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A REMARK ON: LOWER BOUNDS FOR EIGENVALUES OF HYPERSURFACE DIRAC OPERATORS

Xiao Zhang

Let N be an $(n+1)$-dimensional Riemannian manifold and M be an n-dimensional spin hypersurface in N. Let S be the hypersurface spinor bundle of M and $\widetilde{D}$ be the hypersurface Dirac operator of M. Denote R and H as the scalar curvature and the mean curvature of M respectively. Suppose e^0 is the unit normal covector of M. Then $\widetilde{D} = D + \frac{H}{2}e^0$, $\widetilde{D}^* = D - \frac{H}{2}e^0$. In [Z], we establish the following lower bound estimate for eigenvalue of operator $\widetilde{D}^*\widetilde{D}$.

Theorem 3.1 [Z]. *Let $M \subset N$ be a compact spin hypersurface, and λ be the eigenvalue of $\widetilde{D}^*\widetilde{D}$. Then*

$$\lambda \geq \frac{1}{4} \sup_a \inf_M \left(\frac{R}{1 + na^2 - 2a} - \frac{(n-1)H^2}{(1-na)^2} \right), \tag{1}$$

where a is any real number, $a \neq \frac{1}{n}$ if $H \neq 0$. If λ achieves its minimum, M must have constant Ricci and mean curvatures,

$$R_{ij} = \frac{(n-1)(1+na_0^2 - 2a_0)^2}{(1-na_0)^4} H^2 \delta_{ij}, \tag{2}$$

with eigenvalue $\lambda = \frac{(n-1)^2}{4(1-na_0)^4} H^2$, where a_0 is chosen such that the right side of (1) achieves its maximum.

Denote $x = (1-na)^2$. Then (1) becomes

$$\lambda \geq \frac{1}{4} \sup_{x \in R^+} \inf_M \left(\frac{nR}{x+n-1} - \frac{(n-1)H^2}{x} \right). \tag{3}$$

Remark 1. *We observe that, from the proof of Theorem 3.1 in* [Z], *a or x, in fact, can be chosen as a real function. Therefore, by choosing a special a or x, Theorem 3.1 will be replaced by the one (Theorem 3.1' below) in a much more precise form which is nearly optimal (see Remark 2 below).*

We assume $nR > (n-1)H^2$ since, otherwise, the right-hand side of (3) is negative and the estimate of eigenvalue is meaningless.

Now we can prove the following result.

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Theorem 3.1'. *Let $M \subset N$ be a compact spin hypersurface, and λ be the eigenvalue of $\widetilde{D}^*\widetilde{D}$. Suppose $nR > (n-1)H^2$. Then*

$$\lambda \geq \frac{1}{4}\inf_M \left(\sqrt{\frac{n}{n-1}R} - |H|\right)^2. \tag{4}$$

If λ achieves its minimum, M must have constant Ricci and mean curvatures.

Proof. Define a modified covariant derivative on $\Gamma(S)$ by

$$L_i = \nabla_i + \frac{1-a}{2(1-na)}He^0e^i + ae^i\widetilde{D},$$

where $(1-na)^2 = \frac{(n-1)|H|}{\sqrt{\frac{n}{n-1}R}-|H|}$. Same as [Z], we obtain

$$\int_M |\widetilde{D}\phi|^2 = \int_M \frac{|L\phi|^2}{1+na^2-2a} + \frac{1}{4}(\sqrt{\frac{n}{n-1}R} - |H|)^2|\phi|^2. \tag{5}$$

Therefore (4) is proved. When λ achieves its minimum, we know, from [Z], that $\widetilde{H} = \frac{1+na^2-2a}{(1-na)^2}H$ is constant. And the Ricci curvature $R_{ij} = (n-1)\widetilde{H}^2\delta_{ij}$ is constant also. On the other hand, (5) implies that $\sqrt{\frac{n}{n-1}R} - |H|$ is constant. Therefore the mean curvature H is constant. The proof of theorem is complete. □

Remark 2. *Note that, on R^+, the real function*

$$f(x) = \frac{C^2}{x+n-1} - \frac{1}{x},$$

where constant $C > 1$, achieves its maximum $\frac{(C-1)^2}{n-1}$ at point $x = \frac{n-1}{C-1}$. Therefore, if there exists constant $C > 1$ such that $nR \geq C^2(n-1)H^2$, then (4) is optimal.

References

[Z] X. Zhang, *Lower bounds for eigenvalues of hypersurface Dirac operators*, Math. Res. Lett. **5** (1998), 199–210.

Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China
E-mail address: xzhang@math08.math.ac.cn