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SOME REMARKS ON RATIONAL PERIODIC POINTS

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Abstract. Let M be a finitely generated field over $\mathbb{Q}$ and X a variety defined over M. We study when the set $\{P \in X(K) \mid f^{\circ n}(P) = P$ for some $n \geq 1\}$ is finite for any finite extension fields K of M and for any dominant K-morphisms $f : X \to X$ with $\deg f \geq 2$.

Introduction

To define a variety, we infer an integral separated scheme of finite type over a ground field. Let M be a finitely generated field over $\mathbb{Q}$ and X a variety defined over M. Let K be a finite extension field of M and $f : X \to X$ a dominant morphism defined over K. We say that a point $P \in X(K)$ is periodic with respect to f if there is a positive integer n with $f^{\circ n}(P) = P$. Let $X(K)_{per,f}$ be the set of periodic K-points with respect to f. We say that X is periodically finite if $X(K)_{per,f}$ is a finite set for any finite extension fields K of M and any dominant K-morphisms $f : X \to X$ with $\deg f \geq 2$.

In this paper, we study when X is periodically finite. In order to show the finiteness of $X(K)_{per,f}$, we introduce the set of backward K-orbits of f, denoted by $\varprojlim_f X(K)$, which is defined by

$$\varprojlim{}_f X(K) = \left\{ (x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} X(K) \mid f(x_{n+1}) = x_n \quad (n \geq 0) \right\}.$$

It is easy to see that if $\varprojlim_f X(K)$ is a finite set, then so is $X(K)_{per,f}$ and $\# \varprojlim_f X(K) = \# X(K)_{per,f}$ (cf. Lemma 2.2).

We obtain the following results.

Theorem A (cf., Corollary 2.5 and §6)**.** *Let X be a geometrically irreducible normal projective variety defined over a finitely generated field over $\mathbb{Q}$. Assume that the Picard number of X is* 1 (*for example, X is $\mathbb{P}^n$ or a geometrically irreducible normal projective curve*). *Then X is periodically finite.*

We prove this by using Northcott's finiteness theorem of height functions. More precisely, this result is a corollary of the fact that if there is an ample line bundle L such that $f^*(L) \otimes L^{-1}$ is also ample, then $\varprojlim_f X(K)$ is a finite set (Theorem 2.4).

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We also show the following theorems.

Theorem B (cf., Corollary 3.4 and §6)**.** *Let C be a curve defined over a finitely generated field over $\mathbb{Q}$. Then C is periodically finite.*

Theorem C (cf., Theorem 4.4 and §6)**.** *Let A be an abelian variety defined over a finitely generated field over $\mathbb{Q}$. Then A is periodically finite if and only if A is simple.*

Theorem D (cf., Theorem 5.6 and §6)**.** *Let X be a smooth projective surface with the non-negative Kodaira dimension such that X is defined over a finitely generated field over $\mathbb{Q}$. Then X is* not *periodically finite if and only if X is one of the following types;*

(i) *X is an abelian surface which is* not *simple, or*
(ii) *X is a hyperelliptic surface.*

In order to clarify the argument, M is assumed to be a number field before §6, where in §6, we deal with a finitely generated field over $\mathbb{Q}$ in general.

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1. Quick review of height theory

In this section, we recall some properties of height functions. We refer to [13] for details. Let $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the logarithmic height function. Namely, for a point $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$, $h(x)$ is defined by

$$h(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \log \left(\max_{1 \le i \le n} \{|x_i|_v\} \right),$$

where $x = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(K)$ is its coordinate over a sufficiently large number field K, and M_K is the set of all places of K.

Now let X be a projective variety defined over $\overline{\mathbb{Q}}$, $\phi : X \to \mathbb{P}^n$ a morphism over $\overline{\mathbb{Q}}$. For a point $x \in X(\overline{\mathbb{Q}})$, we define the height of x with respect to ϕ, denoted by $h_\phi(x)$, to be $h_\phi(x) = h(\phi(x))$.

Then the following theorem holds.

Theorem 1.1 (Height Machine)**.** *For every line bundle L on a projective variety X defined over $\overline{\mathbb{Q}}$, there exists a unique function $h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ modulo bounded functions with the following property;*

(i) *For any two line bundles L_1, L_2, $h_{L_1 \otimes L_2} = h_{L_1} + h_{L_2} + O(1)$.*
(ii) *If $f : X \to Y$ is a morphism of projective varieties over $\overline{\mathbb{Q}}$, then $h_{f^*(L)} = f^*(h_L) + O(1)$.*
(iii) *If $\phi : X \to \mathbb{P}^n$ is a morphism over $\overline{\mathbb{Q}}$, then $h_{\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))} = h_\phi + O(1)$.*

We also recall some properties of height functions.

Theorem 1.2.

(i) (*positiveness*) *If we denote* Supp $\left(\mathrm{Coker}(H^0(X,L)\otimes\mathcal{O}_X)\to L\right)$ *by* $\mathrm{Bs}(L)$, *then* h_L *is bounded below on* $(X\setminus\mathrm{Bs}(L))(\overline{\mathbb{Q}})$.

(ii) (*Northcott*) *Assume* L *be ample. Then for any* $d\geq 1$ *and* $M\geq 0$,

$$\{x\in X(\overline{\mathbb{Q}})\mid h_L(x)\leq M,\quad [\mathbb{Q}(x):\mathbb{Q}]\leq d\}$$

is a finite set.

For Theorem 1.1, we refer to [13, Theorem 3.3]. For Theorem 1.2, we refer to [13, Corollary 3.4 and Proposition 3.5]. Although in [13] Theorem 1.1 (ii) is written for a morphism of smooth projective varieties, it also holds for not necessarily smooth projective varieties.

2. Finiteness

Let X be a variety defined over a number field M. Let K be a finite extension of M and $f: X\to X$ a dominant morphism defined over K.

We say that a point $P\in X(K)$ is *periodic* with respect to f if there is a positive integer n with $f^{\circ n}(P)=P$. Let $X(K)_{per,f}$ be the set of periodic K-points with respect to f.

We also define the set of *backward* K*-orbits* of f, denoted by $\varprojlim{}_f X(K)$, to be

$$\varprojlim{}_f X(K)=\{(x_n)_{n=0}^{\infty}\in\prod_{n=0}^{\infty}X(K)\mid f(x_{n+1})=x_n\quad(n\geq 0)\}.$$

We say that X is *periodically finite* if for any finite extension fields K of M and for any dominant K-morphisms $f: X\to X$ with $\deg f\geq 2$, $X(K)_{per,f}$ is a finite set. Note that if there is no morphism $f: X\to X$ with $\deg f\geq 2$, then X is periodically finite. For example, a variety of general type is periodically finite.

In this paper, we would like to study what kind of X is periodically finite. We first remark elementary properties of $X(K)_{per,f}$ and $\varprojlim{}_f X(K)$.

Lemma 2.1. *Let* $S\subset X(K)$ *be a finite set and* $(x_n)_{n=0}^{\infty}\in\varprojlim{}_f X(K)$. *Assume that there is a subsequence* $(x_{n_i})_{i=0}^{\infty}$ *consisting of elements in* S. *Then* $(x_n)_{n=0}^{\infty}$ *is periodic, i.e., there is a positive integer* p *with* $x_{n+p}=x_n$ *for* $n\geq 0$. *Moreover,* $(x_n)_{n=0}^{\infty}$ *is uniquely determined by* x_0.

Proof. Since S is a finite set, there is an element $s\in X(K)$ such that, for infinitely many n, x_n equals to s. Let $(x_{n_j})_{j=0}^{\infty}$ be the subsequence of $(x_n)_{n=0}^{\infty}$ with $x_{n_j}=s$ for $j\geq 0$. Let us set $p=n_1-n_0$. We show that $n_2-n_1=p$. Indeed, since $f^{\circ q}(x_{n_2})=x_{n_1}$, if we set $q=n_2-n_1$, then we have $f^{\circ q}(s)=s$. If we assume $q>p$, then $n_2>n_2-p>n_1$ and $x_{n_2}=x_{n_2-p}=x_{n_1}=s$. This is a contradiction. If we assume $p>q$, then we similarly have a contradiction. Thus $n_2-n_1=n_1-n_0=p$. In the same way, $n_{j+1}-n_j=p$ for any $j\geq 0$.

Now let us take any $n \geq 0$. We fix an n_j with $n_j > n$ and set $r = n_j - n$. Then $n_j + p = n_{j+1}$ and $n_{j+1} - (n+p) = r$. Therefore, we get

$$x_{n+p} = f^{\circ r}(x_{n_{j+1}}) = f^{\circ r}(s) = f^{\circ r}(x_{n_j}) = x_n.$$

This shows that $(x_n)_{n=0}^{\infty}$ is periodic. Moreover if we divide n by p and write $n = qp + l$ with $0 \leq l \leq p-1$, then it is easy to see that $x_n = f^{\circ(p-l)}(x_0)$. This shows the latter assertion of the lemma. □

The next lemma gives the relationship between $\varprojlim_f X(K)$ and $X(K)_{per,f}$.

Lemma 2.2.

(i) *If P is a K-periodic point, then there is an element $(x_n)_{n=0}^{\infty} \in \varprojlim_f X(K)$ such that $P = x_0$. By this correspondence, $X(K)_{per,f}$ can be seen as a subset of $\varprojlim_f X(K)$. We say an element of $\varprojlim_f X(K)$ which lies in the image of $X(K)_{per,f}$ is periodic.*

(ii) *If $X(K)_{per,f} \subsetneq \varprojlim_f X(K)$ in the above correspondence, then $\varprojlim_f X(K)$ is an infinite set.*

(iii) *If $\varprojlim_f X(K)$ is a finite set, then $X(K)_{per,f} = \varprojlim_f X(K)$ in the above correspondence. In particular, $X(K)_{per,f}$ is also a finite set.*

Proof. (i) Let $f^{\circ p}(P) = P$. For any $n \geq 0$, we divide n by p and write $n = qp + l$ with $0 \leq l \leq p-1$. Then if we put $x_n = f^{\circ(p-l)}(P)$, $(x_n)_{n=0}^{\infty}$ is an element of $\varprojlim_f X(K)$.

(ii) Suppose $(x_n)_{n=0}^{\infty} \in \varprojlim_f X(K)$ is not periodic. By lemma 2.1, for any fixed m, there are only finitely many k with $x_k = x_m$. Then $\{(x_n)_{n=m}^{\infty} \mid m \geq 0\} \subset \varprojlim_f X(K)$ is an infinite set.

(iii) If $\varprojlim_f X(K)$ is a finite set, then every $(x_n)_{n=0}^{\infty} \in \varprojlim_f X(K)$ is periodic by (ii). In particular, x_0 is periodic. Therefore, the correspondence of (i) becomes bijective. □

Next lemma shows that finiteness still holds if we change f to some powers of f.

Lemma 2.3. *Let k be a positive integer.*

(i) *$X(K)_{per,f^{\circ k}}$ is a finite set if and only if $X(K)_{per,f}$ is a finite set.*

(ii) *$\varprojlim_{f^{\circ k}} X(K)$ is a finite set if and only if $\varprojlim_f X(K)$ is a finite set.*

Proof. (i) Suppose P satisfies $f^{\circ m}(P) = P$. Then P satisfies $(f^{\circ k})^{\circ m}(P) = P$. This shows that $X(K)_{per,f} = X(K)_{per,f^{\circ k}}$.

(ii) We have only to prove the "only if"part. If $\varprojlim_{f^{\circ k}} X(K)$ is a finite set, its elements are all periodic by Lemma 2.2(ii). Thus if we set

$$S = \{x \in X(K) \mid \text{there is an } (x_n)_{n=0}^{\infty} \in \varprojlim_{f^{\circ k}} X(K) \text{ and an } m \text{ such that } x = x_m.\},$$

then S is a finite set. Now the finiteness of $\varprojlim_f X(K)$ follows from Lemma 2.1. □

Now we prove the following theorem.

Theorem 2.4. *Let X be a projective variety defined over a number field K and $f : X \to X$ a surjective morphism defined over K. Let d be a positive integer. Assume that there is an ample line bundle L such that $f^*(L) \otimes L^{-1}$ is ample. Then $\bigcup_{[K':K]\leq d} \varprojlim_f X(K')$ is a finite set. In particular, $\bigcup_{[K':K]\leq d} X(K')_{per,f}$ is also a finite set and $\# \bigcup_{[K':K]\leq d} \varprojlim_f X(K') = \# \bigcup_{[K':K]\leq d} X(K')_{per,f}$.*

Proof. If we take a positive rational number ϵ' which is sufficiently small, then $f^*(L) \otimes L^{-(1+\epsilon')}$ is still ample as a $\mathbb{Q}$-line bundle. Then by Theorem 1.2(i), and by the fact that $h_{f^*(L)}(P) - h_L(f(P))$ is a bounded function, we have a constant C such that

$$h_L(f(P)) - (1+\epsilon')h_L(P) \geq C.$$

for all $P \in X(\overline{K})$. Let us take an ϵ with $0 < \epsilon < \epsilon'$. Then there is a constant M such that if $h_L(P) > M$, then

$$h_L(f(P)) - (1+\epsilon)h_L(P) > 0.$$

Now let us define a set S to be

$$S = \{x \in X(K') \mid [K' : K] \leq d \text{ and } h_L(x) \leq M\}.$$

Since L is ample, S is a finite set by Northcott.

In the following we show that, if $(x_n)_{n=0}^{\infty} \in \varprojlim_f X(K')$, then there is a subsequence $(x_{n_i})_{i=0}^{\infty}$ consisting of elements in S. In fact, suppose on the contrary that there is an m such that, for any $n \geq m$, x_n does not belong to S. Since $h_L(x_n) > M$ for $n \geq m$, we have

$$\cdots < (1+\epsilon)^2 h_L(x_{m+2}) < (1+\epsilon)h_L(x_{m+1}) < h_L(x_m).$$

This is a contradiction because

$$h_L(x_n) < \frac{1}{(1+\epsilon)^{n-m}} h_L(x_m) \to 0 \qquad (n \to \infty).$$

Now by applying Lemma 2.1, we get that $(x_n)_{n=0}^{\infty}$ is periodic and uniquely determined by x_0. We also get that the number of $\bigcup_{[K':K]\leq d} \varprojlim_f X(K')$ does not exceed the number of S. This proves the first assertion. The second assertion follows from Lemma 2.2. □

As a corollary, we obtain the finiteness for a certain class of varieties.

Corollary 2.5. *Let X be a geometrically irreducible normal projective variety defined over a number field M. Assume that the Picard number of X is* 1 (*for example, X is $\mathbb{P}^n$ or a geometrically irreducible normal projective curve*). *Then X is periodically finite.*

Proof. Let K be a finite extension field of M and $f : X \to X$ be a surjective K-morphism of $\deg f \geq 2$. We take an arbitrary ample line bundle L on X. Then by our hypothesis, there is a integer $d \geq 2$ such that $f^*(L)$ is numerically equivalent to $L^{\otimes d}$. In particular, $f^*(L) \otimes L^{-1}$ is ample. □

Let us keep the notation of Theorem 2.4. Assume here that $f^*(L)$ is linearly equivalent to $L^{\otimes d}$. In this case, due to Tate, there exists a unique height function $h_{L,f}$ such that $h_{L,f} = h_L + O(1)$ and that $h_{L,f}(f(P)) = dh_L(P)$ (cf., [6, Chap 4. Proposotion 1.9] or [2, Corollary 1.1.1]). Then for any periodic points with respect to f, their height must be 0 with respect to $h_{L,f}$. An example for this is the following corollary.

Corollary 2.6. *Let K be a number field, A an Abelian variety defined over K and $[m] : A \to A$ the m-plication map with $m \geq 2$. Then $\varprojlim_{[m]} A(K)$ is a finite set and the number of $\varprojlim_{[m]} A(K)$ does not exceed the number of torsion K-points.*

Proof. Extending K if necessary, we may assume that there is an ample symmetric line bundle L on A. Then $f^*(L) \simeq L^{\otimes m^2}$ and we can apply the theorem. In this case, if x is a periodic point, then x is a torsion point. □

We finish this section by giving examples such that $X(K)_{per,f}$ is infinite.

Example 2.7. We give an example such that $X(K)_{per,f}$ (and thus $\varprojlim_f X(K)$) is infinite. Let E be an elliptic curve defined over a number field K such that $E(K)$ is an infinite set. Let X be $E \times E$ and $f : X \to X$ map (P, Q) to $(P, [2](Q))$. Then f is finite of degree 4 and the points of the form $(P, 0)$ are all periodic points.

Example 2.8. We give an example such that $X(K)_{per,f}$ is finite but $\varprojlim_f X(K)$ is infinite. Let E be an elliptic curve defined over a number field K for which $E(K)$ contains non-torsion points. Let $P_0 \in E(K)$ be a non-torsion point. Let X be $E \times E$ and $f : X \to X$ map (P, Q) to $(P + P_0, [2](Q))$. Then f is finite of degree 4 and contains a sequence $(x_n)_{n=0}^{\infty} \in \varprojlim_f X(K)$ with $x_n = (-[n](P_0), 0)$. Thus by Lemma 2.2, $\varprojlim_f X(K)$ is not finite. On the other hand, there are no periodic points.

We note that we can give examples similar to the above two examples by using $\mathbb{P}^1$.

3. Curves

By a curve, we mean an integral separated scheme of finite type over a ground field. In this section, we prove that a curve is periodically finite. Since there is no surjective morphism $f : C \to C$ with $\deg f \geq 2$ if C is a smooth projective curve of genus ≥ 2, we are mainly concerned with a curve C such that $C \otimes \overline{\mathbb{Q}}$ is a reduced scheme consisting of rational curves and elliptic curves. First we prove two lemmas.

Lemma 3.1. *Let C be a curve defined over $\overline{\mathbb{Q}}$, and $f : C \to C$ a morphism over $\overline{\mathbb{Q}}$. Then there is a completion $\overline{C}$ of C and a morphism $\overline{f} : \overline{C} \to \overline{C}$ which is an extension of f.*

Proof. Let us take an arbitrary complete curve $\overline{C}'$ which is a completion of C and set $T' = \overline{C}' \setminus C(\overline{\mathbb{Q}})$. If $t \in T'$ is a singular point of $\overline{C}'$, then we blow it up. Iterating this procedure, we get a completion $\overline{C}$ such that every point in $T = \overline{C} \setminus C(\overline{\mathbb{Q}})$ is a smooth point of $\overline{C}$. Now f defines a rational map $\overline{f} : \overline{C} \dashrightarrow \overline{C}$. Since it is defined over T and C, $\overline{f}$ is actually a morphism. □

Lemma 3.2. *Let C be a curve defined over a number field M which is geometrically irreducible. Then C is periodically finite.*

Proof. Let K be a finite extension of M and $f : X \to X$ a surjective morphism defined over K with $\deg f \geq 2$. By taking a finite extension of K if necessary, Lemma 3.1 indicates that there is a completion $\overline{C}$ of C and a extension $\overline{f}$ of f which are defined over K. Then $\varprojlim_f C(K)$ can be seen as a subset of $\varprojlim_{\overline{f}} \overline{C}(K)$. For a general point $P \in \overline{C}(\overline{\mathbb{Q}})$, let $L = \mathcal{O}_{\overline{C}}(P)$. Then, since $\deg \overline{f} \geq 2$, $f^*(L) \otimes L^{-1}$ is ample. Thus, by Theorem 2.4, $\varprojlim_{\overline{f}} \overline{C}(K)$ is a finite set. This proves the lemma. □

Now we prove the following proposition.

Proposition 3.3. *Let C be a reduced scheme which is a chain of geometrically irreducible curves over $\overline{\mathbb{Q}}$. Let $f : C \to C$ be a surjective morphism such that, for every irreducible component C_i of C, $f|_{C_i} : C_i \to f(C_i)$ has degree ≥ 2. Then for a number field $K \subset \overline{\mathbb{Q}}$ such that C and f are defined over K, $\varprojlim_f C(K)$ is a finite set.*

Proof. If K' is a extension field of K, then the finiteness of $\varprojlim_f C(K')$ implies the finiteness of $\varprojlim_f C(K)$. Thus to prove the proposition, we may take a finite extension of K if necessary. Let $C_1, C_2, \ldots, C_l$ be the irreducible components of C. Since f is surjective, the dimension of $f(C_\alpha)$ is 1 for every α. Thus f is seen to induce a transposition of the set $C_1, C_2, \ldots, C_l$. Then $f^{\circ l!}$ maps C_α to C_α for $1 \leq i \leq l$. Let us set $S = (\cup_{\alpha \neq \beta} C_\alpha \cap C_\beta)_{red}$. By Lemma 2.3, we have only to show that $\varprojlim_{f^{\circ l!}} C(K)$ is a finite set. We may take a sufficiently large K, so that C_α's and S are all defined over K. Now let $(x_n)_{n=0}^\infty \in \varprojlim_{f^{\circ l!}} X(K)$.

Case 1 Suppose that there exists a subsequence $(x_{n_i})_{i=0}^\infty$ consisting of elements in S. Then by Lemma 2.1, the number of $(x_n)_{n=0}^\infty$ in this case is finite.

Case 2 Suppose that there is no subsequence $(x_{n_i})_{i=0}^\infty$ consisting of elements in S. Then there is an α such that every x_n belongs to C_α. By Lemma 3.1, $\varprojlim_{f^{\circ l!}} C_\alpha(K)$ is a finite set. Thus the number of $(x_n)_{n=0}^\infty$ in this case is also finite. □

As a corollary, we get

Corollary 3.4. *Let C be a curve defined over a number field M. Then C is periodically finite.*

Proof. Let K be a finite extension of M and $f : C \to C$ be a surjective K-morphism with $\deg f \geq 2$. Let us consider $C_{\overline{\mathbb{Q}}}$ and let $C_1, C_2, \ldots, C_l$ be its irreducible components. By abbreviation, f also denotes the induced morphism $C_{\overline{\mathbb{Q}}} \to C_{\overline{\mathbb{Q}}}$. Since $C_1, C_2, \dots, C_l$ are all conjugate to each other, the degree of $f|_{C_\alpha}$ is greater or equal to 2 for each $1 \le \alpha \le l$. Now the assertion follows from Proposition 3.3. □

4. Abelian varieties

Let A be an abelian variety defined over a number field M. Recall that A is said to be simple if $End(A)_{\mathbb{Q}}$ is simple. In this section, we show that A is periodically finite if and only if A is simple. First we show that if an abelian variety is simple, then it is periodically finite.

Proposition 4.1. *Let A be a simple abelian variety defined over a number field M. Then A is periodically finite.*

Proof. Let K be a finite extension field of M and $f : X \to X$ a finite K-morphism with $\deg f \ge 2$. Let us set $B_n = \{P \in A(K) \mid f^{\circ n}(P) = P\}$. We prove the finiteness of $A(K)_{per,f}$ in two steps.

Step 1 We assume here that f is a homomorphism. Let us denote by $A(K)_{tor}$ the set of K-valued torsion points on A. It is well known that $A(K)_{tor}$ is a finite set (cf., Corollary 2.6). Since A is simple and $f^{\circ n} \neq 1$, $B_n = \mathrm{Ker}(f^{\circ n} - 1)(K)$ is a finite abelian group. In particular, $B_n \subset A(K)_{tor}$. Thus $A(K)_{per,f} = \cup_{n=1}^{\infty} B_n \subset A(K)_{tor}$ is a finite set.

Step 2 Here we treat a general f. If $B_n = \emptyset$ for $n \ge 1$, then we have nothing to prove. Thus we assume that there is an k with $B_k \neq \emptyset$ and we shall prove $A(K)_{per,f}$ is a finite set. Since $A(K)_{per,f^{\circ k}} = A(K)_{per,f}$ by Lemma 2.3, we may assume that $B_1 \neq \emptyset$. We take $x_0 \in B_1$, i.e., $f(x_0) = x_0$. We give A another group structure such that the identity is x_0. We denote this abelian variety by A'. Since f maps x_0 to itself, f is a homomorphism of A'. Therefore, $A'(K)_{per,f}$ is a finite set by Step 1. Since A and A' are identical as a set and thus $A(K)_{per,f} = A'(K)_{per,f}$, we are done. □

Next we show that if A is not simple, then A is not periodically finite. First we note the following lemma.

Lemma 4.2. *Let A be an abelian variety defined over a finitely generated field M over $\mathbb{Q}$. Then there exists a finite extension field K of M such that $A(K)$ is an infinite set.*

This is proven by many authors (cf., [12, Theorem 10.1], [8, Theorem 7.6]). We note that this is an easy corollary of Raynaud's theorem [9] (Manin-Mumford conjecture). Indeed, by Bertini's theorem, there is a curve C of genus ≥ 2 on $A_{\overline{M}}$. By Raynaud's theorem, $C(\overline{M}) \cap A(\overline{M})_{tor}$ is a finite set. Thus if we take a sufficiently large extension field K of M, then there exists a point $P \in C(K)$ which is not torsion. This proves the lemma.

Proposition 4.3. *Let A be an abelian variety defined over a number field M. If A is not simple, then A is not periodically finite.*

Proof. Since A is not simple, there is an $\overline{\mathbb{Q}}$-isogeny $g : B \times C \to X$, where B and C are positive-dimensional abelian varieties. Let us set $D = \operatorname{Ker} g$, which is a finite group of order $d = \#D$.

We consider a morphism

$$[d+1] \times [1] : B \times C \longrightarrow B \times C.$$

Since, for a point $(b, c) \in D$, $([d]b, [d]c) = 0$, we get $[d+1] \times [1](b, c) = (b, c)$ for any $(b, c) \in D$. In particular, $[d+1] \times [1]$ induces a morphism

$$f : A \longrightarrow A.$$

By the snake lemma, $\operatorname{Ker}([d+1] \times [1]) = \operatorname{Ker} f$, thus f is a surjective morphism with $\deg f \geq 2$. Now we take a finite extension field K of M such that B and C are defined over K and that $C(K)$ is an infinite set. Then the infinite set

$$g(\{(0, Q) \in B(K) \times C(K)\}),$$

is contained in $A(K)_{per,f}$. □

Combining Proposition 4.1 and Proposition 4.3, we obtain the following theorem.

Theorem 4.4. *Let A be an abelian variety defined over a number field . Then A is periodically finite if and only if A is simple.*

5. surfaces with non-negative Kodaira dimensions

In this section we consider smooth projective surfaces with non-negative Kodaira dimensions.

E. Sato and Y. Fujimoto [10] [11] classified smooth projective varieties of dim = 3 with the non-negative Kodaira dimensions which has a non-trivial surjective endomorphism.

As a test case, they considered the surface case, which is as in the following.

Theorem 5.1 (E. Sato and Y. Fujimoto)**.** *If a smooth projective surface X has a surjective endomorphism $f : X \to X$ with $\deg f \geq 2$, then X must be minimal and is one of the following types;*

(i) *X is an abelian surface,*
(ii) *X is a hyperelliptic surface, or*
(iii) *The Kodaira dimension $\kappa(X)$ of X is 1 and X carries an elliptic fibration $\pi : X \to B$ whose singular fibers are at most multiple of the type ${}_mI_0$ in the sense of Kodaira, where B is a smooth projective curve.*

Proof. For the reader's sake, we give a brief sketch of a proof.

Since X has non-negative Kodaira dimension, $f : X \to X$ must be étale (cf., [4, Theorem 11.7]). Suppose there is an exceptional curve C on X. Then the equality

$$f^*(C) \cdot K_X = f^*(C) \cdot f^*K_X = -(\deg f),$$

shows that there are at least two exceptional curves on X. Iterating this procedure, we get a contradiction.

We note that since f is étale, $\chi_{top}(X) = (\deg f)\chi_{top}(X)$. Then $\deg f \geq 2$ implies $\chi_{top}(X) = 0$. In the same way, we get $\chi(\mathcal{O}_X) = 0$.

If $\kappa(X) = 2$, then there are no surjective morphisms $f : X \to X$ with $\deg f \geq 2$ (cf., [4, Proposition 10.10]); If $\kappa(X) = 1$, then $\chi_{top}(X) = 0$ indicates that X has possibly only multiple singular fibers of type ${}_mI_0$; If $\kappa(X) = 0$, then $\chi(\mathcal{O}_X) = 0$ indicates that X cannot be a K3 surface nor an Enriques surface. □

We determined in the previous section when an abelian surface is periodically finite. Now we study whether a surface of the case (ii) or (iii) is periodically finite.

Proposition 5.2. *Let X be a hyperelliptic surface defined over a number field M. Then X is not periodically finite.*

Proof. Let E, F be arbitrary elliptic curves, G a group of translations of E which operates on F. According to the Bagnera-De Franchis list ([1, Liste VI.20]), all the hyperelliptic curves are one of the following types;

(i) $X \cong (E \times F)/G$, $G = \mathbb{Z}/2$ operating on F by $x \mapsto -x$,
(ii) $X \cong (E \times F)/G$, $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ operating on F by $x \mapsto -x$, $x \mapsto x + \epsilon$ ($\epsilon \in F_2$),
(iii) $X \cong (E \times F_i)/G$, $G = \mathbb{Z}/4$ operating on F_i by $x \mapsto ix$, where $F_i = \mathbb{C}/\mathbb{Z}+i\mathbb{Z}$,
(iv) $X \cong (E \times F_i)/G$, $G = \mathbb{Z}/4$ operating on F_i by $x \mapsto ix$,
(v) $X \cong (E \times F_\rho)/G$, $G = \mathbb{Z}/3$ operating on F_ρ by $x \mapsto \rho x$, where $\rho = \frac{-1+\sqrt{-3}}{2}$ and $F_\rho = \mathbb{C}/\mathbb{Z} + \rho\mathbb{Z}$.
(vi) $X \cong (E \times F_\rho)/G$, $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ operating on F_ρ by $x \mapsto \rho x$, $x \mapsto x + \frac{1-\rho}{3}$
(vii) $X \cong (E \times F_\rho)/G$, $G = \mathbb{Z}/6$ operating on F_ρ by $x \mapsto -\rho x$.

Now we consider the case (i). In this case,

$$[3] \times [1] : E \times F \longrightarrow E \times F$$

induces a surjective morphism

$$f : X \to X$$

with $\deg f \geq 2$. If we take a sufficiently large finite extension field K of M, Then the infinite set $\{(0, Q) \mid Q \in F(K)\}$ is contained in $(E \times F)(K)_{per,[3]\times[1]}$. Thus $X(K)_{per,f}$ is also an infinite set. The other cases can be treated in similar ways. In lieu of $[3] \times [1]$, we have only to consider $[g + 1] \times [1]$ where $g = \#G$. □

Next we treat a case of an elliptic surface. We prove the following lemma in advance.

Lemma 5.3. *Let $\pi : X \to B$ be a flat morphism of projective varieties over with $\dim B = 1$ such that all the fibers are (possibly non-reduced) abelian varieties. If the Kodaira dimension of X is greater or equal to 1, then the geometric genus of every horizontal curve is greater or equal to 2.*

Proof. Suppose there is a horizontal curve C on X such that the geometric genus of C is 0 or 1. Then there is an elliptic curve B' with a surjection $u : B' \to C$. Let us set $v = u \circ \pi : B' \to B$. Now we consider the following Cartesian product,

$$\begin{array}{ccc} X' & \xrightarrow{v'} & X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ B' & \xrightarrow{v} & B. \end{array}$$

Since the singular fibers of π' are at most multiple fibers of an abelian variety and since $\pi' : X' \to B'$ has a section, π' must be a smooth morphism. Then there is an elliptic curve B'' and an étale covering $B'' \to B'$ such that its pull-back $\pi'' : X'' = X' \times'_B B''$ is trivial, i.e., X'' is a product of an abelian variety by an elliptic curve (cf. [1, Proposition VI.8]). Thus the Kodaira dimension of X'' is zero. On the other hand, since there is a surjective morphism $X'' \to X$, the Kodaira dimension of X'' must be greater or equal to 1. This is a contradiction. □

Lemma 5.4. *Let K be a number field. Let $\pi : X \to B$ be a flat morphism of projective varieties over K with $\dim B = 1$. Let $f : X \to X$ a surjective morphism over K which commutes with π, i.e., $\pi \circ f = \pi$. We make the following three assumptions.*

(A) *There exist a covering $u : B' \to B$ over K and an abelian variety A over K such that $X \times_B B'$ is isomorphic to $A \times_K B'$ over B':*

$$\begin{array}{ccc} A \times_K B' \simeq X \times_B B' & \xrightarrow{u'} & X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ B' & \xrightarrow{u} & B. \end{array}$$

(B) *If C is a horizontal curve on X, then the geometric genus of C is greater or equal to 2.*

(C) *There is a line bundle L on X such that both $L|_{X_\eta}$ and $f^*(L) \times L^{-1}|_{X_\eta}$ are ample, where η is a generic point of B.*

Then $X(K)_{per,f}$ is a finite set.

Proof. We set $X' = X \times_B B'$ and $f' = f \times_B id : X' \to X'$. We first claim that

$$X(K)_{per,f} \subset u' \left(\bigcup_{[K':K] \leq d} X'(K')_{per,f'} \right),$$

where $d = \deg u$. Indeed, for $P \in X(K)_{per,f}$, we set $Q = \pi(P)$ and take a point Q' on B' with $u(Q') = Q$. We note that Q' is defined over a number field K' with $[K' : K] \leq d$. Then the point $P' = (P, Q')$ on $X' = X \times_B B'$ satisfies $u'(P') = P$ and defined over K'. For n with $f^{\circ n}(P) = P$, we have $f'^{\circ n}(P') = (f^{\circ n}(P), Q) = (P, Q) = P'$. Thus $P' \in X'(K')_{per,f'}$.

Let $p : X' = A \times_K B' \to A$, $q : X' = A \times_K B' \to B'$ be the projections. We next claim that there are finite points $x_1, x_2, \cdots, x_n \in A(\overline{K})$ such that

$$\bigcup_{[K':K]\le d} X'(K')_{per,f'} \subset \{x_1, x_2, \cdots, x_n\} \times_K B'.$$

Indeed, by the rigidity of abelian varieties (cf., [5, Chap 8 Theorem 1]), there exist a finite extension field K_1 of K and a morphism $g : A \to A$ defined over K_1 such that $f'_{K_1} = g \times_{K_1} id : X'_{K_1} \to X'_{K_1}$. (Here, $f'_{K_1} = f \times_K K_1$ and $X'_{K_1} = X' \times_K K_1$.) Now if we set $e = [K_1 : K]$, then we have

$$p\left(\bigcup_{[K':K]\le d} X'(K')_{per,f'}\right) \subset \bigcup_{[K'':K]\le de} A(K'')_{per,g}.$$

On the other hand, if we set $L' = u'^*(L)|_A$, then, by the assumption (C), both L' and $g^*(L') \otimes L'^{-1}$ are ample. Thus, by Theorem 2.4, $\bigcup_{[K'':K]\le de} A(K'')_{per,g}$ is a finite set. Therefore, we get the claim.

From the above two claims, we get

$$X(K)_{per,f} \subset u'\left(\{x_1, x_2, \cdots, x_n\} \times_K C'\right).$$

On the other hand, $u'(x_i \times_K C')$ is a horizontal curve on X and thus by the assumption (B), its geometric genus is greater or equal to 2. Then $u'(x_i \times_K C')(K)$ is a finite set by Mordell-Faltings' theorem. Thus $X(K)_{per,f}$ is a finite set. □

Proposition 5.5. *Let M be a number field. Let X be a smooth projective surface defined over M with the Kodaira dimension* 1*. We assume that X carries an elliptic fibration $\pi : X \to B$ with at most multiple singular fibers of the type ${}_mI_0$ in the sense of Kodaira, where B is a smooth projective curve of genus* 0 *or* 1*. Then X is periodically finite.*

Proof. Let $f : X \to X$ be a surjective morphism with $\deg f \ge 2$. Since X has a unique structure of an elliptic fibration up to isomorphisms, there is an automorphism $g : B \to B$ with $\pi \circ f = g \circ \pi$. Let K be a sufficiently large number field such that X, B, f, π, g are all defined over K.

Case 1 Suppose that for any $k \ge 1$, $g^{\circ k}$ is not the identity morphism. In this case, the genus of B is 0 or 1. Let us set

$$S = \{b \in B(\overline{\mathbb{Q}}) \mid g^{\circ k}(b) = b \text{ for some } k \ge 1\}.$$

We claim that S consists at most two points. Indeed, suppose S contains three points $b_1, b_2, b_3 \in B(\overline{\mathbb{Q}})$ such that $g^{\circ k_i}(b_i) = b_i$ for $i = 1, 2, 3$. Then for $k = k_1k_2k_3$ we get $g^{\circ k}(b_i) = b_i$ for $i = 1, 2, 3$. Since B is $\mathbb{P}^1$ or an elliptic curve, this shows that $g^{\circ k}$ is the identity morphism, which contradicts our assumption of Case 1.

We take l such that $g^{\circ l}(b) = b$ for any $b \in S$. Now we prove the finiteness of $X(K)_{per,f}$ by showing the finiteness of $\varprojlim_{f^{\circ l}} X(K)$ (cf., Lemma 2.2 and Lemma 2.3). Let $(x_n)_{n=0}^{\infty}$ be an element of $\varprojlim_{f^{\circ l}} X(K)$. Since $\pi(x_n)$ belongs to

S, x_n are all contained in the fiber $X_{\pi(x_0)}$. Since $f^{\circ l}$ is an étale morphism (cf., [4, Theorem 11.7]), $\varprojlim_{f^{\circ l}|(X_b)_{red}}(X_b)_{red}(K)$ is a finite set for $b \in S$ by Lemma 3.2. Using the finiteness of S, we obtain the finiteness of $\varprojlim_{f^{\circ l}} X(K)$.

Case 2 Suppose that there is a $k \geq 1$ such that $g^{\circ k}$ is the identity morphism. To prove the finiteness of $X(K)_{per,f}$, we may (and will) assume by interchanging f with $f^{\circ k}$ that g is the identity morphism (cf., Lemma 2.3).

Now by re-taking sufficiently large K, we show that X satisfies all the assumptions of Lemma 5.4. Indeed, a similar argument of the proof of Lemma 5.3 yields the assumption (A); The assumption (B) is a consequence of Lemma 5.3. Moreover, if we take an ample line bundle L on X, then L satisfies the assumption (C), because the fiber is one-dimensional. Thus by Lemma 5.4, $X(K)_{per,f}$ is a finite set. □

Combining all the results of this section, we obtain the following theorem.

Theorem 5.6. *Let X be a smooth projective surface with the non-negative Kodaira dimension such that X is defined over a number field. Then X is* not *periodically finite if and only if X is one of the following types;*

(i) *X is an abelian surface which is* not *simple, or*
(ii) *X is a hyperelliptic surface.*

6. Finitely generated fields over $\mathbb{Q}$

In this section, we work over a finitely generated field over $\mathbb{Q}$. A. Moriwaki has recently constructed the theory of height functions over a finitely generated field over $\mathbb{Q}$. We first recall a part of his theory. We refer to [7] for details.

Let K be a finitely generated field over $\mathbb{Q}$ with $\operatorname{tr.deg}_{\mathbb{Q}}(K) = d$. Let B be a normal variety which is projective and flat over $\mathbb{Z}$ such that the field of rational functions of B is K. Let $\overline{H} = (H, h_H)$ be a nef C^∞-hermitian line bundle on B, i.e., H is a line bundle on B and h_H is a C^∞-hermitian line bundle such that for any curve on C on B, $\widehat{\deg}\left(\widehat{\mathrm{c}}_1(\overline{H}|_C)\right) \geq 0$ (in the sense of the Arakelov geometry) and that the Chern form $\mathrm{c}_1(\overline{H})$ is semi-positive. There exist many such $\overline{B} = (B, \overline{H})$. We pick up a $\overline{B}$ and fix it in the following.

Now, for a point $x \in \mathbb{P}^n(\overline{K})$, let us define $h^{\overline{B}}(x)$ to be

$$h^{\overline{B}}(x) = \sum_{\Gamma} \log\left(\max_{1\leq i\leq n}\{-\operatorname{ord}_\Gamma(x_i)\}\,\widehat{\deg}\left(\widehat{\mathrm{c}}_1(\overline{H}|_\Gamma)^d\right)\right) + \int_{B(\mathbb{C})} \log\left(\max_{1\leq i\leq n}\{|x_i|\}\right)\mathrm{c}_1(\overline{H})^d,$$

where $x = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(K')$ is its coordinate over a sufficiently large extension field K' of K, and Γ runs through all prime divisors on B. This gives rise to a function $h^{\overline{B}} : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$.

Now let X be a projective variety defined over K, $\phi : X \to \mathbb{P}^n$ a morphism over K. For a point $x \in X(\overline{K})$, we define the height of x with respect to ϕ, denoted by $h_{\phi}^{\overline{B}}(x)$, to be $h_{\phi}^{\overline{B}}(x) = h(\phi(x))$.

Then the following theorem holds as is the number field case (cf., [7, §3–§4]).

Theorem 6.1. *For every line bundle L on a projective variety X defined over K, there exists a unique function $h_L^{\overline{B}} : X(\overline{K}) \to \mathbb{R}$ modulo bounded functions with the following property;*

(i) *For any two line bundles L_1, L_2, $h_{L_1 \otimes L_2}^{\overline{B}} = h_{L_1}^{\overline{B}} + h_{L_2}^{\overline{B}} + O(1)$.*

(ii) *If $f : X \to Y$ is a morphism of projective varieties over K, then $h_{f^*(L)}^{\overline{B}} = f^*(h_L^{\overline{B}}) + O(1)$.*

(iii) *If $\phi : X \to \mathbb{P}^n$ is a morphism over K, then $h_{\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))}^{\overline{B}} = h_{\phi}^{\overline{B}} + O(1)$.*

Moreover the following properties hold.

(a) (*positiveness*) *If we denote* $\mathrm{Supp}\left(\mathrm{Coker}(H^0(X,L) \otimes \mathcal{O}_X) \to L\right)$ *by* $\mathrm{Bs}(L)$, *then $h_L^{\overline{B}}$ is bounded below on $(X \setminus \mathrm{Bs}(L))(\overline{K})$.*

(b) (*Northcott*) *Assume L is ample. Then for any $e \geq 1$ and $M \geq 0$,*

$$\{x \in X(\overline{K}) \mid h_L^{\overline{B}}(x) \leq M, \quad [K(x) : K] \leq e\}$$

is a finite set.

Aside from the Northcott finite theorem, we used Mordell-Faltings' theorem (cf., Lemma 5.4). It is known that this is also true for a finitely generated field over $\mathbb{Q}$ (cf., [3, Chapter VI]).

Now it is clear that all the results before this section also hold for a finitely generated field over $\mathbb{Q}$.

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