superconformal vertex algebra, in the sense of [K], 5.8.

We have the Künneth formula: for any two smooth varieties X, Y a canonical isomorphism of conformal vertex superalgebras

$$(1.0) \qquad H^*(X \times Y, \Omega_{X \times Y}^{ch}) = H^*(X, \Omega_X^{ch}) \otimes H^*(Y, \Omega_Y^{ch}).$$

From now on we assume that X *is complete*, unless specified otherwise. The "chiral Hodge — de Rham spectral sequence" degenerates not at E_1 but at E_2. Namely, the chiral de Rham differential d_{DR}^{ch} on Ω_X^{ch} induces a differential

$$Q: \ H^q(X, \Omega_i^{ch,p}) \longrightarrow H^q(X, \Omega_i^{ch,p+1}),$$

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and the cohomology of $H^*(X, \Omega^{ch})$ with respect to Q is equal to $H^*(X, \Omega_X)$. Indeed, as in the proof of [MSV], Theorem 2.4, the operator G_0 is a zero homotopy on the components of nonzero conformal weight. The "chiral de Rham cohomology" coincides with the usual de Rham cohomology.

We can consider the similar sheaf $\Omega^{ch,an}_X$ over the corresponding analytic variety X^{an}. We have canonical isomorphism

$$H^*(X, \Omega^{ch}_X) \xrightarrow{\sim} H^*(X^{an}, \Omega^{ch,an}_X).$$

Indeed, we have an obvious map from the left hand side to the right hand side, compatible with the above mentioned filtrations, and it is an isomorphism by GAGA and five-lemma.

The aim of this note is to prove

1.1. Theorem. *The space $H^*(X, \Omega^{ch}_X)$ carries a canonical non-degenerate symmetric bilinear form*

$$(1.1) \qquad \langle\ ,\ \rangle:\ H^*(X, \Omega^{ch}_X) \times H^*(X, \Omega^{ch}_X) \longrightarrow \mathbb{C}.$$

This form makes the components of different conformal weights orthogonal and identifies

$$(1.2) \qquad H^q(X, \Omega^{ch,p}_i)^* = H^{n-q}(X, \Omega^{ch,n-p}_i).$$

Here $n = \dim(X)$. The restriction of this form to the conformal weight zero component coincides with the usual Poincaré pairing.

The symmetry of (1.1) is understood in the $\mathbb{Z}/2$-graded sense.

1.2. Remark. Recall that we have defined in [MS], Part II, 1.4 for every conformal vertex algebra V a canonical Lie algebra antiinvolution

$$(1.3) \qquad \eta : \mathrm{Lie}(V) \longrightarrow \mathrm{Lie}(V),$$

Here $\mathrm{Lie}(V)$ denotes the Lie algebra of Fourier components of the fields of V.

It follows from the construction of (1.1) that this pairing is η-contravariant, i.e.,

$$(1.4) \qquad \langle x_{(n)}y, z \rangle = (-1)^{\tilde{x}\tilde{y}} \langle y, \eta(x_{(n)})z \rangle.$$

Of course the Poincaré duality in the usual Hodge cohomology is an immediate consequence of the Serre duality $H^i(X, E)^* = H^{n-i}(X, E^o)$ where for a vector bundle E on X, E^o denotes the dual bundle $\mathcal{Hom}_{\mathcal{O}_X}(E, \omega_X)$, $\omega_X := \Omega^n_X$. Similarly, we deduce Theorem 1.1 from the corresponding local statement, see Theorem 8.1. To formulate it, we need to define the dual of the sheaf Ω^{ch}_X; this

is not immediate since Ω_X^{ch} is not an $\mathcal{O}_X$-module. We define the dual using M. Saito's language of induced D-modules, [S].

In fact we do more: we introduce a suitable category of "restricted" (in the sense of [MSV]) Ω_X^{ch}-modules along with a duality functor on it and prove a general statement, which can be thought of as a chiral analogue of Serre duality, see no. 11, formula (11.6).

Both the chiral Serre duality and Theorem 8.1 are consequences of Theorem 10.1, which may be of independent interest. It says that the "Weyl module" functor is an equivalence between the categories of $\mathcal{D}_{\Omega_X}$-modules and restricted Ω_X^{ch}-modules. Theorem 11.2 adds that this equivalence preserves the duality functor.

In no. 12 we present an alternative proof of 1.1 for $X = \mathbb{P}^1$ and in no. 13 derive some consequences about the structure of $H^*(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch})$.

2. First let us recall Saito's theory. By a D-module on X we mean a right $\mathcal{D}_X$-module quasicoherent over $\mathcal{O}_X$. The category of D-modules on X will be denoted $\mathcal{M}(X)$. For a D-module M, let $DR(M)$ denote its de Rham complex

$$(2.1) \quad DR(M): \ 0 \longrightarrow M \otimes_{\mathcal{O}_X} \Lambda^n \Theta_X \longrightarrow \cdots \longrightarrow M \otimes_{\mathcal{O}_X} \Theta_X \longrightarrow M \longrightarrow 0,$$

(we regard it as sitting in degrees $-n, \ldots, 0$). Here Θ_X is the tangent sheaf. The de Rham cohomology $H_{DR}^*(X, M)$ is defined as the hypercohomology $H^*(X, DR(M))$. Set

$$(2.2) \qquad\qquad h(M) = H^0(DR(M)) = M/M\Theta_X.$$

For a quasicoherent $\mathcal{O}_X$-module P, set $P^\sim := P \otimes_{\mathcal{O}} \mathcal{D}_X$; this is a D-module (the action of $\mathcal{D}_X$ is induced by the right action of $\mathcal{D}_X$ on itself). A D-module isomorphic to $P^\sim$ for some P is called *induced*.

The de Rham complex $DR(P^\sim)$ is a left resolution of P; more precisely, we have a canonical arrow

$$(2.3) \qquad\qquad \nu_P: \ DR(P^\sim) \longrightarrow P,$$

sending $p \otimes \partial \in P \otimes \mathcal{D}_X = DR^0(P^\sim)$ to $p\partial$, and ν_P is a quasiisomorphism. In particular, $h(P^\sim) = P$. As a consequence, we have a canonical isomorphism

$$(2.4) \qquad\qquad H_{DR}^*(X, P^\sim) = H^*(X, P),$$

A morphism of D-modules $f: \ P^\sim \longrightarrow Q^\sim$ induces a morphism of sheaves $h(f): \ P \longrightarrow Q$. One checks that $h(f)$ is a differential operator and this gives an isomorphism

$$(2.5) \qquad\qquad \mathrm{Hom}_{\mathcal{D}_X}(P^\sim, Q^\sim) = \mathrm{Diff}(P, Q),$$

where $\mathrm{Diff}(P, Q)$ denotes the space of differential operators, in the sense of Grothendieck, acting from P to Q, cf., [S], 1.20.

3. Definition. *A D-bundle on X is a locally free right $\mathcal{D}_X$- module of finite rank.*

The D-bundles form a full subcategory $\mathcal{D} - \mathcal{B}un(X)$ of $\mathcal{M}(X)$.

4. A *differential bundle* on X is a quadruple $(P, \mathcal{U}, \{E^U\}, \{s_U\})$ where P is a sheaf of $\mathbb{C}$-vector spaces on X, $\mathcal{U} = \{U\}$ is a Zariski open covering of X, E^U ($U \in \mathcal{U}$) are vector bundles (:= locally free $\mathcal{O}_U$-modules of finite rank),

$$(4.1) \qquad\qquad s_U : \ P_U \overset{\sim}{\longrightarrow} E^U$$

are $\mathbb{C}$-linear sheaf isomorphisms. Here P_U denotes the restriction of P to U.

The isomorphisms s_U should satisfy the property

(Diff). For each $U, V \in \mathcal{U}$, the isomorphism

$$(4.2) \qquad\qquad c_{UV} := s_V s_U^{-1} : \ E_V^U \overset{\sim}{\longrightarrow} E_U^V,$$

should be a differential operator.

We will regard a differential bundle as a sheaf P equipped with certain complementary structure, and sometimes, abusing the notation, will denote it simply by P.

5. Let M be a D-bundle on X. Choose a trivialization of M, i.e., a Zariski open covering $\mathcal{U}$ of X and isomorphisms of $\mathcal{D}_U$-modules

$$t_U : \ M_U \overset{\sim}{\longrightarrow} \mathcal{D}_U^n \ (U \in \mathcal{U}).$$

Set $P = h(M)$, $E^U = \mathcal{O}_U^n$, $s_U = h(t_U)$. Then $(P, \mathcal{U}, \{E^U\}, \{s_U\})$ is a differential bundle.

Conversely, let $(P, \mathcal{U}, \{E^U\}, \{s_U\})$ be a differential bundle. Over each $U \in \mathcal{U}$, one can from the induced module $E^{U\sim}$. Let us glue them together using the Cech 1-cocycle $c^\sim = (c_{UV}^\sim)$ where

$$c_{UV}^\sim \in \mathrm{Hom}_{\mathcal{M}(U \cap V)}(E_V^{U\sim}, E_U^{V\sim})$$

corresponds to c_{UV} via (2.5). We get a D-bundle $P^\sim$.

For two differential bundles P, P', we define the space $\mathrm{Diff}(P, P')$ of differential operators as

$$\mathrm{Diff}(P, P') = \mathrm{Hom}_{\mathcal{M}(X)}(P^\sim, P'^\sim).$$

Note that $\mathrm{Diff}(P, P')$ is canonically a subspace of $\mathrm{Hom}_{\mathbb{C}}(P, P')$. This way we get a category $\mathcal{D}iffbun(X)$ of differential bundles, with

$$\mathrm{Hom}_{\mathcal{D}iffbun(X)}(P, P') = \mathrm{Diff}(P, P').$$

The correspondence $P \mapsto P^\sim$ give rise to the equivalence of categories

$$\mathcal{D}iffbun(X) \overset{\sim}{\longrightarrow} \mathcal{D} - \mathcal{B}un(X).$$

We will denote by h the quasiinverse equivalence.

The obvious morphism $\nu : DR(P^\sim) \longrightarrow P$ is a quasiisomorphism for each differential bundle P. Consequently, we have canonically

$$(5.1) \qquad\qquad H_{DR}^*(X, P^\sim) = H^*(X, P).$$

6. For a finite non-empty set I, an I-family $\{P_i\}$ of differential bundles and a differential bundle Q let $\mathrm{Diff}_I(\{P_i\}, Q)$ denote the subspace of the space of maps $\otimes_{I\mathbb{C}} P_i \longrightarrow Q$ which are differential operators by each argument (when all but one arguments are fixed). These spaces define a pseudo tensor structure on $\mathcal{D}iffbun(X)$, in the sense of Beilinson-Drinfeld, [BD].

On the other hand, the category $\mathcal{D}-\mathcal{B}un(X)$ carries a pseudo tensor structure induced from the $*$-pseudo tensor structure on $\mathcal{M}(X)$ introduced in *op. cit.,* 2.2.3. More precisely, for an I-family of D-bundles $\{L_i\}$ and a D-bundle M, we define the space of polylinear operations $P_I^*(\{L_i\}, M)$ as the space of all D-module maps

$$\boxtimes_I : \ L_i \longrightarrow \Delta_*^{(I)} M.$$

Here $\Delta^{(I)} : \ X \longrightarrow X^{(I)}$ is the diagonal embedding.

The functor h identifies both pseudo tensor structures: we have canonically

$$(6.1) \qquad\qquad \mathrm{Diff}_I(\{P_i\}, Q) = P_I^*(\{P_i^{\sim}\}, Q^{\sim}).$$

7. Duality. Recall the duality functor for D-modules. Consider the sheaf $\omega_X^{\sim} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. It carries two commuting structures of a right $\mathcal{D}_X$-module: the first one coming from the tensor product of a right and a left $\mathcal{D}_X$-module, and the second one appearing from the right $\mathcal{D}_X$-module structure on $\mathcal{D}_X$. Note that according to a lemma by Saito, [S], Lemma 1.7, there is a canonical involution on the above sheaf which interchages two $\mathcal{D}_X$-module structures.

Let M be a D-bundle. Set

$$(7.1) \qquad\qquad M^o := \mathcal{H}om_{\mathcal{D}_X}(M, \omega_X^{\sim}),$$

where $\mathcal{H}om$ is taken with respect to the first $\mathcal{D}_X$-module structure on $\omega_X^{\sim}$, and the right $\mathcal{D}_X$-action on it is induced by the second structure; it is also a D-bundle.

7.1. Lemma. *One has canonical isomorphisms* $H_{DR}^i(X, M)^* = H_{DR}^{n-i}(X, M^o)$ *where* $n = \dim(X)$.

This is the duality theorem for D-modules, cf., [S].

Note that if $M = E^{\sim}$ where E is a vector bundle then $M^o = E^{o\sim}$ where $E^o = \mathcal{H}om_{\mathcal{O}_X}(E, \omega_X)$.

7.2. Definition. *Let P be a differential bundle. The dual differential bundle P^o is defined by $P^o = h(P^{\sim o})$.*

One can give a direct definition of P^o using the gluing functions. Let $\{c_{UV}\}$ be as in (4.2). Recall that for two vector bundles E, F we have a canonical isomorphism

$$(7.2) \qquad\qquad \mathrm{Diff}(E, F) = \mathrm{Diff}(F^o, E^o).$$

For example, if $E = F = \mathcal{O}_X$ then (7.2) amounts to the usual correspondence between left and right $\mathcal{D}_X$-modules.

Now, the dual differential bundle P^o is glued by means of the dual Cech cocycle $c^o = (c^o_{UV})$ where $c^o_{UV} \in \mathrm{Diff}(E^{Vo}_U, E^{Uo}_V)$ corresponds to c_{UV} via (7.2).

Note that we have canonically

$$(7.3) \qquad\qquad\qquad\qquad E^{oo} = E.$$

Lemma 7.1 along with (5.1) implies

7.3. Lemma. *(Serre duality) One has canonical isomorphisms $H^i(X, P)^* = H^{n-i}(X, P^o)$.*

8. The arguments of [MSV], 6.10 (cf., also [MS], Part I) show that a choice of étale coordinates on a Zariski open $U \subset X$ gives a trivialization of the sheaves $\Omega^{ch,p}_{iU}$, and one sees that the transition functions are differential operators. It follows that the sheaves $\Omega^{ch,p}_i$ carry a canonical structure of differential bundles.

8.1. Theorem. *For all p, i there exist canonical isomorphisms*

$$(8.1) \qquad\qquad\qquad \chi^p_i : (\Omega^{ch,p}_i)^o \xrightarrow{\sim} \Omega^{ch,n-p}_i.$$

For $i = 0$ the isomorphisms (8.1) are induced by the wedge products of differential forms.

These isomorphisms are symmetric in the following sense:

$$(8.2) \qquad\qquad\qquad\qquad \chi^{po}_i = \chi^{n-p}_i.$$

The proof is given after the proof of Theorem 11.2 below.

Theorem 8.1 and Lemma 7.2 3 immediately imply Theorem 1.1.

9. Note that the sheaf Ω^{ch}_X is a vertex algebra in the Diff-pseudo tensor structure defined in 6 (a *vertex*$^{\mathrm{Diff}}$ *algebra*). This means that the operations

$$_{(n)} : \Omega^{ch}_X \times \Omega^{ch}_X \longrightarrow \Omega^{ch}_X,$$

belong to $\mathrm{Diff}_{\{1,2\}}(\{\Omega^{ch}_X, \Omega^{ch}_X\}, \Omega^{ch}_X)$.

Equivalently, the D-module $\Omega^{ch\sim}_X$ is a *vertex** *algebra* (a vertex algebra in the $*-$ pseudo tensor structure).

10. Let us consider the de Rham algebra of differential forms $\Omega_X = \oplus \, \Omega^p_X$. Let $\mathcal{D}_{\Omega_X}$ be the superalgebra of differential operators on Ω_X. Let $\mathcal{D}_{\Omega_X} - \mathrm{Mod}$ denote the category of left $\mathcal{D}_{\Omega_X}$-modules (everything here is $\mathbb{Z}/2$-graded). Let $\Omega^{ch}_X - \mathrm{Mod}$ denote the category of restricted Ω^{ch}_X-modules ("restricted" means that the modules are graded by conformal weight and there are no components of negative weight).

Recall that we defined in [MSV], 6.11, [MS], I.4.5 the *Weyl* functor

$$(10.1) \qquad\qquad W_\Omega : \mathcal{D}_{\Omega_X} - \mathrm{Mod} \longrightarrow \Omega^{ch}_X - \mathrm{Mod} .$$

This functor is the left adjoint to the functor of taking the conformal weight zero component.

For example, $W_\Omega(\Omega_X) = \Omega^{ch}_X$. In general, we refer to $W_\Omega(V)$ as a Weyl module over Ω^{ch}_X, or simply a chiral Weyl module.

10.1. Theorem. *The functor W_Ω is an equivalence of categories.*

Proof. By definition, for any $\mathcal{D}_{\Omega_X}$-module V, $W_\Omega(V)_0 = V$. Therefore, it suffices to show that for any $\mathcal{V} \in \Omega_X^{ch} - \mathrm{Mod}$ the adjunction map

$$\widehat{id} : W_\Omega(\mathcal{V}_0) \to \mathcal{V}.$$

is an isomorphism.

To prove injectivity and surjectivity of $\widehat{id}$ we introduce, for any $\mathcal{V} \in \Omega_X^{ch} - \mathrm{Mod}$, the subsheaf of *singular vectors*, $\mathrm{Sing}(\mathcal{V})$, to be defined as follows:

$$(10.2) \quad \Gamma(U, \mathrm{Sing}(\mathcal{V})) = \{v \in \Gamma(U, \mathcal{V}) : x_i v = 0 \text{ for all } x \in \Gamma(U, \Omega_X^{ch}), i > 0\}.$$

10.2. Lemma.
  (i) $\mathrm{Sing}(\mathcal{V}) \neq 0$ *for any* $\mathcal{V} \in \Omega_X^{ch} - \mathrm{Mod}$.
  (ii) $\mathrm{Sing}(\mathcal{V}) \subset \mathcal{V}_0$.

This lemma allows us to complete the proof of the theorem at once. By (ii) $\mathrm{Sing}(\mathrm{Ker}(\widehat{id})) = \mathrm{Ker}(\widehat{id})_0$, which equals $\mathrm{Ker}(id)$ and is, therefore, 0. Hence, by (i), $\mathrm{Ker}(\widehat{id}) = 0$. Replacing Ker with Coker in this argument, we get that $\mathrm{Coker}(\widehat{id}) = 0$.

Let us finally prove the lemma. Item (i) is an obvious consequence of the restrictedness property: it is enough to observe that $\mathcal{V}_i \subset \mathrm{Sing}(\mathcal{V})$ if $i \geq 0$ is the minimum number satisfying $V_i \neq 0$.

As to (ii), we remind the reader that conformal weights are eigenvalues of one of the Virasoro generators, L_0, which is given locally (over, say, a formal polydisk) by the formula

$$L_0 = \sum_{i,k} i : a_i^k b_{-i}^k : + i : \phi_{-i}^k \psi_i^k : .$$

Comparing with (10.2) we see that because of the coefficient i, L_0 acts as 0 on $\mathrm{Sing}(\mathcal{V})$.

11. Chiral Serre duality. We define a Ω_X^{ch}-*bundle* to be an Ω_X^{ch}-module $\mathcal{E}$ equipped with a structure of a differential bundle such that all operations

$$_{(n)} : \Omega_X^{ch} \otimes \mathcal{E} \longrightarrow \mathcal{E},$$

belong to $\mathrm{Diff}_{\{1,2\}}(\{\Omega_X^{ch}, \mathcal{E}\}, \mathcal{E})$.

Let $\mathcal{E}$ be a Ω_X^{ch}-bundle. Consider the *restricted* dual differential bundle $\mathcal{E}^o$. By this we mean the following: being graded by conformal weight $\mathcal{E}$ is a direct sum of differential bundles $\mathcal{E} = \oplus_{i \geq 0} \mathcal{E}_i$ and we set

$$(11.1) \qquad\qquad\qquad \mathcal{E}^o = \oplus_{i \geq o} \mathcal{E}_i^o.$$

(We do not change the notation hoping that this will not lead to a confusion: it is a general principle that in the realm of modules over a vertex algebra a dual means a restricted dual.)

We want to introduce a canonical structure of an Ω_X^{ch}-bundle on it.

Note that if M is a module over a vertex algebra V then M is automatically a $\mathrm{Lie}(V)$-module; as for the converse, it ain't necessarily so. However, a $\mathrm{Lie}(V)$-module structure on M may come from at most one V-module srtucture.

Let us endow $\mathcal{E}^o$ with a $\mathrm{Lie}(\Omega_X^{ch})$-module structure. First of all, by (2.5) the D-bundle $\mathcal{E}^{\sim o} = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{E}^\sim, \omega^\sim)$ equals $\mathcal{D}iff(\mathcal{E}, \omega)$. Therefore it carries a canonical structure of a right $\mathrm{Lie}(\Omega_X^{ch})$-module defined by the formula

$$(11.2) \qquad (x_n f)(.) = (-1)^{\tilde{x}\tilde{f}} f(x_n .),$$

and hence a canonical structure of a left $\mathrm{Lie}(\Omega_X^{ch})$-module defined by the formula

$$(11.3) \qquad (x_n f)(.) = (-1)^{\tilde{x}\tilde{f}} f(\eta(x_n).),$$

where x is a local section of Ω_X^{ch}. Here

$$\eta : \mathrm{Lie}(\Omega_X^{ch}) \longrightarrow \mathrm{Lie}(\Omega_X^{ch}),$$

is the canonical antiinvolution (see 1.2).

Second of all, this $\mathrm{Lie}(\Omega_X^{ch})$-module structure descends to the quotient $\mathcal{E}^o = \mathcal{D}iff(\mathcal{E}, \omega)/\mathcal{D}iff(\mathcal{E}, \omega)\Theta_X$; this is because the action of the tangent sheaf Θ_X commutes with the action of $\mathrm{Lie}(\Omega_X^{ch})$: Θ_X acts on the value of the function $f(.)$, while $\mathrm{Lie}(\Omega_X^{ch})$ acts on its argument, see (11.2).

11.1. Claim. *The above* $\mathrm{Lie}(\Omega_X^{ch})$*-module structure on* $\mathcal{E}^o$ *comes from the* Ω_X^{ch}*-module structure.*

Proof. Let temporarily V be a vertex algebra. To prove the claim we have to understand what is it that singles out V-modules from the class of $\mathrm{Lie}(V)$-modules. A pair (E, ρ) is a $\mathrm{Lie}(V)$-module if E is a vector space and $\rho : \mathrm{Lie}(V) \to \mathrm{End}(E)$ is a Lie (super)algebra morphism. In particular, for any $x \in V$ we have a family of operators $\rho(x_n) \in \mathrm{End}(E)$. For (E, ρ) to be a V-module the two additional conditions are to be satisfied:

(A) For any $x \in V$ and $e \in E$, $\rho(x_n) = 0$ for all $n >> 0$.
(B) The operators $\rho(x_n) \in \mathrm{End}(E)$ satisfy the Borcherds identities.

It follows from [K] Proposition 4.8 that those Borcherds identities that do not follow from the Lie algebra structure on $\mathrm{Lie}(V)$ follow from the the relations:

$$(x_{-\Delta_x} y)_n = \sum_{i \in \mathbb{Z}} : x_i y_{n-i} :, \ n \in \mathbb{Z},$$

where as usual Δ_x is a number such that $x \in V_{\Delta_x}$.

Therefore (B) is equivalent to

(B$_0$). (E, ρ) is a Lie(V)-module and

$$(11.4) \qquad \rho((x_{-\Delta_x} y)_n) = \sum_{i \in \mathbb{Z}} \, : \rho(x_i)\rho(y_{n-i}) : \, .$$

The last formula means that each product $: \rho(x_i)\rho(y_{n-i}) :$ is ordered in the standard way and applied to any $e \in E$ from the right to the left.

In the same way one compares right V-modules and right Lie(V)-modules and concludes that a right V-module is a pair (E, ρ) as above satisfying the following conditions

(A^r) For any $x \in V$ and $e \in E$, $\rho(x_n) = 0$ for all $n << 0$.
(B$^r_0$) (E, ρ) is a right Lie(V)-module and

$$(11.5) \qquad \rho((x_{-\Delta_x} y)_n) = \sum_{i \in \mathbb{Z}} \, : \rho(x_i)\rho(y_{n-i}) : \, ,$$

where each product $: \rho(x_i)\rho(y_{n-i}) :$ is again ordered in the standard way but applied to any $e \in E$ from the left to the right.

Having reviewed this undoubtedly well-known material we cast a glance at (11.2) and convince ourselves that (11.2) indeed determines a right Ω_X^{ch}-module structure on $\mathcal{E}^o$: the restrictedness guarantees the condition (A^r), while (14.5) holds simply because (B$_0$) holds for $\mathcal{E}$.

Finally approaching (11.3) we see that (A) holds because (A^r) is satisfied for the right module structure determined by (11.2) and the fact that η changes the conformal weight to the opposite one. As to (B$_0$), it is implied by the following easily checked property of the antiinvolution η:

$$\eta(\sum_{i \in \mathbb{Z}} : x_i y_{n-i} :) = \sum_{i \in \mathbb{Z}} \eta(: x_i y_{n-i} :),$$

where the action of η on each monomial is as follows:

$$\eta(x_s y_t) = \eta(y_t)\eta(x_s), \eta(y_s x_t) = \eta(x_t)\eta(y_s).$$

This defines an Ω_X^{ch}-bundle structure on $\mathcal{E}^o$. By Lemma 7.3 we have

$$(11.6) \qquad H^i(X, \mathcal{E}^o) = H^{n-i}(X, \mathcal{E})^*.$$

The wedge product $\Omega_X \times \Omega_X \longrightarrow \omega_X$ induces an isomorphism of $\mathcal{O}_X$-modules

$$(11.7) \qquad \Omega_X \xrightarrow{\sim} \Omega_X^o.$$

There is a unique left $\mathcal{D}_{\Omega_X}$-module structure on Ω_X^o such that (11.7) is an isomorphism of $\mathcal{D}_{\Omega_X}$-modules.

More generally, for a left $\mathcal{D}_{\Omega_X}$-module E, the dual sheaf E^o is canonically a right $\mathcal{D}_{\Omega_X}$-module. Indeed, we have an algebra homomorphism $\mathcal{D}_{\Omega_X} \longrightarrow \mathcal{D}iff(E)$, hence $\mathcal{D}^o_{\Omega_X} \longrightarrow \mathcal{D}iff(E)^o = \mathcal{D}iff(E^o)$ (cf., (7.2)). On the other hand, the isomorphism (11.7) induces an antiautomorphism

$$(11.8) \qquad \mathcal{D}_{\Omega_X} \xrightarrow{\sim} \mathcal{D}^o_{\Omega_X}.$$

Therefore, E^o gets a canonical structure of a left $\mathcal{D}_{\Omega_X}$-module. This is the conformal weight zero part of the definition of duality at the beginning of this no.

11.2. Theorem. *The functor W_Ω commutes with duality, i.e. we have natural isomorphisms of Ω^{ch}_X-modules*

$$(11.9) \qquad W_\Omega(E^o) \xrightarrow{\sim} W_\Omega(E)^o$$

Proof. By construction $W_\Omega(E)^o_0 = E^o$. On the other hand, due to (10.2) $W_\Omega(E^o)_0 = E^o$. Hence $W_\Omega(E^o)_0 = W_\Omega(E)^o_0$. But what Theorem 10.1 tells us is that an Ω^{ch}_X-module is uniquely determined by its conformal weight zero component. Therefore (11.9) immediately follows from Theorem 10.1. $\triangle$

The isomorphisms (8.1) are a particular case of (11.9) with $E = \Omega_X$. By 10.1, it suffices to check the symmetry (8.2) on the conformal weight zero level, where it is evident. This proves Theorem 8.1.

Theorem 1.1 is an immediate consequence of Theorem 8.1 and Lemma 7.3.

12. Let $X = \mathbb{P}^1$. In this case (1.2) reduces to the following

$$(12.1) \qquad H^0(\mathbb{P}^1, \Omega^{ch,p}_{\mathbb{P}^1})^* = H^1(\mathbb{P}^1, \Omega^{ch,1-p}_{\mathbb{P}^1}).$$

For the sake of a mistrustful reader we present here a direct proof of (12.1).

12.1. First of all, we explicitly describe the space $\Gamma(\mathbb{C}^*, \Omega^{ch}_{\mathbb{P}^1})$. Consider the Lie (super)algebra Γ on the even generators a_i, b_i, $i \in \mathbb{Z}$ odd generators ϕ_i, ψ_i, $i \in \mathbb{Z}$ and relations:

$$(12.2) \qquad [a_i, b_{-i}] = [\psi_i, \phi_{-i}] = 1,$$

all other brackets being equal 0.

This algebra is $\mathbb{Z}$-graded (by conformal weight)

$$\Gamma = \oplus_{i \in \mathbb{Z}} \Gamma_i,$$

so that Γ_i is linearly spanned by x_i, where x is a, b, ϕ, or ψ. There arise four subalgebras

$$\Gamma_> = \oplus_{i>0}\Gamma_i, \quad \Gamma_< = \oplus_{i<0}\Gamma_i, \quad \Gamma_\geq = \oplus_{i\geq 0}\Gamma_i, \quad \Gamma_0,$$

and the decomposition

$$(12.3) \qquad \Gamma = \Gamma_< \oplus \Gamma_0 \oplus \Gamma_>.$$

For any Lie (super)algebra $\mathfrak{g}$, denote by $U(\mathfrak{g})$ its universal enveloping algebra. We have the extension of Lie algebras

$$(12.4) \qquad 0 \to \Gamma_> \to \Gamma_\geq \to \Gamma_0 \to 0,$$

If we fix b, a coordinate on $\mathbb{C}^*$, and identify b_0 with b, a_0 with d/db, ϕ_0 with db, and finally ψ_0 with the odd vector field $d/d(db)$, then $U(\Gamma_0)$ gets identified with the algebra of differential operators acting on $\Gamma(\mathbb{C}, \Omega_{\mathbb{P}^1}) = \Gamma(\mathbb{C}, \Omega^0_{\mathbb{P}^1} \oplus \Omega^1_{\mathbb{P}^1})$. Hence the latter space, as well as the space $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})$, becomes a Γ_0-module, and, by pull-back due to (12.4), a $\Gamma_\geq$-module. The inspection of the relevant definitions in [MSV] shows that

$$(12.5) \qquad \Gamma(\mathbb{C}^*, \Omega^{ch}_{\mathbb{P}^1}) = \mathrm{Ind}_{\Gamma_\geq}^{\Gamma} \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}).$$

12.2. There is a natural pairing

$$(12.6) \qquad <.\,,.>: \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}) \otimes \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}) \to \mathbb{C}, \quad <\nu, \mu> = \mathrm{Res}_{b=0}\nu \wedge \mu.$$

It enjoys the following 'contravariance' properties

$$(12.7) \qquad <b_0\nu, \mu> = <\nu, b_0\mu>, \quad <a_0\nu, \mu> = - <\nu, a_0\mu>.$$

(Similar equalities hold for the odd elements ϕ_0, ψ_0.) The pairing (12.6) induces the following map

$$(12.8) \qquad \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}) \to \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})^*.$$

Well-known in representation theory is the operation of taking a contragredient module. If we have a Lie algebra $\mathfrak{g}$ with an antiinvolution and a $\mathfrak{g}$-module M graded by finite dimensional subspaces, then the contragredient $\mathfrak{g}$-module, M^c, is defined as the restricted dual of M, the action of $\mathfrak{g}$ being equal to the canonical right action twisted by the antiinvolution. Apply this construction to Γ_0 operating on $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})$. The two necessary structures are as follows: the antiinvolution is defined by
$$(12.9)$$
$$\eta: \Gamma_0 \to \Gamma_0, \quad \eta(b_0) = b_0, \quad \eta(a_0) = -a_0, \quad \eta(\phi_0) = \phi_0, \quad \eta(\psi_0) = -\psi_0,$$

and the grading on $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})$ is determined by the condition

$$(12.10) \qquad \deg(b_0) = 1, \quad \deg(a_0) = -1.$$

In this way we get the contragredient module $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})^c$. It is obvious that the map (12.8) gives an isomorphism of Γ_0-modules

$$(12.11) \qquad \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}) \to \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})^c.$$

The same construction applies to the Γ-module $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})$. We, first, define an antiinvolution
(12.12)
$$\eta : \Gamma \to \Gamma, \quad \eta(b_i) = b_{-i}, \quad \eta(a_i) = -a_{-i}, \quad \eta(\phi_i) = \phi_{-i}, \quad \eta(\psi_i) = -\psi_{-i}.$$

As to the grading on $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})$, we notice that for any i the subspace of conformal weight i, $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})$, is infinite dimensional and we cure this by setting

$$\deg(x_i) = i \text{ if } i \neq 0,$$
$$\deg(x_0) = 0 \text{ unless } x = b \text{ or } a,$$
$$\deg(b_0) = -\deg(a_0) = 1.$$

In this way we get the contragredient module $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})^c$. By definition $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})^c$ inherits the grading by conformal weight:

$$(12.13) \qquad \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})^c = \oplus_{i \geq 0} \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})_i^c.$$

Due to (12.5) $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})_0 = \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1})$ and by (12.13) the map (12.11) is actually an isomorphism of $\Gamma_{\geq}$-modules

$$(12.14) \qquad \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})_0 \to \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})_0^c.$$

The universality property of induced modules implies that the map (12.14) uniquely extends to a morphism of Γ-modules

$$(12.15) \qquad \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch}) \to \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})^c.$$

The latter map gives rise to the pairing

$$(12.16) \qquad < . , . >: \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch}) \otimes \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch}) \to \mathbb{C},$$

which has the following contravariance property (cf., (12.7))

$$(12.17) \qquad < x\nu, \mu >= (-1)^{\tilde{x}\tilde{\nu}} < \nu, \eta(x)\mu > .$$

It is easy to see that the contragredient form (12.16) is uniquely determined by the property (12.17). Thus we could have defined this form by demanding that (12.17) be valid, but then a certain argument proving existence would have been required; we chose instead to construct the map (12.15) using the properties of induction.

A closer look at the process of calculating the form (12.16) by the repeated application of (12.17) shows that is is commutative:

$$(12.18) \qquad < \nu, \mu >= (-1)^{\tilde{\nu}\tilde{\mu}} < \mu, \nu > .$$

12.3. We have everything ready for the proof of (12.1). The space $\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})$ has two subspaces: $\Gamma(\mathbb{C}, \Omega_{\mathbb{P}^1}^{ch})$ and $\Gamma(\mathbb{P}^1 - \{0\}, \Omega_{\mathbb{P}^1}^{ch})$. To somewhat simplify the notation set $V = \Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch})$, $V_0 = \Gamma(\mathbb{C}, \Omega_{\mathbb{P}^1}^{ch})$, and $V_\infty = \Gamma(\mathbb{P}^1 - \{0\}, \Omega_{\mathbb{P}^1}^{ch})$. One easily checks that the map (12.15) is actually an isomorphism; therefore the pairing (12.16) is non-degenerate and we use it to identify V with V^*. By construction V_0 equals its annihilator, $\mathrm{Ann}\, V_0$. Since Γ_0-invariance allows us to interchange 0 and ∞, V_∞ is also equal to $\mathrm{Ann}\, V_\infty$. We now compute in a rather standard manner

$$H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch})^* = (V_0 \cap V_\infty)^* = V/(\mathrm{Ann}\, V_0 + \mathrm{Ann}\, V_\infty)$$
$$= V/(V_0 + V_\infty) = H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch}).$$

13. Let us deduce some corollaries from the previous construction. The form (12.16) has several attractive properties. We have already noted that it is symmetric (12.18) and contravariant (12.17).

Let $\mathfrak{g} = sl(2)$. We have proven in [MS], Part III, §1 that the affine Lie algebra $\widehat{\mathfrak{g}}$ acts canonically on the sheaf $\Omega_{\mathbb{P}^1}^{ch}$.

13.1. Claim.   *The pairing* (12.16) *is $\widehat{\mathfrak{g}}$-contravariant.*

In fact, (12.17) implies that (12.16) is contravariant with respect to $\mathrm{Lie}(\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch}))$; therefore it is also with respect to $\widehat{\mathfrak{g}}$ since the latter acts by means of an embedding $\widehat{\mathfrak{g}} \hookrightarrow \mathrm{Lie}(\Gamma(\mathbb{C}^*, \Omega_{\mathbb{P}^1}^{ch}))$.

As we noted in *op. cit.*, Part III, 2.1, the space $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch})$ is the maximal $\mathfrak{g}$-integrable $\widehat{\mathfrak{g}}$-submodule of $H^0(\mathbb{C}^*, \Omega_{\mathbb{C}^*}^{ch})$.

13.2. Corollary.   *The $\widehat{\mathfrak{g}}$-module $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch})$ is the maximal $\mathfrak{g}$-integrable quotient of $H^0(\mathbb{C}^*, \Omega_{\mathbb{C}^*}^{ch})$.*

This fact, conjectured by B. Feigin, was the starting point of the present note.

13.3. We presented in *op. cit.*, Part III, 2.2 a rather explicit description of $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch})$ as a $\widehat{\mathfrak{g}}$-module. Now (12.1) and 13.1 provide us with no less explicit description of $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{ch})$. We leave the details for the interested reader.

Acknowledgement

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