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WEIGHT REPRESENTATIONS OF THE POLYNOMIAL CARTAN TYPE LIE ALGEBRAS W_n AND $\bar{S}_n$

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To the memory of Moshe Flato (1937–1998)

Abstract. We give an explicit description of the support of an arbitrary irreducible weight module of the infinite-dimensional Lie algebra of polynomial vector fields W_n, as well as of its subalgebra $\bar{S}_n$ of vector fields with constant divergence.

Introduction

The Lie algebras of Cartan type are certain infinite-dimensional simple Lie algebras of vector fields with formal power series coefficients. They are the Lie algebras of "infinite Lie groups" which arose in the work of Sophus Lie around 1870 and were further studied by Elie Cartan in 1904–1908. The work of Lie and Cartan has been continued and explained in modern terms by I.M. Singer and S. Sternberg, [?SS], in 1964. The general theory of representations of the Cartan type infinite-dimensional Lie algebras was initiated only in 1973 when A.N. Rudakov began the study of topological irreducible representations of these Lie algebras, [?R1, ?R2]. Rudakov's main result, roughly speaking, is that all irreducible representations which satisfy a natural continuity condition can be described explicitly as induced modules or quotients of induced modules.

In this paper we take a different approach. We restrict ourselves to the polynomial Cartan type Lie algebras W_n and $\bar{S}_n$ (see their definition in section 3) and, without imposing any continuity condition, we investigate arbitrary irreducible weight representations of W_n and $\bar{S}_n$ in the spirit of the recent papers [?DMP, ?PS, ?DP2]. Our main result is an explicit description of all possible sets of weights of such representations, i.e., a description of the supports of all irreducible weight representations of W_n and $\bar{S}_n$. Although a relatively small part of these modules belongs to Rudakov's class, the answer turns out to be surprisingly simple and resembles very much the answer for the finite-dimensional Lie algebra sl_{n+1}, see [?DMP]. A key technical feature that makes our description possible (and which is not shared by the polynomial Hamiltonian and contact Lie algebras) is that any parabolic subalgebra of sl_{n+1} has a certain canonical extension to a parabolic subalgebra of W_n or $\bar{S}_n$ (see Lemma ?lm22? below). The main problem in the theory of weight W_n- and $\bar{S}_n$-modules now is to gain a better understanding of each class of irreducible weight modules with fixed

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support. O. Mathieu has informed us that his method from [?M], together with the result of the present paper, should lead to a classification of all irreducible weight W_n- and $\bar{S}_n$-modules with finite-dimensional weight spaces.

Section 1 is devoted to preliminaries: Lie algebras admitting a root decomposition with respect to a finite-dimensional Cartan subalgebra, parabolically induced weight modules over such algebras, shadow decompositions for reductive Lie algebras. In section 2 we introduce our main object of study, the Lie algebras W_n and $\bar{S}_n$, and discuss some properties of their root decomposition. We then state our main theorem and give examples. The proof of the theorem is presented in section 3. The reader could go directly to sections 2 and 3 and then use section 1 as reference only.

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0. Conventions

The ground field is $\mathbb{C}$. If A and B are two sets, $A\backslash B$ is by definition the set $\{a \in A \mid a \notin B\}$. The superscript * always stands for dual space. We denote by $\mathbb{R}_+$ (respectively by $\mathbb{R}_-$) the set of non-negative (resp. non-positive) real numbers; $\mathbb{Z}_\pm \stackrel{\text{def}}{=} \mathbb{Z} \cap \mathbb{R}_\pm$. Linear span is denoted by $\langle\ \rangle_{\mathbb{R}}$, $\langle\ \rangle_{\mathbb{R}_+}$, $\langle\ \rangle_{\mathbb{Z}_+}$etc., the subscript indicating the coefficients. We use the term cone as a synonym for an additive subset of a vector space. If $\mathfrak{g}$ is a Lie algebra, the terms a "representation of $\mathfrak{g}$" and a "$\mathfrak{g}$-module" are synonyms. We assume that a $\mathfrak{g}$-module is automatically non-zero; a trivial $\mathfrak{g}$-module is a 1-dimensional vector space with the zero action of $\mathfrak{g}$.

1. Lie algebras with root decomposition and generalized weight modules

Let $\mathfrak{g}$ denote a (complex) Lie algebra with generalized root decomposition, i.e., with a fixed self-normalizing nilpotent Lie subalgebra, called a *Cartan subalgebra*, such that as an $\mathfrak{h}$-module $\mathfrak{g}$ decomposes as $\mathfrak{h} \oplus (\oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha)$, where

$$\mathfrak{g}^\alpha \stackrel{\text{def}}{=} \{g \in \mathfrak{g} \mid (\mathrm{ad}\,(h) - \alpha\,(h))^N\,(g) = 0 \text{ for every } h \in \mathfrak{h} \text{ and some } N > 0\}.$$

The set of non-zero linear functions $\alpha \in \mathfrak{h}^*$ with $\mathfrak{g}^\alpha \neq 0$ is the set of *roots* Δ *of* $\mathfrak{g}$. In what follows we assume $\mathfrak{h}$ to be finite-dimensional. Some results for the case when $\mathfrak{h}$ is infinite-dimensional see in [?DP2]. A $\mathfrak{g}$-module M is a *generalized weight* $\mathfrak{g}$*-module* iff as an $\mathfrak{h}$-module M decomposes as $\oplus_{\mu \in \mathfrak{h}^*} M^\mu$, where

$$M^\mu \stackrel{\text{def}}{=} \{m \in M \mid (h - \mu\,(h))^N \cdot m = 0 \text{ for every } h \in \mathfrak{h} \text{ and some } N > 0\}.$$

The set of all linear functions $\mu \in \mathfrak{h}^*$ such that $M^\mu \neq 0$ is the *support* of M and will be denoted by $\operatorname{supp} M$. It is not difficult to verify (using the fact that $\dim \mathfrak{h} < \infty$) that the requirement for M to be a generalized weight module is equivalent to the requirement that $\mathfrak{h}$ act locally finitely on M. Furthermore, obviously, $\mathfrak{g}$ has a root decomposition iff the adjoint module is a generalized weight $\mathfrak{g}$-module, i.e., iff the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is locally finite.

A generalized weight $\mathfrak{g}$-module M is a *weight* $\mathfrak{g}$*-module* iff $\mathfrak{h}$ acts semi-simply on M, i.e., iff $h \cdot m = \mu(h)\, m$ for any $m \in M^\mu$ and any $h \in \mathfrak{h}$. We would like to recall

Proposition 1. *Let* $\dim \mathfrak{g}$ *be countable and* $\mathfrak{g}$ *be semi-simple as* $\mathfrak{h}$*-module (in particular, let* $\mathfrak{h}$ *be abelian). Then every irreducible generalized weight* $\mathfrak{g}$*-module* M *is a weight module.*

Proof. It is based on the following infinite-dimensional version of Schur's lemma.

Lemma 1. *Let* A *be an associative* $\mathbb{C}$*-algebra of countable dimension and* B *be an irreducible* A*-module. Then* $\operatorname{End}_A(B) = \mathbb{C}$*.*

Proof of Lemma **?lm1?**. Note that $\operatorname{End}_A(B)$ is a division algebra over $\mathbb{C}$, and therefore $\operatorname{End}_A(B) \neq \mathbb{C}$ implies that $\operatorname{End}_A(B)$ contains a field isomorphic to $\mathbb{C}(x)$ (the field of rational functions of an indeterminate x). On the other hand, $\dim_{\mathbb{C}}(\operatorname{End}_A(B)) \leq \dim_{\mathbb{C}} B \leq \dim_{\mathbb{C}} A$, and thus $\dim_{\mathbb{C}}(\operatorname{End}_A(B))$ is finite or countable. But $\dim_{\mathbb{C}} \mathbb{C}(x)$ is uncountable. Hence $\operatorname{End}_A(B) = \mathbb{C}$. □

To prove Proposition **?prop1?** now, let U^0 denote the subalgebra in $U(\mathfrak{g})$ generated by all monomials of weight zero with respect to the adjoint action of $\mathfrak{h}$. Since $\mathfrak{h}$ acts semi-simply on $\mathfrak{g}$, the subalgebra $S(\mathfrak{h})$ belongs to the center of U^0. Furthermore the irreducibility of M implies that M^μ is an irreducible U^0-module for any $\mu \in \operatorname{supp} M$. By Lemma **?lm1?** (applied to $A = U^0$ and $B = M^\mu$), $S(\mathfrak{h})$ acts via scalars on M^μ, i.e., M^μ, and therefore also M, is a semi-simple $\mathfrak{h}$-module.

□

Recall next, see [**?DP3**], that a subdivision $\Delta = \Delta^+ \sqcup \Delta^-$ is called a *triangular decomposition* iff $\langle -\Delta^+ \sqcup \Delta^- \rangle_{\mathbb{R}_+} \cap \langle \Delta^+ \sqcup -\Delta^- \rangle_{\mathbb{R}_+} = \{0\}$. Every triangular decomposition of Δ is determined by some (in general not unique) maximal flag of real vector subspaces in $\langle \Delta \rangle_{\mathbb{R}}$,

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{\dim \langle \Delta \rangle_{\mathbb{R}}} = \langle \Delta \rangle_{\mathbb{R}},$$

and by a choice of labeling by $+$ and $-$ of the two connected components of $F_i \backslash F_{i-1}$ for each $i = 1, \ldots, n$, via the formula

$$\Delta^\pm = \sqcup_{i=1}^{\dim \langle \Delta \rangle_{\mathbb{R}}} \left((F_i \backslash F_{i-1})^\pm \cap \Delta \right).$$

A *generalized triangular decomposition*, or a *parabolic decomposition*, of Δ is by definition a subdivision

$$\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$$

such that, if $p\colon \langle\Delta\rangle_{\mathbb{R}} \to \langle\Delta\rangle_{\mathbb{R}} / \langle\Delta^0\rangle_{\mathbb{R}}$ is the natural projection, then $p(\Delta^+) \cap p(\Delta^-) = \varnothing$, $0 \notin p(\Delta^\pm)$ and $p(\Delta\backslash\Delta^0) = p(\Delta^+) \sqcup p(\Delta^-)$ is a triangular decomposition of $p(\Delta\backslash\Delta^0)$. Clearly, a parabolic decomposition is determined by a (non-unique) flag

$$G_0 \subset G_1 \subset \cdots \subset G_k = \langle\Delta\rangle_{\mathbb{R}}, \ \dim G_i/G_{i-1} = 1, \ k \leq \dim\langle\Delta\rangle_{\mathbb{R}},$$

together with a labeling by $+$ and $-$ of the two connected components of $G_i\backslash G_{i-1}$, via the formulae

$$\Delta^0 = \Delta \cap G_0,$$

$$\Delta^\pm = \sqcup_{i=1}^k \left((G_i\backslash G_{i-1})^\pm \cap \Delta\right).$$

A more general statement see in the Appendix of [?DP2].

If $\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$ is a parabolic decomposition, let $\mathfrak{g}^0$ and $\mathfrak{g}^\pm$ be the following Lie subalgebras:

$$\mathfrak{g}^0 \stackrel{\text{def}}{=} \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta^0}\mathfrak{g}^\alpha), \mathfrak{g}^\pm \stackrel{\text{def}}{=} \oplus_{\alpha\in\Delta^\pm}\mathfrak{g}^\alpha.$$

A Lie subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is called *parabolic* if, as a vector space, $\mathfrak{p}$ equals $\mathfrak{g}^0 \oplus \mathfrak{g}^+$ for some parabolic decomposition.

In this paper we will consider parabolic subalgebras associated with general parabolic decompositions in which Δ^- does not necessarily equal $-\Delta^+$. The following lemma gives some rough but important information about the irreducible quotient of a corresponding generalized Verma module.

Lemma 2. *Let $\mathfrak{p}$ be the parabolic subalgebra associated with a given parabolic decomposition $\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$.*

(*a*) *Let $M^{\mathfrak{p}}$ be an irreducible generalized weight $\mathfrak{p}$-module. Then the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{p}}$ has a unique irreducible quotient M and*

$$\operatorname{supp} M \subset \mu + (\langle\Delta^0 \sqcup \Delta^-\rangle_{\mathbb{Z}_+} \cap \langle\Delta^0 \sqcup \Delta^+\rangle_{\mathbb{Z}_-})$$

for any $\mu \in \operatorname{supp} M^{\mathfrak{p}}$.

(*b*) *Let M be an irreducible generalized weight $\mathfrak{g}$-module and $\lambda \in \operatorname{supp} M$ be such that $\lambda + \alpha \notin \operatorname{supp} M$ for any $\alpha \in \Delta^+$. Then M is the unique quotient of the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{p}}$, $M^{\mathfrak{p}} \stackrel{def}{=} U(\mathfrak{p}) \cdot M_\lambda$ being an irreducible $\mathfrak{p}$-module.*

Proof. (*a*) The existence of a unique quotient follows from the fact that any proper submodule of the induced module is a generalized weight module whose support does not intersect $\operatorname{supp} M^{\mathfrak{p}}$. Note next that $\mathfrak{g}^+ \cdot M^{\mathfrak{p}} = 0$ by the irreducibility of $M^{\mathfrak{p}}$ as a $\mathfrak{p}$-module, and thus $M^{\mathfrak{p}}$ is irreducible as a $\mathfrak{g}^0$-module. This implies

$$\operatorname{supp} M^{\mathfrak{p}} \subseteq \mu + (\langle\Delta^0\rangle_{\mathbb{Z}_+} \cap \langle\Delta^0\rangle_{\mathbb{Z}_-}).$$

Furthermore, as M is irreducible, for any $\mu' \in \operatorname{supp} M$ one can find $\beta_1, \dots, \beta_r \in \Delta^+$ such that $\mu' + \beta_1 + \cdots + \beta_r \in \operatorname{supp} M^{\mathfrak{p}}$. Therefore

$$\operatorname{supp} M \subseteq \operatorname{supp} M^{\mathfrak{p}} + \left\langle \Delta^+ \right\rangle_{\mathbb{Z}_-}.$$

On the other hand, obviously,

$$\operatorname{supp} M \subseteq \operatorname{supp} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{p}} \subseteq \operatorname{supp} M^{\mathfrak{p}} + \left\langle \Delta^- \right\rangle_{\mathbb{Z}_+}.$$

Hence

$$\operatorname{supp} M \subseteq \mu + (\left\langle \Delta^0 \sqcup \Delta^- \right\rangle_{\mathbb{Z}_+} \cap \left\langle \Delta^0 \sqcup \Delta^+ \right\rangle_{\mathbb{Z}_-}).$$

The proof of (b) is left to the reader. □

We need to introduce also the notion of a shadow decomposition. Let now $\mathfrak{g}$ be a reductive Lie algebra with root system Δ and M be an irreducible weight $\mathfrak{g}$-module. Fix a point $\mu \in \operatorname{supp} M$ and for any $\alpha \in \Delta$ consider the set $n_\alpha^\mu \stackrel{\text{def}}{=} \{q \in \mathbb{R} \mid \mu + q\alpha \in \operatorname{supp} M\}$. There are four possible types of sets n_α^μ: bounded in both directions, unbounded in both directions, bounded from above and unbounded from below, and unbounded from above but bounded from below. The main point is, see [?DMP], that the type of n_α^μ depends only on α and not on μ, and therefore the module M itself determines a decomposition of Δ into four mutually disjoint subsets:

$$\Delta = \Delta_M^+ \sqcup \Delta_M^I \sqcup \Delta_M^F \sqcup \Delta_M^-, \tag{1}$$

where

$$\Delta_M^+ \stackrel{\text{def}}{=} \{\alpha \in \Delta \mid n_\alpha^\mu \text{ is bounded only from above}\},$$
$$\Delta_M^- \stackrel{\text{def}}{=} \{\alpha \in \Delta \mid n_\alpha^\mu \text{ is bounded only from below}\},$$
$$\Delta_M^F \stackrel{\text{def}}{=} \{\alpha \in \Delta \mid n_\alpha^\mu \text{ is bounded in both directions}\},$$
$$\Delta_M^I \stackrel{\text{def}}{=} \{\alpha \in \Delta \mid n_\alpha^\mu \text{ is unbounded in both directions}\}.$$

We call (?equ1?) the *shadow of* M and its existence means simply that the type of n_α^μ does not depend on a choice of $\mu \in \operatorname{supp} M$.

Any subdivision of the form (?equ1?) can be characterized abstractly by the property that, for $\Delta_M^0 \stackrel{\text{def}}{=} \Delta_M^I \sqcup \Delta_M^F$, $\Delta = \Delta_M^+ \sqcup \Delta_M^0 \sqcup \Delta_M^-$ is a parabolic decomposition and all roots of Δ_M^I are orthogonal to all roots of Δ_M^F. A subdivision $\Delta = \Delta^+ \sqcup \Delta^I \sqcup \Delta^F \sqcup \Delta^-$ satisfying this property is by definition a *shadow decomposition of* Δ. Moreover, the Fernando-Futorny parabolic induction theorem, see [?DMP], implies that the shadow of M determines $\operatorname{supp} M$ up to the support of a finite dimensional $\mathfrak{g}^F$-module, where $\mathfrak{g}^F \stackrel{\text{def}}{=} \mathfrak{h} \oplus \left(\oplus_{\alpha \in \Delta_M^F} \mathfrak{g}^\alpha \right)$. More precisely, $\operatorname{supp} M = \operatorname{supp} M^F + \left\langle \Delta_M^I \sqcup \Delta_M^- \right\rangle_{\mathbb{Z}_+}$ for some finite-dimensional irreducible $\mathfrak{g}^F$-module M^F.

In our study of weight modules over W_n and $\bar{S}_n$ (the Lie algebras W_n and $\bar{S}_n$ are introduced in the next section) we will use certain shadow decompositions of the root system A_n, and furthermore our main result will imply that, if M is an irreducible weight module over W_n or $\bar{S}_n$, then M itself has a well-defined shadow.

2. The Lie algebras W_n and $\bar{S}_n$ and the main theorem

Let $P_n = \mathbb{C}[x_1, \dots, x_n]$ be the ring of polynomials in the indeterminates $x_1, \dots, x_n$. Then W_n is by definition the Lie algebra of derivations of P_n, i.e.,

$$W_n \stackrel{\text{def}}{=} Der\,\mathbb{C}[x_1, \dots, x_n] = \left\{ \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i} \mid p_i \in P_n \right\},$$

and $\bar{S}_n \subset W_n$ is its Lie subalgebra of all derivations with constant divergence, i.e.,

$$\bar{S}_n \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i} \mid \sum_{i=1}^{n} \frac{\partial^2 p_i}{\partial x_i \partial x_j} = 0 \text{ for all } j = 1, \dots, n \right\}.$$

It is known that W_n is simple for $n \geq 1$, and that $S_n \stackrel{\text{def}}{=} \left[\bar{S}_n, \bar{S}_n\right]$ is a simple ideal of codimension 1 in $\bar{S}_n$ for $n \geq 2$.

Throughout the rest of this paper $\mathfrak{g}$ will denote W_n for $n \geq 1$ or $\bar{S}_n$ for $n \geq 2$. Fix the Cartan subalgebra $\mathfrak{h} = \langle \{x_i \frac{\partial}{\partial x_i} \mid i = 1, \dots, n\} \rangle_{\mathbb{C}}$ of $\mathfrak{g}$ and set $\mathfrak{h}' \stackrel{\text{def}}{=} \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$. Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis in $\mathfrak{h}^*$ dual to the basis $x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n}$ in $\mathfrak{h}$. We introduce a symmetric bilinear form on $\mathfrak{h}^*$ by requiring $\varepsilon_1, \dots, \varepsilon_n$ to be orthonormal. The set Δ of roots of $\mathfrak{g}$ is the same for W_n and $\bar{S}_n$ and consists of the vectors $\sum_{j=1}^{n} m_j \varepsilon_j$ and $-\varepsilon_i + \sum_{j \neq i} m_j \varepsilon_j$, where all $m_j \in \mathbb{Z}_+$ and $i = 1, \dots, n$. In what follows we always assume that $m_j \in \mathbb{Z}_+$.

Consider $\Delta' \stackrel{\text{def}}{=} \Delta \cap -\Delta = \{\pm\varepsilon_i, \varepsilon_i - \varepsilon_j \mid i, j \leq n, i \neq j\}$. Clearly, Δ' is a root system of type A_n. When $\mathfrak{g} = W_n$ there is a subalgebra isomorphic to sl_{n+1} in $\mathfrak{g}$ (arising from the infinitesimal action of the action of PSL_{n+1} on $\mathbb{C}P^n$) , but when $\mathfrak{g} = \bar{S}_n$ the root subsystem Δ' does not correspond to a subalgebra of $\mathfrak{g}$. Let $\Delta'' \stackrel{\text{def}}{=} \{\varepsilon_i - \varepsilon_j \mid i, j \leq n, i \neq j\}$. In both cases the weight spaces $\mathfrak{g}^\alpha$ for all $\alpha \in \Delta''$ generate a subalgebra of $\mathfrak{g}$ isomorphic to sl_n.

In the following proposition we summarize some properties of $\mathfrak{g}$ whose proof is straightforward.

Proposition 2. (*a*) *The Cartan subalgebra* $\mathfrak{h}$ *acts semi-simply on* $\mathfrak{g}$*. Therefore, by Proposition* **?prop1?***, every irreducible generalized weight* $\mathfrak{g}$*-module is a weight module.*

(*b*) *For any nonproportional* $\alpha, \beta \in \Delta$ *we have* $\mathfrak{g}^{\alpha+\beta} = \left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right]$.

(*c*) *For any* $\alpha \in \Delta'' \sqcup \{-\varepsilon_i \mid 1 \leq i \leq n\}$ *we have* $\dim \mathfrak{g}^\alpha = 1$*, and the adjoint action of* $\mathfrak{g}^\alpha$ *on* $\mathfrak{g}$ *is locally nilpotent.*

(*d*) *For every* $i \leq n$ *we have* $[\mathfrak{g}^{-\varepsilon_i}, \mathfrak{g}^{\varepsilon_i}] = \mathfrak{h}'$*, and the Lie subalgebra* $\mathfrak{g}(\mathbb{Z}\varepsilon_i) \overset{def}{=} \mathfrak{g}^{-\varepsilon_i} \oplus \mathfrak{h} \oplus \mathfrak{g}^{\varepsilon_i} \oplus \mathfrak{g}^{2\varepsilon_i} \oplus \dots$ *contains a subalgebra isomorphic to* W_1. □

The following property is not quite obvious and is very important in our description of the supports of irreducible weight $\mathfrak{g}$-modules.

Lemma 3. *Let* $\Delta' = (\Delta')^+ \sqcup (\Delta')^0 \sqcup (\Delta')^-$ *be a parabolic decomposition of* Δ'. *Set* $\Delta^0 \overset{def}{=} \Delta \cap \langle (\Delta')^0 \rangle_{\mathbb{Z}}$, $\Delta^+ \overset{def}{=} (\Delta \cap \langle (\Delta')^+ \sqcup (\Delta')^0 \rangle_{\mathbb{Z}_+}) \backslash \Delta^0$, $\Delta^- \overset{def}{=} \Delta \backslash \left(\Delta^0 \sqcup \Delta^+\right)$. *Then* $\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$ *is a parabolic decomposition of* Δ.

Proof. We start by observing that any sequence $\gamma_1, \dots, \gamma_k$ of linearly independent vectors in $\langle \Delta \rangle_{\mathbb{R}}$ determines a parabolic decomposition of Δ, as well as a parabolic decomposition of Δ'. Indeed, define the following cones in $\langle \Delta \rangle_{\mathbb{R}}$:

$$C^0(\gamma_1, \dots, \gamma_k) = \{\xi \in \langle \Delta \rangle_{\mathbb{R}} \mid (\gamma_1, \xi) = \dots = (\gamma_k, \xi) = 0\},$$

$$C^{\pm}(\gamma_1, \dots, \gamma_k) = \\ \left\{\xi \in \langle \Delta \rangle_{\mathbb{R}} \mid (\gamma_1, \xi) = \dots = (\gamma_{i-1}, \xi) = 0, (\gamma_i, \xi) >_{(<)} 0 \text{ for some } i \leq k\right\}.$$

Then $\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$, where

$$\Delta^0 \overset{\text{def}}{=} C^0(\gamma_1, \dots, \gamma_k) \cap \Delta, \quad \Delta^{\pm} \overset{\text{def}}{=} C^{\pm}(\gamma_1, \dots, \gamma_k) \cap \Delta,$$

is a parabolic decomposition of Δ, and respectively $\Delta' = (\Delta')^+ \sqcup (\Delta')^0 \sqcup (\Delta')^-$ is a parabolic decomposition of Δ', where $(\Delta')^0 \overset{\text{def}}{=} \Delta' \cap \Delta^0$, $(\Delta')^{\pm} \overset{\text{def}}{=} \Delta' \cap \Delta^{\pm}$.

Fix now a parabolic decomposition of Δ', $\Delta' = (\Delta')^+ \sqcup (\Delta')^0 \sqcup (\Delta')^-$. Using the fact that any maximal parabolic subalgebra of a finite-dimensional Lie algebra corresponds to a fundamental weight, we can check that our fixed parabolic decomposition is determined by a sequence of the form $\omega_1^-, \dots, \omega_k^-, \omega_1^+, \dots, \omega_l^+$, where

$$\omega_r^- = -\sum_{s \in S_r^-} \varepsilon_s \text{ for a decreasing sequence of sets } S_1^- \supset \dots \supset S_k^-,$$

$$\omega_r^+ = \sum_{s \in S_r^+} \varepsilon_s \text{ for an increasing sequence of sets } S_1^+ \subset \dots \subset S_l^+,$$

and $S_l^+ \cap S_1^- = \emptyset$. All it remains is to verify that the parabolic decomposition of Δ determined by the sequence $\omega_1^-, \dots, \omega_k^-, \omega_1^+, \dots, \omega_l^+$ coincides with the decomposition of Δ defined in the Lemma. For this it suffices to check that any $\alpha \in \Delta^+ \sqcup \Delta^0$ can be written as a sum $\beta_1 + \dots + \beta_p$ for some $\beta_j \in (\Delta')^0 \sqcup (\Delta')^+$. Note that, by definition, either $\left(\omega_1^-, \alpha\right) > 0$ or $\left(\omega_1^-, \alpha\right) = 0$. In the former case

$$\alpha = -\varepsilon_i + \sum_{j \notin S_1^-} m_j \varepsilon_j$$

for some i. Since $-\varepsilon_i \in \Delta^+$ and $\varepsilon_j \in (\Delta')^+ \sqcup (\Delta')^0$ for all j, in this case the Lemma is proved.

The latter case splits into two:

(1) $(\omega_1^-, \alpha) = \ldots = (\omega_{r-1}^-, \alpha) = 0$, $(\omega_r^-, \alpha) > 0$ for some $r \leq k$. Then $\alpha = \varepsilon_i - \varepsilon_{i'} + \sum_{j \notin S_1^-} m_j \varepsilon_j$ for some $i' \in S_{r-1}^- \backslash S_r^-$ and some $i \in S_r^-$. Since $\varepsilon_i - \varepsilon_{i'} \in (\Delta')^+$ and $\varepsilon_j \in (\Delta')^+ \sqcup (\Delta')^0$, this case is also settled.

(2) $(\omega_1^-, \alpha) = \ldots = (\omega_k^-, \alpha) = 0$. Then either $\alpha = \varepsilon_i - \varepsilon_{i'} + \sum_{j \notin S_1^-} m_j \varepsilon_j$ with $i, i' \in S_{r-1}^- \backslash S_r^-$ (here $\varepsilon_i - \varepsilon_{i'} \in (\Delta')^0$, $\varepsilon_j \in (\Delta')^+ \sqcup (\Delta')^0$), or $\alpha = \sum_{j \notin S_1^-} m_j \varepsilon_j$ (here $\varepsilon_j \in (\Delta')^+ \sqcup (\Delta')^0$), or finally $\alpha = -\varepsilon_i + \sum_{j \notin S_1^-} m_j \varepsilon_j$ for some $i \notin S_1^-$. In the first two subcases there is nothing to check. In the last subcase $(\omega_1^+, \alpha) = \ldots = (\omega_{p-1}^+, \alpha) = 0$, $(\omega_p^+, \alpha) > 0$ for some $p \leq l+1$, where, if $p = l+1$, we define S_p^+ as $\{1, \ldots, n\} \backslash S_1^-$ and ω_p^+ as $\sum_{s \in S_p^+} \varepsilon_s$. Assume first that $i \in S_{p-1}^+$. Choose $r < p$ such that $i \in S_r^+ \backslash S_{r-1}^+$. Then there is $q \in S_r^+ \backslash S_{r-1}^+$ such that $\alpha = \varepsilon_q - \varepsilon_i + \sum_{j \notin S_1^- \sqcup S_{p-1}^+} m_j \varepsilon_j$. Here all $\varepsilon_j \in (\Delta')^+ \sqcup (\Delta')^0$ and $\varepsilon_q - \varepsilon_i \in (\Delta')^0$. Let now $i \notin S_{p-1}^+$. Then one can find $q \in S_p^+ \backslash S_{p-1}^+$ with $m_q > 0$ and rewrite α as the sum $(\varepsilon_q - \varepsilon_i) + \sum_{j \notin S_1^-} m'_j \varepsilon_j$. This settles the last subcase, as $\varepsilon_q - \varepsilon_i \in (\Delta')^+ \sqcup (\Delta')^0$ for all $i \notin S_{p-1}^+$, and $\varepsilon_j \in (\Delta')^+ \sqcup (\Delta')^0$ for all $j \notin S_1^-$.

□

We say that a parabolic decomposition of Δ *is induced from a parabolic decomposition* of Δ' if it can be obtained as in Lemma **?lm22?**.

In what follows we assume that M is an irreducible weight $\mathfrak{g}$-module, unless stated otherwise. For any $\lambda \in \operatorname{supp} M$, set

$$\Gamma_\lambda \stackrel{\text{def}}{=} \langle \alpha \in \Delta' \mid \lambda + \mathbb{Z}_+ \alpha \subseteq \operatorname{supp} M \rangle_{\mathbb{Z}_+}$$

and

$$\begin{aligned}
(\Delta')_\lambda^I &\stackrel{\text{def}}{=} \{\alpha \in \Delta' \mid \alpha, -\alpha \in \Gamma_\lambda\}, \\
(\Delta')_\lambda^F &\stackrel{\text{def}}{=} \{\alpha \in \Delta' \mid \alpha, -\alpha \notin \Gamma_\lambda\}, \\
(\Delta')_\lambda^+ &\stackrel{\text{def}}{=} \{\alpha \in \Delta' \mid -\alpha \in \Gamma_\lambda, \alpha \notin \Gamma_\lambda\}, \\
(\Delta')_\lambda^- &\stackrel{\text{def}}{=} -(\Delta')_\lambda^+.
\end{aligned}$$

Clearly,

$$\Delta' = (\Delta')_\lambda^+ \sqcup (\Delta')_\lambda^I \sqcup (\Delta')_\lambda^F \sqcup (\Delta')_\lambda^- . \tag{3}$$

Furthermore, given a shadow decomposition $\Delta' = (\Delta')^+ \sqcup (\Delta')^I \sqcup (\Delta')^F \sqcup (\Delta')^-$ and a weight $\lambda \in \mathfrak{h}^*$, define the parabolic decomposition ${}_\lambda\Delta' = {}_\lambda(\Delta')^+ \sqcup$

${}_\lambda(\Delta')^0 \sqcup {}_\lambda(\Delta')^-$ by putting

$${}_\lambda(\Delta')^\pm \overset{\text{def}}{=} (\Delta')^\pm \sqcup \left\{\alpha \in (\Delta')^F \mid (\lambda,\alpha) >_{(<)} 0\right\},$$

$${}_\lambda(\Delta')^0 \overset{\text{def}}{=} (\Delta')^I \sqcup \left\{\alpha \in (\Delta')^F \mid (\lambda,\alpha) = 0\right\}.$$

We call a root $\alpha \in (\Delta')^+$ λ-*indecomposable* if it can not be decomposed as a sum $\beta+\beta'$ for some $\beta,\beta' \in (\Delta')^I \sqcup {}_\lambda(\Delta')^+$. A weight $\lambda \in \mathfrak{h}^*$ is *compatible* with the given shadow decomposition if the following two conditions hold:

(1) $(\lambda,\alpha) \in \mathbb{Z}$ for any $\alpha \in \Delta'' \cap (\Delta')^F$;

(2) $\lambda(\mathfrak{h}') \neq 0$ whenever there is at least one λ-indecomposable root $\alpha_0 \in (\Delta')^+$, and moreover $(\lambda,\alpha) \notin \mathbb{Z}_+$ for all λ-indecomposable roots $\alpha \in \Delta'' \cap (\Delta')^+$.

Finally we set $\mathfrak{g}^F \overset{\text{def}}{=} \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta''\cap(\Delta')^F}\mathfrak{g}^\alpha)$ and note that $\mathfrak{g}^F$ is a reductive subalgebra, $\mathfrak{g}^F \subseteq gl_n \subset \mathfrak{g}$. For any weight λ, compatible with the given shadow decomposition, there is a suitable set of positive roots in $(\Delta'') \cap (\Delta')^F$ such that λ is the highest weight of a finite-dimensional irreducible $\mathfrak{g}^F$-module M_λ^F.

The following theorem is our main result.

Theorem 1. *Let M be an irreducible weight $\mathfrak{g}$-module.*

(*a*) *$\Gamma_\lambda = \Gamma_\mu$ for any $\lambda,\mu \in \operatorname{supp} M$, and therefore the subdivision* (?equ2?) *does not depend on $\lambda \in \operatorname{supp} M$. We shall denote it by*

$$\Delta' = (\Delta')^+ \sqcup (\Delta')^I \sqcup (\Delta')^F \sqcup (\Delta')^-. \tag{4}$$

(*b*) *The subdivision* (?equ3?) *is a shadow decomposition of Δ', and moreover $(\Delta')^F \subseteq \Delta''$.*

(*c*) *There exists $\lambda \in \operatorname{supp} M$, such that it is compatible with* (?equ3?) *and*

$$\operatorname{supp} M = \operatorname{supp} M_\lambda^F + \langle(\Delta')^I \sqcup (\Delta')^-\rangle_{\mathbb{Z}_+}. \tag{5}$$

(*d*) *For any shadow decomposition $\Delta' = (\Delta')^+ \sqcup (\Delta')^I \sqcup (\Delta')^F \sqcup (\Delta')^-$ such that $(\Delta')^F \subseteq \Delta''$, and for any compatible weight λ, there is an irreducible weight $\mathfrak{g}$-module M whose support is given by* (?equ4?).

The proof is presented in section 3. In the rest of this section we comment on the result and discuss examples. Note, first of all, that the Theorem implies that any irreducible weight $\mathfrak{g}$-module M has a well-defined shadow. Indeed, using the explicit description of $\operatorname{supp} M$ given in claim (c), the reader will verify that for any $\beta \in \Delta$ and $\mu \in \operatorname{supp} M$ the type of $n_\beta^\mu \overset{\text{def}}{=} \{q \in \mathbb{R} \mid \mu + q\beta \in \operatorname{supp} M\}$ does not depend on $\mu \in \operatorname{supp} M$, and therefore M defines a decomposition of Δ:

$$\Delta = \Delta_M^+ \sqcup \Delta_M^I \sqcup \Delta_M^F \sqcup \Delta_M^-. \tag{6}$$

Furthermore, it is natural to ask if the decomposition (?equ5?) inherits the properties of the shadow decomposition (?equ3?), and in particular is it true that, if we set $\Delta_M^0 \overset{\text{def}}{=} \Delta_M^I \sqcup \Delta_M^F$, the decomposition $\Delta_M^+ \sqcup \Delta_M^0 \sqcup \Delta_M^-$ is a parabolic decomposition? The answer turns out to be no as for instance $\mathfrak{g}^0 \overset{\text{def}}{=} \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_M^0} \mathfrak{g}^\alpha)$ may not be a Lie subalgebra. Indeed, consider the adjoint representation of $\mathfrak{g} = W_n$, $n \geq 2$. Then $-\varepsilon_1 + 2\varepsilon_2$, $-\varepsilon_2 + 2\varepsilon_1 \in \Delta_M^F$ but $(-\varepsilon_1 + 2\varepsilon_2) + (-\varepsilon_2 + 2\varepsilon_1) = \varepsilon_1 + \varepsilon_2 \in \Delta_M^-$, and in this case $\mathfrak{g}_M^0$ is not a subalgebra.

Note also that, although the subdivision (?equ3?) is a shadow decomposition for the root system Δ' of the Lie algebra sl_{n+1}, not all expressions of the form (?equ4?) are equal to supports of irreducible weight sl_{n+1}-modules. This is already clear in the case when $\mathfrak{g} = W_1$, since if $\lambda \neq 0$ is dominant integral, the support of the irreducible highest weight module with highest weight is not equal to the support of the finite-dimensional irreducible sl_2-module with the same highest weight, see case 2 in Lemma ?lm23? below.

Finally, it is easy to establish an explicit relationship between the above theorem and the class of modules studied by Rudakov, see [?R1]. Indeed, note that Rudakov's continuity condition ([?R1], Lemma 2.1) applies to the Lie algebras W_n and $\bar{S}_n$ considered in the present paper. Furthermore, a simple computation based on the Theorem shows that the non-trivial irreducible weight modules from Rudakov's class are precisely those, whose shadow decomposition satisfies $\varepsilon_i \in (\Delta')^+$ for all i.

Example 1. We start with the simplest example when $\mathfrak{g} = W_1$. This case is almost trivial, but we would like to discuss it first as it will be used in the proof of the general case. The following lemma is a slightly stronger version of the main theorem for $\mathfrak{g} = W_1$. The cases 1–4 below correspond to all possible shadow decompositions of $\Delta' = \{\varepsilon_1, -\varepsilon_1\}$.

Lemma 4. *Let $\mathfrak{g} = W_1$ and M be an indecomposable weight $\mathfrak{g}$-module generated by a weight vector. Then one of the following is true:*

(1) $\operatorname{supp} M = \{0\}$ *and M is a trivial module;*
(2) $\mathfrak{g}^{-\varepsilon_1}$ *acts locally nilpotently on M and then* $\operatorname{supp} M = \lambda + \mathbb{Z}_+\varepsilon_1$ *for some $\lambda \in \mathfrak{h}^*$;*
(3) $\mathfrak{g}^{-\varepsilon_1}$ *acts freely on M and* $\operatorname{supp} M = \lambda + \mathbb{Z}_-\varepsilon_1$ *for some $\lambda \in \mathfrak{h}^*$;*
(4) $\mathfrak{g}^{-\varepsilon_1}$ *acts freely on M and* $\operatorname{supp} M = \lambda + \mathbb{Z}\varepsilon_1$ *for some $\lambda \in \mathfrak{h}^*$.*

Furthermore, all above cases are possible.

Proof. Note first that M must be a trivial module whenever $\operatorname{supp} M$ is bounded. Indeed, in this case the annihilator of M is a non-zero ideal in $\mathfrak{g}$, and therefore coincides with $\mathfrak{g}$ as $\mathfrak{g}$ is simple.

If now $\mathfrak{g}^{-\varepsilon_1}$ acts locally nilpotently on a non-trivial M one can find $\lambda \in \operatorname{supp} M$ and $v \in M_\lambda$ such that $\mathfrak{g}^{-\varepsilon_1} \cdot v = 0$. Then M is a highest weight module with highest weight λ and $\operatorname{supp} M = \lambda + \mathbb{Z}_+\varepsilon_1$. If $\mathfrak{g}^{-\varepsilon_1}$ acts freely on M, then

$\operatorname{supp} M$ is $\mathbb{Z}_{-}\varepsilon_1-$ invariant. Therefore $\operatorname{supp} M$ equals either $\lambda+\mathbb{Z}_{-}\varepsilon_1$ or $\lambda+\mathbb{Z}\varepsilon_1$, and in the former case M is again a highest weight module.

Finally, note that all supports listed in the Lemma can be realized. Indeed cases 2 and 3 correspond simply to highest weight modules (in this case, if M is irreducible, $\operatorname{supp} M$ determines M uniquely up to isomorphism). To construct a module with support as in 4 consider the space $M \stackrel{\text{def}}{=} x_1^{\mu}\mathbb{C}\left[x_1, x_1^{-1}\right](dx_1)^{\lambda-\mu}$ with the natural action of W_1. One can easily check that M is an irreducible W_1-module whenever $\mu \notin \mathbb{Z}$. □

Example 2. Let $\mathfrak{g} = W_2$. Here $\Delta' = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm(\varepsilon_1-\varepsilon_2)\}$ and $\Delta'' = \{\pm(\varepsilon_1-\varepsilon_2)\}$. The Theorem implies that the following are all possible (up to a permutation of indices) supports of irreducible $\mathfrak{g}$-modules:

(1) $\lambda+\mathbb{Z}\varepsilon_1+\mathbb{Z}\varepsilon_2$, λ being an arbitrary weight;
(2) $\lambda+\mathbb{Z}\varepsilon_1\pm\mathbb{Z}_{+}\varepsilon_2$, λ being an arbitrary weight;
(3) $\lambda+\mathbb{Z}(\varepsilon_1-\varepsilon_2)\pm\mathbb{Z}_{+}\varepsilon_2$, λ being an arbitrary weight;
(4) $\lambda+\mathbb{Z}_{+}\varepsilon_1+\mathbb{Z}_{-}\varepsilon_2$, $\lambda\neq 0$;
(5) $\lambda+\mathbb{Z}_{+}(\varepsilon_1-\varepsilon_2)+\mathbb{Z}_{+}\varepsilon_2$, $\lambda=\lambda_1\varepsilon_1+\lambda_2\varepsilon_2$ being such that $\lambda_2-\lambda_1\notin\mathbb{Z}_{+}$;
(6) $\lambda+\mathbb{Z}_{+}(\varepsilon_1-\varepsilon_2)+\mathbb{Z}_{-}\varepsilon_1$, $\lambda=\lambda_1\varepsilon_1+\lambda_2\varepsilon_2$ being such that $\lambda_2-\lambda_1\notin\mathbb{Z}_{+}$;
(7) $(\lambda_1\varepsilon_1+\lambda_2\varepsilon_2+\mathbb{Z}_{+}(\varepsilon_1-\varepsilon_2)+\mathbb{Z}_{+}\varepsilon_2)\cap(\lambda_2\varepsilon_1+\lambda_1\varepsilon_2+\mathbb{Z}_{+}(\varepsilon_2-\varepsilon_1)+\mathbb{Z}_{+}\varepsilon_1)$, $\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2\setminus\{0\}$ being such that $\lambda_2-\lambda_1\in\mathbb{Z}_{+}$;
(8) $(\lambda_1\varepsilon_1+\lambda_2\varepsilon_2+\mathbb{Z}_{+}(\varepsilon_1-\varepsilon_2)+\mathbb{Z}_{-}\varepsilon_1)\cap(\lambda_2\varepsilon_1+\lambda_1\varepsilon_2+\mathbb{Z}_{+}(\varepsilon_2-\varepsilon_1)+\mathbb{Z}_{-}\varepsilon_2)$, $\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2\setminus\{0\}$ being such that $\lambda_2-\lambda_1\in\mathbb{Z}_{+}$;
(9) $\{0\}$.

The next example shows that the main theorem does not extend to the polynomial contact Lie algebras.

Example 3. Let $\mathfrak{g}$ be the polynomial contact algebra K_5. We define it as the space $\mathbb{C}[t, x_1, y_1, x_2, y_2]$ with Lie bracket

$$[f,g] \stackrel{\text{def}}{=} -f'Dg+g'Df-\sum_{i=1}^{2}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}-\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}\right),$$

where

$$Dh \stackrel{\text{def}}{=} 2h-\sum_{i=1}^{2}\left(x_i\frac{\partial h}{\partial x_i}+y_i\frac{\partial h}{\partial y_i}\right).$$

Here $\mathfrak{h} \stackrel{\text{def}}{=} \langle t, z_1, z_2\rangle_{\mathbb{C}}$, where $z_1 \stackrel{\text{def}}{=} x_1y_1$, $z_2 \stackrel{\text{def}}{=} x_2y_2$. Furthermore,

$$\Delta = \{k\delta+a\varepsilon_1+b\varepsilon_2 \mid k+2\in\mathbb{Z}_{+}, a,b\in\mathbb{Z}, |a|+|b|\le k+2, a+b+k\in 2\mathbb{Z}\}.$$

The set $\Delta' \stackrel{\text{def}}{=} \Delta\cap-\Delta = \{\pm 2\delta, \pm\delta\pm\varepsilon_1, \pm\delta\pm\varepsilon_2, \pm\varepsilon_1\pm\varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ is a root system of type C_3 and its C_2-subsystem $\{\pm\varepsilon_1\pm\varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ plays the role of Δ''. One can check that all statements of Proposition **?prop21?** hold after suitable reformulation. However, Lemma **?lm22?** is no longer true. As a result there are irreducible weight $\mathfrak{g}$-modules M such that the faces of the closure of the convex hull of $\operatorname{supp} M$ are not generated by roots. Here is an example. Let $\omega = \varepsilon_1 + t\varepsilon_2$ for some irrational real number t. Consider the parabolic decomposition $\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$,

$$\Delta^0 \overset{\text{def}}{=} \{\alpha \in \Delta \mid (\omega, \alpha) = 0\} = \{k\delta \mid k + 2 \in \mathbb{Z}_+\},$$
$$\Delta^\pm \overset{\text{def}}{=} \left\{\alpha \in \Delta \mid (\omega, \alpha) >_{(<)} 0\right\}.$$

Then $\mathfrak{g}^0 = \mathbb{C}[t, z_1, z_2]$. Fix $\lambda \notin \mathbb{Z}$ and define a $\mathfrak{g}^0$-module structure on $M' \overset{\text{def}}{=} t^\lambda \mathbb{C}\left[t^{\pm 1}\right]$ by letting $t^m z_1^p z_2^q$ act on M' as the operator $2t^m \left(\frac{\partial}{\partial t}\right)^{1-p-q}$ (where $\left(\frac{\partial}{\partial t}\right)^{-1}$ denotes antiderivative in M'). Obviously M' is an irreducible $\mathfrak{g}^0$-module and $\operatorname{supp} M' = \lambda\delta + \mathbb{Z}\delta$. Make M' a $\mathfrak{p}$-module by putting $\mathfrak{g}^+ \cdot M' \overset{\text{def}}{=} 0$ and consider the irreducible quotient M of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M'$. One can check that

$$\operatorname{supp} M = \{\lambda\delta + l\delta + k_1\varepsilon_1 + k_2\varepsilon_2 \mid l, k_1, k_2 \in \mathbb{Z}, k_1 + tk_2 < 0\},$$

and therefore the single face of the closure of the convex hull of $\operatorname{supp} M$ is not generated by its intersection with the root lattice.

3. Proof of the Theorem

The proof is subdivided into 14 Lemmas. The main idea is to show that one can find a certain special weight $\lambda \in \operatorname{supp} M$, called minimal, such that (3) is a shadow decomposition of Δ' and moreover (5) holds for this shadow decomposition. The purpose of Lemmas 1 through 10 is to establish the existence of a minimal λ. Then, in Lemma 11 we prove that there is a minimal λ which is compatible with the shadow decomposition (3) and that indeed (5) holds for the decomposition (3). Lemmas 1–11 yield more or less immediately the claims (a), (b), and (c) in the Theorem. Finally, Lemmas 12 through 14 are necessary for the proof of claim (d).

Let now $\Gamma \overset{\text{def}}{=} \cap_{\lambda \in \mathrm{suppM}} \Gamma_\lambda$. Clearly, $\operatorname{supp} M$ is Γ-invariant. For any $\lambda \in \operatorname{supp} M$ define K_λ,

$$K_\lambda \overset{\text{def}}{=} \{\alpha \in \Delta' \mid \lambda + \alpha \notin \operatorname{supp} M\},$$

and put $\bar{K}_\lambda \overset{\text{def}}{=} \Delta' \backslash K_\lambda$. We call $\lambda \in \operatorname{supp} M$ *extremal* if K_λ is maximal (i.e., if K_λ is not a proper subset of K_μ for any $\mu \in \operatorname{supp} M$).

Lemma 5. (*a*) *If* $\alpha \in \Delta''$ *and* $\alpha, -\alpha \notin \Gamma$, *then* $\mathfrak{g}^\alpha$ *and* $\mathfrak{g}^{-\alpha}$ *act locally nilpotently on* M, *and* $\operatorname{supp} M$ *is invariant with respect to the Weyl group reflection* r_α. *Moreover, if* $\alpha \in K_\lambda$, *then* $(\lambda, \alpha) \in \mathbb{Z}_+$.

(*b*) *If* $\lambda, \mu \in \operatorname{supp} M$ *and* $\varepsilon_i \in K_\lambda$, $-\varepsilon_i \in K_\mu$, *then* $\lambda(\mathfrak{h}') = 0$ *and* $-\varepsilon_i \in K_\lambda$.

(*c*) *If* $\alpha \in \Gamma$ *and* $\lambda, \lambda - \alpha \in \operatorname{supp} M$*, then* $K_\lambda \subseteq K_{\lambda-\alpha}$*. In particular, if* λ *is extremal, then* $\lambda - \alpha$ *is also extremal with* $K_\lambda = K_{\lambda-\alpha}$*.*

(*d*) *(Convexity property) If* $\alpha \in \Delta'$ *and* $\lambda, \lambda + k\alpha \in \operatorname{supp} M$ *for some* $k \in \mathbb{Z}_+$*, then* $\lambda + j\alpha \in \operatorname{supp} M$ *for any* $0 \leq j \leq k$*.*

Proof. (*a*) Note that if $\alpha \in \Delta''$ then $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ generate an sl_2-subalgebra. Since furthermore $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ act locally nilpotently on $\mathfrak{g}$, the existence of a vector $v \in M$ with $\mathfrak{g}^\alpha \cdot v = 0$ (which is a consequence of the fact that $\alpha \notin \Gamma$) implies via a standard argument ([?K], Lemma 3.4) that $\mathfrak{g}^\alpha$ acts locally nilpotently on M. Similarly $\mathfrak{g}^{-\alpha}$ acts locally nilpotently on M. Therefore the sl_2 -subalgebra generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ acts locally finitely on M and in particular $\operatorname{supp} M$ is r_α-invariant. Finally, if $\alpha \in K_\lambda$, then $\mathfrak{g}^\alpha \cdot M_\lambda = 0$ and λ is a highest sl_2-weight, i.e., $(\lambda, \alpha) \in \mathbb{Z}_+$.

(*b*) Since $\mathfrak{g}^{-\varepsilon_i}$ acts nilpotently on $\mathfrak{g}$, for the same reason as in (*a*) $\mathfrak{g}^{-\varepsilon_i}$ acts locally nilpotently on M. By Lemma ?prop21? (*d*) and Lemma ?lm23?, $(\lambda + \mathbb{Z}\varepsilon_i) \cap \operatorname{supp} M = \{\lambda\}$. Therefore $\mathfrak{g}^{\varepsilon_i} \cdot M_\lambda = \mathfrak{g}^{-\varepsilon_i} \cdot M_\lambda = 0$ and $[\mathfrak{g}^{\varepsilon_i}, \mathfrak{g}^{-\varepsilon_i}] \cdot M_\lambda = \mathfrak{h}' \cdot M_\lambda = 0$. This shows that $\lambda(\mathfrak{h}') = 0$ and therefore also that $-\varepsilon_i \in K_\lambda$.

(*c*) Since $\alpha \in \Gamma$ and since $\operatorname{supp} M$ is Γ-invariant, the inclusion $\lambda - \alpha + \delta \in \operatorname{supp} M$ implies $\lambda + \delta \in \operatorname{supp} M$. Therefore $\delta \in \bar{K}_{\lambda-\alpha}$ gives $\delta \in \bar{K}_\lambda$, i.e., $K_\lambda \subseteq K_{\lambda-\alpha}$.

(*d*) Assume to the contrary that $\lambda + j\alpha \notin \operatorname{supp} M$ for some $j < k$. Then $\alpha \notin \Delta''$ as otherwise the sl_2-subalgebra generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ would have to act locally finitely on M and we immediately obtain a contradiction. Therefore $\alpha = -\varepsilon_i$ for some i. Then $\mathfrak{g}^{-\varepsilon_i}$ acts locally nilpotently on M, and Lemma ?lm23?, applied to a W_1-subalgebra of $\mathfrak{g}(\mathbb{Z}\varepsilon_i)$ (see Proposition ?prop21? (*d*)), provides a contradiction.

□

Lemma 6. *Let* $\lambda \in \operatorname{supp} M$ *be an extremal point and let* $\alpha, \beta, \alpha + \beta \in \Delta'$*.*

(*a*) *If* $\alpha, \beta \in K_\lambda$*, then* $\alpha + \beta \in K_\lambda$*.*

(*b*) *If* $\alpha, \beta \in \bar{K}_\lambda$*, then* $\alpha + \beta \in \bar{K}_\lambda$*.*

Proof. (*a*) First, assume that at least one of the two roots α, β (say α) belongs to Δ''. Assume also that $\alpha, \beta \in K_\lambda$ and $\alpha + \beta \notin K_\lambda$. Then $\mu = \lambda + \alpha + \beta \in \operatorname{supp} M$, $\alpha \in K_\lambda$, $-\alpha \in K_\mu$. By Lemma ?lm31? (*a*), we have $(\lambda, \alpha) \in \mathbb{Z}_+$ and $(\mu, \alpha) \in \mathbb{Z}_-$. Since $(\mu - \lambda, \alpha) = (\alpha + \beta, \alpha) = 1$, this is impossible.

Let now $\alpha, \beta \notin \Delta''$. Then without loss of generality we can assume that $\alpha = -\varepsilon_i$ and $\beta = \varepsilon_j$. Suppose again that $\alpha, \beta \in K_\lambda$ and $\alpha + \beta \notin K_\lambda$. Then $\mu = \lambda + \alpha + \beta \in \operatorname{supp} M$, $-\varepsilon_i, \varepsilon_j \in K_\lambda$, $\varepsilon_i, -\varepsilon_j \in K_\mu$. By Lemma ?lm31? (*b*), $\lambda(\mathfrak{h}') = \mu(\mathfrak{h}') = 0$, which is impossible since $(\lambda - \mu)(\mathfrak{h}') = (\varepsilon_j - \varepsilon_i)(\mathfrak{h}') \neq 0$.

(*b*) As in the proof of (*a*) we consider two cases. First, let $\alpha \in \Delta''$. Assume that $\alpha, \beta \in \bar{K}_\lambda$ but $\alpha + \beta \in K_\lambda$, i.e., $\lambda + \alpha, \lambda + \beta \in \operatorname{supp} M$, $\lambda + \alpha + \beta \notin \operatorname{supp} M$. Note that $-\alpha \notin \Gamma$ by Lemma ?lm31? (*c*). On the other hand, $\alpha \notin \Gamma$ because $\alpha \in K_{\lambda+\beta}$. Therefore, $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ act locally nilpotently on M, and $(\lambda + \beta, \alpha) \in \mathbb{Z}_+$. Since $(\beta, \alpha) = -1$, we have $(\lambda, \alpha) \geq 1$. Furthermore, $\beta \in K_{\lambda+\alpha} \backslash K_\lambda$.

Since λ is extremal, one can find $\delta \in K_\lambda \backslash K_{\lambda+\alpha}$. Then $\lambda+\alpha+\delta \in \operatorname{supp} M$ and $\lambda+\delta \notin \operatorname{supp} M$, which implies $(\lambda+\alpha+\delta,\alpha) \in \mathbb{Z}_-$. But $(\delta,\alpha) \geq -1$ and therefore $(\alpha+\delta,\alpha) > 0$, while we showed already that $(\lambda,\alpha) \geq 1$. Contradiction.

Now let $\alpha,\beta \notin \Delta''$, for example let $\alpha = -\varepsilon_i, \beta = \varepsilon_j$. Assume that $\alpha,\beta \in \bar{K}_\lambda$ but $\alpha+\beta \in K_\lambda$, i.e., $\lambda-\varepsilon_i, \lambda+\varepsilon_j \in \operatorname{supp} M$, $\lambda+\varepsilon_j-\varepsilon_i \notin \operatorname{supp} M$. Notice that $-\varepsilon_j, \varepsilon_i \notin \Gamma$. Since $\varepsilon_j \in K_{\lambda-\varepsilon_i}$, by Lemma **?lm31?** (b), we have $(\lambda-\varepsilon_i)(\mathfrak{h}') = 0$. On the other hand, $\varepsilon_j \in K_{\lambda-\varepsilon_i} \backslash K_\lambda$. Since λ is extremal, there exists $\delta \in K_\lambda \backslash K_{\lambda-\varepsilon_i}$, i.e., $\lambda-\varepsilon_i+\delta \in \operatorname{supp} M$ and $\lambda+\delta \notin \operatorname{supp} M$. Then again, by Lemma **?lm31?** (b), $(\lambda-\varepsilon_i+\delta)(\mathfrak{h}') = 0$. Thus $\delta(\mathfrak{h}') = 0$ for some $\delta \in \Delta'$, which is impossible. □

Corollary 1. *Let $\lambda \in \operatorname{supp} M$ be extremal. Put*

$$
\begin{aligned}
{}_\lambda(\Delta')^+ &\stackrel{def}{=} K_\lambda \backslash (K_\lambda \cap -K_\lambda),\\
{}_\lambda(\Delta')^- &\stackrel{def}{=} \bar{K}_\lambda \backslash \left(\bar{K}_\lambda \cap -\bar{K}_\lambda\right),\\
{}_\lambda(\Delta')^0 &\stackrel{def}{=} (K_\lambda \cap -K_\lambda) \sqcup \left(\bar{K}_\lambda \cap -\bar{K}_\lambda\right).
\end{aligned}
$$

Then

$$ {}_\lambda(\Delta')^+ \sqcup {}_\lambda(\Delta')^0 \sqcup {}_\lambda(\Delta')^- $$

is a parabolic decomposition of Δ'. □

Lemma 7. *If $\operatorname{supp} M$ contains more than one point, then $\{\varepsilon_i, -\varepsilon_i\} \cap \Gamma \neq \varnothing$ for all i.*

Proof. Assume that $\varepsilon_i, -\varepsilon_i \notin \Gamma$ for some i. Then, by Lemma **?lm31?** (b), one can choose $\lambda \in \operatorname{supp} M$ such that $\varepsilon_i, -\varepsilon_i \in K_\lambda$ and $\lambda(\mathfrak{h}') = 0$. Furthermore, λ can be chosen to be extremal. Consider the parabolic decomposition ${}_\lambda(\Delta')^+ \sqcup_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^-$ from Corollary **?cor34?**. We claim that ${}_\lambda(\Delta')^0 = K_\lambda \cap -K_\lambda$. This is equivalent to saying that for any $\alpha \in \Delta'$ at least one of the roots α and $-\alpha$ belongs to K_λ. The latter however is obvious because otherwise $\lambda+\alpha, \lambda-\alpha \in \operatorname{supp} M$ implies $\lambda+\alpha+\varepsilon_i, \lambda-\alpha+\varepsilon_i \in \operatorname{supp} M$, and then, by Lemma **?lm31?** (d), one has $\lambda+\varepsilon_i \in \operatorname{supp} M$. Consider now the parabolic decomposition $\Delta = {}_\lambda\Delta^+ \sqcup {}_\lambda\Delta^0 \sqcup {}_\lambda\Delta^-$ induced by the parabolic decomposition ${}_\lambda(\Delta')^+ \sqcup {}_\lambda(\Delta')^0 \sqcup {}_\lambda(\Delta')^-$, and let ${}_\lambda\mathfrak{p}$ be the corresponding parabolic subalgebra. Then ${}_\lambda\mathfrak{p}$ is generated by $\mathfrak{h}$ and $\mathfrak{g}^\alpha$ for all $\alpha \in_\lambda (\Delta')^+ \sqcup_\lambda (\Delta')^0$. Therefore $\mathfrak{p} \cdot M^\lambda = M^\lambda$ and M is the unique irreducible quotient of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^\lambda$. Since $\lambda(\mathfrak{h}') = 0$, we have $\dim M = 1$ and $\operatorname{supp} M = \{\lambda\}$. □

Lemma 8. *Let $\lambda \in \operatorname{supp} M$.*

(a) $(\Delta')^-_\lambda \subseteq \Gamma$.

(b) *The set $(\Delta')^F_\lambda$ is a root subsystem of Δ''.*

(c) *If $\alpha \in (\Delta')^F_\lambda$ and $\beta \in (\Delta')^I_\lambda$, then $(\alpha,\beta) = 0$.*

Proof. (*a*) Let $\alpha \in (\Delta')^-_\lambda$. If $\alpha \notin \Delta''$, Lemma ?lm35? implies that $\alpha \in \Gamma$. Let now $\alpha \in \Delta''$. Assume that $\alpha \notin \Gamma$. Then $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ act locally nilpotently on M and each sl_2-string $\lambda + \mathbb{Z}\alpha \cap \operatorname{supp} M$ is either bounded or unbounded in both directions. Therefore $\alpha, -\alpha \in (\Delta')^F_\lambda$ or $\alpha, -\alpha \in (\Delta')^I_\lambda$. Contradiction. Hence $\alpha \in \Gamma$.

(*b*) By Lemma ?lm35?, $(\Delta')^F_\lambda \subseteq \Delta''$ and, by definition, $-(\Delta')^F_\lambda = (\Delta')^F_\lambda$. All we need to check is that, for any $\alpha, \beta \in (\Delta')^F_\lambda$, $\alpha+\beta \in \Delta''$ implies $\alpha+\beta \in (\Delta')^F_\lambda$.

Assume that $\alpha+\beta \notin (\Delta')^F_\lambda$. Then $\alpha+\beta \in (\Delta')^I_\lambda$ and $\lambda + k(\alpha+\beta) \in \operatorname{supp} M$ for all $k \in \mathbb{Z}_+$. By possibly changing α to $-\alpha$ and and β to $-\beta$, we can assume that $(\lambda, \alpha) \geq 0$. Therefore $r_\alpha(\lambda + k(\alpha+\beta)) = \lambda - m\alpha + k\beta$ for some nonnegative m. Then, by Lemma ?lm31? (*d*), $\lambda + k\beta \in \operatorname{supp} M$ for any $k \in \mathbb{Z}_+$, which contradicts to the fact that $\beta \in (\Delta')^F_\lambda$.

(*c*) Assume that the statement is not true, i.e., that one can find $\alpha \in (\Delta')^F_\lambda$ and $\beta \in (\Delta')^I_\lambda$ such that $(\alpha, \beta) > 0$. Then the roots $\pm\alpha$, $\pm\beta$ and $\pm r_\alpha(\beta)$ form a root system R of type A_2. The intersection of Γ_λ with R either coincides with $\{\beta, -\beta\}$ or contains all roots from the half-plane of $\langle R\rangle_{\mathbb{R}}$ bounded by the line $\mathbb{R}\beta$. In the former case $(\Delta')^F_\lambda \cap R = R \backslash \{\beta, -\beta\}$, which contradicts to (*b*). In the latter case at least one of α and $-\alpha$ belongs to Γ_λ, which is also impossible. □

Lemma 9. *Let $\lambda \in \operatorname{supp} M$ be extremal. Then*

$$K_\lambda = (\Delta')^+_\lambda \sqcup \{\alpha \in (\Delta')^F_\lambda \mid (\lambda, \alpha) \geq 0\}.$$

Proof. Obviously,

$$K_\lambda \subseteq (\Delta')^+_\lambda \sqcup \{\alpha \in (\Delta')^F_\lambda \mid (\lambda, \alpha) \geq 0\}.$$

We will prove that also

$$(\Delta')^+_\lambda \sqcup \{\alpha \in (\Delta')^F_\lambda \mid (\lambda, \alpha) \geq 0\} \subseteq K_\lambda.$$

Let us show first that $(\Delta')^+_\lambda \subseteq K_\lambda$. Let $\alpha \in (\Delta')^+_\lambda$. Choose $k \in \mathbb{Z}_+$ such that $\lambda + k\alpha \in \operatorname{supp} M$ but $\lambda + (k+1)\alpha \notin \operatorname{supp} M$. By Lemma ?lm36? (*a*), $-\alpha \in \Gamma$. By Lemma ?lm31? (*c*), $K_\lambda \subseteq K_{\lambda+k\alpha}$. Since λ is extremal, $k = 0$ and $\alpha \in K_\lambda$.

It remains to show that $\{\alpha \in (\Delta')^F_\lambda \mid (\lambda, \alpha) \geq 0\} \subseteq K_\lambda$. Let $\alpha \in (\Delta')^F_\lambda$ and $(\lambda, \alpha) \geq 0$. Observe that $K_\lambda \subseteq K_{\lambda+\alpha}$. Indeed, if $\delta \in \bar{K}_{\lambda+\alpha}$ then $\lambda + \alpha + \delta \in \operatorname{supp} M$ and $(\lambda + \alpha + \delta, \alpha) > 0$. Therefore $\lambda + \alpha + \delta - \alpha = \lambda + \delta \in \operatorname{supp} M$ and $\delta \in \bar{K}_\lambda$. We complete the proof now by using the same argument as in the case when $\alpha \in (\Delta')^+_\lambda$. □

Corollary ?cor34?, Lemma ?lm36? and Lemma ?lm32? imply

Corollary 2. *If $\lambda \in \operatorname{supp} M$ is extremal, the decomposition*

$$\Delta' = (\Delta')^+_\lambda \sqcup (\Delta')^F_\lambda \sqcup (\Delta')^I_\lambda \sqcup (\Delta')^-_\lambda$$

is a shadow decomposition of Δ'. □

Set $S^{\pm} \overset{\text{def}}{=} \{i \mid \pm\varepsilon_i \notin \Gamma\}$ and $S \overset{\text{def}}{=} S^+ \sqcup S^-$. We say that $\lambda \in \operatorname{supp} M$ is *minimal* if λ is extremal and $\pm\varepsilon_i \in K_\lambda$ for all $i \in S^{\pm}$.

Lemma 10. *There exists a minimal* $\lambda \in \operatorname{supp} M$*.*

Proof. It consists of three steps.

Step 1. We will show that there is an extremal point $\lambda \in \operatorname{supp} M$ such that ε_l (respectively, $-\varepsilon_l$) belongs to K_λ for every $l \in S^+$ (resp. for every $l \in S^-$). Consider the case of S^+ (for S^- the proof is the same). Assume that the statement is false. This implies that one can find $i, j \in S^+$ and extremal weights $\lambda, \mu \in \operatorname{supp} M$ such that $\varepsilon_i \in K_\lambda$, $\varepsilon_i \in \Gamma_\mu$, $\varepsilon_j \in K_\mu$, $\varepsilon_j \in \Gamma_\lambda$. Then $\lambda + k\varepsilon_j$, $\mu + k\varepsilon_i \in \operatorname{supp} M$ and $\lambda + \varepsilon_i + k\varepsilon_j$, $\mu + \varepsilon_j + k\varepsilon_i \notin \operatorname{supp} M$ for any $k \in \mathbb{Z}_+$. Furthermore, if $\alpha \overset{\text{def}}{=} \varepsilon_i - \varepsilon_j$, then $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ act locally nilpotently on M. Therefore, for sufficiently large k we have $\mu + k\varepsilon_i - \alpha \in \operatorname{supp} M$ as $(\mu + k\varepsilon_i, \alpha) > 0$. Contradiction.

Step 2. Choose an extremal point $\mu \in \operatorname{supp} M$ such that $-\varepsilon_j \in K_\mu$ for all $j \in S^-$. Let $\omega^- = -\sum_{j \in S^-} \varepsilon_j$. Consider the parabolic decomposition $\Delta = \Delta^- \sqcup \Delta^0 \sqcup \Delta^+$, where $\Delta^{\pm} \overset{\text{def}}{=} \left\{\alpha \in \Delta \mid (\omega^-, \alpha) >_{(<)} 0\right\}$. Any $\alpha \in \Delta^+$ can be written as $-\varepsilon_i + \sum_{j \notin S^-} m_j \varepsilon_j$, and, since $-\varepsilon_j \in \Gamma$ for all $j \notin S^-$, we obtain that $\mu + \alpha \notin \operatorname{supp} M$. Then, by Lemma **?lm2?** (b), M is the unique irreducible quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{p}}$.

Step 3. Since $\operatorname{supp} M \subseteq \operatorname{supp} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{p}}$ for any $\nu \in \operatorname{supp} M$, we have $(\omega^-, \nu) \leq (\omega^-, \mu)$. Therefore $-\varepsilon_j \in (\Delta')^+_\nu$ for all $j \in S^-$ and all $\nu \in \operatorname{supp} M$. In particular, if ν is extremal, then by Lemma **?lm32?** $-\varepsilon_j \in K_\nu$ for all $j \in S^-$. Thus, an extremal point λ, such that $\varepsilon_l \in K_\lambda$ for all $l \in S^+$, is necessarily minimal. □

Let $\lambda \in \operatorname{supp} M$ be a minimal point. Consider the parabolic decomposition $\Delta = {}_\lambda\Delta^+ \sqcup_\lambda \Delta^0 \sqcup_\lambda \Delta^+$ induced by the parabolic decomposition ${}_\lambda(\Delta')^+ \sqcup_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^-$ of Corollary **?cor34?**, and set ${}_\lambda\mathfrak{p} \overset{\text{def}}{=} \mathfrak{h} \oplus (\oplus_{\alpha \in {}_\lambda(\Delta')^0 \sqcup_\lambda \Delta^+} \mathfrak{g}^\alpha)$.

Lemma 11. *One can find a minimal* $\lambda \in \operatorname{supp} M$ *such that* $\lambda + \alpha \notin \operatorname{supp} M$ *for any* $\alpha \in {}_\lambda\Delta^+$*. Therefore, by Lemma* **?lm2?** (b)*,* M *is the unique irreducible quotient of the induced module* $U(\mathfrak{g}) \otimes_{U({}_\lambda\mathfrak{p})} M'$ *for some irreducible* ${}_\lambda\mathfrak{p}$*-module* M' *with* $\lambda \in \operatorname{supp} M'$*. Moreover,*

$$\operatorname{supp} M = \operatorname{supp} M^F_\lambda + \left\langle (\Delta')^I_\lambda \sqcup (\Delta')^-_\lambda \right\rangle_{\mathbb{Z}_+}, \tag{7}$$

and λ *is compatible with the shadow decomposition*

$$\Delta' = (\Delta')^+_\lambda \sqcup (\Delta')^I_\lambda \sqcup (\Delta')^F_\lambda \sqcup (\Delta')^-_\lambda . \tag{8}$$

Proof. Set

$$K_\lambda^1 \stackrel{\text{def}}{=} \{\beta = \varepsilon_i + \varepsilon_j - \varepsilon_k \mid k \neq i, j, \lambda + \beta \notin \operatorname{supp} M\}$$

and

$$\bar{K}_\lambda^1 \stackrel{\text{def}}{=} \{\beta = \varepsilon_i + \varepsilon_j - \varepsilon_k \mid k \neq i, j, \lambda + \beta \in \operatorname{supp} M\}.$$

We will prove that any minimal λ with maximal K_λ^1 satisfies the requirement of the Lemma. By Lemma **?lm2?** (a) it is sufficient to show that $\lambda + \alpha \notin \operatorname{supp} M$ for any $\alpha \in_\lambda \Delta^+$.

Recall the proof of Lemma **?lm22?**. It implies that any $\alpha \in_\lambda \Delta^+$ can be written as $\alpha = \alpha_0 + \sum_{j \notin S^-} m_j \varepsilon_j$, (here $S^- = S_1^-$, see Lemma **?lm22?**), where $\alpha_0 \in \Delta'' \cap \left({}_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^+\right)$ or $\alpha_0 = 0$. Since $-\varepsilon_j \in \Gamma$ for all $j \notin S^-$, it is obvious that $\lambda + \alpha \notin \operatorname{supp} M$ whenever $\alpha_0 \in_\lambda (\Delta')^+$ or $\alpha_0 = 0$. Consider the case when $\alpha_0 \in_\lambda (\Delta')^0$. In this case $m_i > 0$ for at least one $i \in S^+$. Assume, to the contrary, that $\lambda + \alpha \in \operatorname{supp} M$. Then $\lambda + \alpha_0 + \varepsilon_i, \lambda + \alpha_0 \in \operatorname{supp} M$. Therefore $\alpha_0 \notin K_\lambda$, which implies by Lemma 9 that $\alpha \in (\Delta')^I_\lambda$. Thus $\mu = \lambda - \alpha_0 \in \operatorname{supp} M$. We claim that μ is minimal and that K_λ^1 is strictly contained in K_μ^1. To prove this claim consider two cases: $\alpha_0 \in \Gamma$ and $\alpha_0 \notin \Gamma$. In the former case the claim is trivial. In the latter case $\mathfrak{g}^{\alpha_0}$ acts locally nilpotently on M and, since $-\alpha_0 \in K_{\lambda+\alpha_0+\varepsilon_i}$, $\mathfrak{g}^{-\alpha_0}$ also acts locally nilpotently on M. Furthermore, $(\lambda + \alpha_0 + \varepsilon_i, \alpha_0) \in \mathbb{Z}_-$, i.e., $(\lambda, \alpha_0) \leq -2$. Fix an arbitrary element $\delta \in \bar{K}_\mu \cup \bar{K}_\mu^1$. Then $\lambda - \alpha_0 + \delta \in \operatorname{supp} M$, and $(\lambda - \alpha_0 + \delta, \alpha_0) = (\lambda, \alpha_0) - 2 + (\delta, \alpha_0) < 0$ since $(\delta, \alpha_0) \leq 2$. Therefore $\lambda + \delta \in \operatorname{supp} M$ and $\delta \in \bar{K}_\lambda \cup \bar{K}_\lambda^1$. Hence $K_\lambda \subseteq K_\mu$. Moreover, K_λ^1 is strictly contained in K_μ^1 as $\alpha_0 + \varepsilon_i \in K_\mu^1 \backslash K_\lambda^1$. This contradicts the maximality of K_λ^1, and hence $\lambda + \alpha \notin \operatorname{supp} M$.

To prove (**?equ90?**) we use Lemma **?lm2?** (a). Indeed, Lemma **?lm22?** (applied to the parabolic decomposition ${}_\lambda (\Delta')^+ \sqcup_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^-$) yields

$$\left\langle {}_\lambda\Delta^0 \sqcup_\lambda \Delta^- \right\rangle_{\mathbb{Z}_+} \cap \left\langle {}_\lambda\Delta^0 \sqcup_\lambda \Delta^+ \right\rangle_{\mathbb{Z}_-} = \left\langle {}_\lambda(\Delta')^0 \sqcup_\lambda (\Delta')^- \right\rangle_{\mathbb{Z}_+}.$$

Therefore $\operatorname{supp} M \subseteq \lambda + \left\langle {}_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^- \right\rangle_{\mathbb{Z}_+}$. Furthermore, $\operatorname{supp} M$ is W_λ-invariant, where W_λ denotes the Weyl group of the root system $(\Delta')^F_\lambda$. This gives

$$\operatorname{supp} M \subseteq \cap_{w \in W_\lambda} w(\lambda + \left\langle {}_\lambda(\Delta')^0 \sqcup_\lambda (\Delta')^- \right\rangle_{\mathbb{Z}_+}). \tag{9}$$

On the other hand, $\Gamma_\lambda = \left({}_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^-\right) \setminus (\Delta')^F_\lambda$, i.e., $\lambda + \mathbb{Z}_+\alpha \subseteq \operatorname{supp} M$ for any $\alpha \in \left({}_\lambda (\Delta')^0 \sqcup_\lambda (\Delta')^-\right) \setminus (\Delta')^F_\lambda$. As $\operatorname{supp} M$ satisfies the convexity property (Lemma **?lm31?** (d)), we obtain that (9) is an equality, i.e.,

$$\operatorname{supp} M = \cap_{w \in W_\lambda} w(\lambda + \left\langle {}_\lambda(\Delta')^0 \sqcup_\lambda (\Delta')^- \right\rangle_{\mathbb{Z}_+}). \tag{10}$$

Furthermore, since

$$\operatorname{supp} M_\lambda^F = \cap_{w\in W_\lambda} w(\lambda + \left\langle (\Delta')_\lambda^F \cap_\lambda (\Delta')^- \right\rangle_{\mathbb{Z}_+}),$$

(10), together with the W_λ-invariance of $(\Delta')_\lambda^I$ and $(\Delta')_\lambda^-$, yields (7).

Finally, we will prove that λ is compatible with the shadow decomposition (?equ91?). First, (?equ90?) and Lemma ?lm31? (a) imply that $(\lambda,\alpha)\in\mathbb{Z}$ for any $\alpha\in(\Delta')_\lambda^F$. To check the second condition of compatibility, choose a λ-indecomposable root $\alpha\in(\Delta')_\lambda^+$. Note that $\beta - p\alpha \notin \langle (\Delta')_\lambda^I \sqcup {}_\lambda(\Delta')^-\rangle_{\mathbb{Z}_+}$ for any $\beta\in {}_\lambda(\Delta')^+$, $\beta\neq\alpha$, and any $p\in\mathbb{Z}_+$, and hence $\mathfrak{g}^\beta\cdot M^{\lambda-p\alpha}=0$. Consider first the case when $\alpha\in\Delta''$. If $k=(\lambda,\alpha)\in\mathbb{Z}_+$, then $\mathfrak{g}^\alpha\cdot M^{\lambda-(k+1)\alpha}=0$, and therefore $M^{\lambda-(k+1)\alpha}$ generates a proper submodule in M. Hence $M^{\lambda-(k+1)\alpha}=0$, which contradicts to the fact that $-\alpha\in\Gamma_\lambda$. Therefore $(\lambda,\alpha)\notin\mathbb{Z}_+$ for $\alpha\in\Delta''$. Let now $\alpha\notin\Delta''$ and assume that $\lambda(\mathfrak{h}')=0$. In the same way we can show that $M^{\lambda-\alpha}$ generates a proper submodule in M, i.e., $M^{\lambda-\alpha}=0$, which contradicts to the fact that $-\alpha\in\Gamma_\lambda$. Thus, we have proved that $\lambda(\mathfrak{h}')\neq 0$. □

Lemma ?lm38? (together with Lemma ?lm36? (b)) implies claims (a), (b) and (c) of the Theorem. Indeed, the explicit expression for $\operatorname{supp} M$ obtained in (7) makes it obvious that the subdivision (3) does not depend on λ, and that moreover $(\Delta')^+\sqcup(\Delta')^I\sqcup(\Delta')^F\sqcup(\Delta')^-$ is a shadow decomposition of Δ'. The fact that $(\Delta')^F\subset\Delta''$ follows from Lemma ?lm36? (b). Finally, as (4) is well-defined, (7) is equivalent to (5). Therefore all that remains is to prove claim (d) of the Theorem.

The following lemma is a straightforward corollary of the definition of the Lie algebras W_n and $\bar{S}_n$ and of Lemma ?lm22?.

Lemma 12. *Let $\Delta=\Delta^+\sqcup\Delta^0\sqcup\Delta^-$ be a parabolic decomposition of Δ induced by some parabolic decomposition of Δ'. If $\mathfrak{g}=W_n$, then the Lie subalgebra $\mathfrak{g}^0$ is isomorphic to a semidirect sum of $W_m=\operatorname{Der}\mathbb{C}[x_{i_1},\dots,x_{i_m}]$ and the ideal $\mathfrak{k}\otimes\mathbb{C}[x_{i_1},\dots,x_{i_m}]$ for some reductive $\mathfrak{h}$-invariant Lie subalgebra $\mathfrak{k}\subseteq gl_n$. If $\mathfrak{g}=\bar{S}_n$, then $\mathfrak{g}^0$ is isomorphic to the intersection of $\bar{S}_n$ with a semidirect sum as above.* □

Lemma 13. *Let $\mathfrak{g}^0$ be as in Lemma ?lm51? and λ be an arbitrary weight. There exists an irreducible weight $\mathfrak{g}^0$-module M^0 with* $\operatorname{supp} M^0=\lambda+\left\langle\Delta^0\right\rangle_{\mathbb{Z}_+}$.

Proof. Let $\lambda=\sum_{i=1}^n\lambda_i\varepsilon_i$, $T=\{i_1,\dots,i_m\}$. There exists an irreducible weight $\mathfrak{k}$-module $M_{\mathfrak{k}}$ with $\operatorname{supp} M_{\mathfrak{k}}=\sum_{i\notin T}\lambda_i\varepsilon_i+\langle\Delta_{\mathfrak{k}}\rangle_{\mathbb{Z}}$. (This follows for example from [?BL].) Consider the space

$$M^0\overset{\text{def}}{=}M_{\mathfrak{k}}\otimes(x_{i_1}^{\lambda_{i_1}-\theta}\cdot\ldots\cdot x_{i_m}^{\lambda_{i_m}-\theta}\mathbb{C}\left[x_{i_1}^{\pm1},\dots,x_{i_m}^{\pm1}\right](dx_{i_1}\cdot\ldots\cdot dx_{i_m})^\theta)$$

with the obvious action of $\mathfrak{g}^0$. One can easily check that, if $\lambda_{i_1}-\theta,\dots,\lambda_{i_m}-\theta\notin\mathbb{Z}$, M^0 is an irreducible $\mathfrak{g}^0$-module which satisfies the requirement of the Lemma. □

Now we are able to construct an irreducible $\mathfrak{g}$-module M whose support is given by (?equ4?).

Let $\Delta' = (\Delta')^+ \sqcup (\Delta')^F \sqcup (\Delta')^I \sqcup (\Delta')^-$ be a shadow decomposition of Δ' and $\lambda \in \mathfrak{h}^*$ be a compatible weight. Consider the parabolic decomposition $\Delta' = (\Delta')^+ \sqcup (\Delta')^0 \sqcup (\Delta')^-$, where $(\Delta')^0 = (\Delta')^F \sqcup (\Delta')^I$, and the corresponding induced parabolic decomposition $\Delta = \Delta^+ \sqcup \Delta^0 \sqcup \Delta^-$. Set $\bar{\mathfrak{h}} \stackrel{\text{def}}{=} \{(a,b) \mid a, b \in \mathfrak{h}, [a, \mathfrak{g}^F] = 0, [b, \mathfrak{g}^I] = 0\}$. Note that $\mathfrak{g}^0$ is isomorphic to $(\mathfrak{g}^F \oplus \mathfrak{g}^I)/\bar{\mathfrak{h}}$. Choose $\mu \in \mathfrak{h}^*$ with $(\lambda \oplus \mu)(\bar{\mathfrak{h}}) = 0$. Let M^I be an irreducible $\mathfrak{g}^I$-module with $\operatorname{supp} M^I = \mu + \langle (\Delta')^I \rangle_{\mathbb{Z}_+}$ (which exists by Lemma ?lm52?) and M^F_λ be the finite-dimensional irreducible $\mathfrak{g}^F$-module with highest weight λ. Then $M^0 \stackrel{\text{def}}{=} M^I \otimes M^F_\lambda$ has a natural structure of a $\mathfrak{g}^0$-module. Furthermore, it is clear that

$$\operatorname{supp} M^0 = \operatorname{supp} M^F_\lambda + \langle (\Delta')^I \rangle_{\mathbb{Z}_+}.$$

Finally, put $\mathfrak{g}^+ \cdot M^0 \stackrel{\text{def}}{=} 0$ and let M be the unique irreducible quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^0$.

Lemma 14.

$$\operatorname{supp} M = \operatorname{supp} M^F_\lambda + \langle (\Delta')^I \sqcup (\Delta')^- \rangle_{\mathbb{Z}_+}.$$

Proof. It follows from Lemma ?lm2? (a) that

$$\operatorname{supp} M \subseteq \operatorname{supp} M^F_\lambda + \langle (\Delta')^I \sqcup (\Delta')^- \rangle_{\mathbb{Z}_+}. \tag{11}$$

To establish equality, note that both sides of (10) are invariant under the Weyl group W of the root system $(\Delta')^F$. Note also that

$$(\Delta')^- = \cap_{w \in W} w({}_\lambda (\Delta')^-).$$

Therefore, by the convexity property of $\operatorname{supp} M$, it is sufficient to show that $\lambda + \mathbb{Z}_- \alpha \subseteq \operatorname{supp} M$ for every simple root $\alpha \in (\Delta')^+$. The latter follows easily from the compatibility of λ. Indeed, if $\alpha \notin \Delta''$ and $\lambda(\mathfrak{h}') \neq 0$, then $-\alpha \in \Gamma$ by Lemma ?lm35?. If $\alpha \in \Delta''$, then $(\lambda, \alpha) \notin \mathbb{Z}_+$ implies that $-\alpha \in \Gamma_\lambda$. □

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