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DISTRIBUTION OF RESONANCES AND LOCAL ENERGY DECAY IN THE TRANSMISSION PROBLEM. II

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1. Introduction and statement of results

This paper is concerned with the resonances of the transmission problem for a transparent bounded strictly convex obstacle $\mathcal{O}$ with a smooth boundary (which may contain an impenetrable body). If the speed of propagation inside $\mathcal{O}$ is bigger than that outside $\mathcal{O}$, we prove under some natural conditions, that there exists a strip in the upper half plane containing the real axis, which is free of resonances. We also obtain an uniform decay of the local energy for the corresponding mixed problem with an exponential rate of decay when the dimension is odd, and polynomial otherwise. It is well known that such a decay of the local energy holds for the wave equation with Dirichlet (Neumann) boundary conditions for any nontrapping obstacle. In our case, however, $\mathcal{O}$ is a trapping obstacle for the corresponding classical system.

Let $\mathcal{O}_1 \subset R^n, n \geq 2$, be a bounded domain with a connected C^∞ boundary Γ_1. Let also $\mathcal{O}_2 \subset R^n$ be a bounded domain with a connected C^∞ boundary Γ and such that $\mathcal{O}_1 \subset \mathcal{O}_2$ and $\Gamma_1 \cap \Gamma = \emptyset$. Set $\Omega = R^n \setminus \overline{\mathcal{O}}_2$ and $\Omega_1 = R^n \setminus \overline{\mathcal{O}}_1$. Consider in $\mathcal{O} = \mathcal{O}_2 \setminus \overline{\mathcal{O}}_1$ the operator

$$\Delta_g := c(x)^2 \sum_{i,j=1}^{n} \partial_{x_i}(g_{ij}(x)\partial_{x_j}),$$

where $c(x), g_{ij}(x) \in C^\infty(\overline{\mathcal{O}})$ and $c(x) \geq c_0 > 0$. We suppose that the principal symbol, $g(x,\xi)$, of $-\Delta_g$ satisfies

$$g(x,\xi) := c(x)^2 \sum_{i,j=1}^{n} g_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \qquad \forall (x,\xi) \in T^*\mathcal{O}, \quad C > 0.$$

Denote by $\mathcal{G}$ the Riemannian metric $\sum_{i,j=1}^n G_{ij}(x)dx_i dx_j$ in $\overline{\mathcal{O}}$ associated with the Hamiltonian g, where $(G_{ij}(x))_{i,j=1}^n$ is the inverse matrix to $(c(x)^2 g_{ij}(x))_{i,j=1}^n$. Given $x' \in \Gamma$, we denote by $\nu'(x')$ the unit inner normal to Γ at x' with respect to the Riemannian metric $\mathcal{G}$.

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Choose a function $f \in C^\infty(R^n)$ such that $f < 0$ in $\mathcal{O}_2$, $f > 0$ in Ω and $df \neq 0$ on Γ. The boundary Γ will be said to be g-strictly concave with respect to $\mathcal{O}$ (or g-strictly convex with respect to Ω) iff for any (x, ξ) satisfying

$$f(x) = 0, \quad g(x, \xi) = 1, \quad \{g, f\}(x, \xi) = 0,$$

we have

$$\{g, \{g, f\}\}(x, \xi) > 0, \tag{1.1}$$

where $\{\cdot, \cdot\}$ denotes the Poisson brackets. Usually, it is required also that $\{f, \{f, g\}\}(x, \xi) > 0$, which is fulfilled automatically in our case since $\{f, \{f, g\}\}(x, \xi) = 2g(x, df(x)) > 0$. In particular, the domain $\mathcal{O}_2$ is strictly convex in the usual sense iff Γ is g^0-strictly concave with respect to $\mathcal{O}$ in the sense of this definition, where $g^0(x, \xi) = |\xi|^2$.

The Hamiltonian g induces a Hamiltonian r on $T^*\Gamma$ as follows. Identifying any $\xi' \in T^*_{x'}\Gamma$ with the covector $\xi = j(\xi') \in T^*_{x'}R^n$, such that $\xi_{|T_{x'}\Gamma} = \xi'$ and $\xi(\nu'(x')) = 0$, we set $r(x', \xi') = g(x', j(\xi'))$. It is easy to describe r in the so called "normal" to the boundary local coordinates $y = (y', y_n) \in \Gamma \times [0, \delta), 0 < \delta \ll 1$, where y' are local coordinates in Γ and

$$x = y'(x) + y_n(x)\nu'(y'(x)).$$

In these coordinates, the principal symbol of $-\Delta_g$ becomes

$$g(y, \eta) = \eta_n^2 + r(y', \eta') + y_n r_1(y, \eta'),$$

where $r(y', \eta')$ is the induced Hamiltonian. Moreover, r is just the principal symbol of the Laplace-Beltrami operator on Γ equipped with the Riemannien metric on Γ induced by the metric $\mathcal{G}$, while

$$2^{-1} r_1(y', 0, \eta', 0) = -4^{-1}\{g, \{g, y_n\}\}(y', 0, \eta', 0) \geq C|\eta'|^2, \quad C > 0,$$

could be identified with the second fundamental form of Γ (associated with ν'). Similarly, we define $r_0(y', \eta')$ for the free Laplacian $\Delta = \sum_{j=1}^n \partial_{x_j}^2$.

Fix a constant $\alpha > 0$ and introduce the Hilbert space $H = L^2(\mathcal{O}; \alpha^{-1}c(x)^{-2}dx) \oplus L^2(\Omega; dx)$. Consider the operator

$$Pu := (\Delta_g u_1, \Delta u_2), \qquad u = (u_1, u_2) \in D(P),$$

with domain of definition

$$D(P) = \{(u_1, u_2) \in H : u_1 \in H^2(\mathcal{O}), u_2 \in H^2(\Omega), Bu_1|_{\Gamma_1} = 0, u_1|_\Gamma = u_2|_\Gamma, \\ \partial_{\nu'} u_1|_\Gamma + \alpha \partial_\nu u_2|_\Gamma = 0\},$$

where ν is the outer unit normal to Γ with respect to the Euclidean metric, B denotes either the Dirichlet or Neumann boundary condition, and we omit the boundary conditions on Γ_1 if $\mathcal{O}_1 = \emptyset$. In the same way as in the case of Dirichlet (Neumann) exterior problems, one can see that P is a selfadjoint, elliptic operator, $P \leq 0$, and the spectrum of P is absolutely continuous with no embedded eigenvalues. As usually we define the resonances associated to P as the poles of the meromorphic continuation of the cutoff resolvent

$$R_\chi(\lambda) := \chi(P+\lambda^2)^{-1}\chi : H \to H$$

from $\mathrm{Im}\,\lambda < 0$ to the whole complex plane C if n is odd, and to the logarithmic Riemann surface if n is even. Here $\chi \in C_0^\infty(R^n)$, $\chi = 1$ on $\mathcal{O}_2$. When $\mathcal{O}_1 = \emptyset$ and $g(x,\xi) = c^2|\xi|^2, c = Const < 1$, it is proved in [5] that there exists an infinite sequence $\{\lambda_j\}$ of different resonances of P such that $\mathrm{Im}\,\lambda_j = O(|\lambda_j|^{-\infty})$, which is due to the existence of the so called totally reflected interior rays in $\mathcal{O}$ near the glancing region $\mathcal{K} = \{\zeta \in T^*\Gamma : c\|\zeta\| = 1\}$. Hereafter, given a $\zeta = (x', \xi') \in T^*\Gamma$, we set $\|\zeta\|^2 := r_0(x', \xi')$. It is easy to see from the proof in [5], however, that this result still holds in the more general setting described above, provided that Γ is g-strictly concave with respect to $\mathcal{O}$ and $r(x', \xi') < r_0(x', \xi')$, $\forall (x', \xi') \in T^*\Gamma \setminus 0$. On the other hand, when $\mathcal{O}_1 = \emptyset$ and $g(x,\xi) = c^2|\xi|^2, c = Const > 1$, (then we have $r(x', \xi') > r_0(x', \xi')$, $\forall (x', \xi') \in T^*\Gamma \setminus 0$), it is proved in [6] that there exists a region free of resonances of the form

$$\{\lambda \in C : \mathrm{Im}\,\lambda \le C_1|\lambda|^{-1}, |\lambda| \ge C_2\}.$$

The purpose of the present work is to improve this result.

Let us have a look at the classical system corresponding to the transmission problem in our case. Any ray coming from the interior splits into two when it hits boundary Γ. One of them reflects by the usual law of the geometric optics and keeps moving in $\mathcal{O}$, and the other one leaves the obstacle. Moving only on the internal rays, we stay in the obstacle, hence, there is a lot of rays trapped by the obstacle. On the other hand, any time when the ray hits Γ, a portion of its energy goes out of the obstacle.

We make the following assumption:

(1.2) There exists $T > 0$ such that for any generalized g-geodesic $\gamma(t)$ with $\gamma(0) \in \overline{\mathcal{O}}$ there is $t = t_\gamma,\ 0 \le t \le T$, such that $\gamma(t) \in \Gamma$.

Recall that any generalized g-geodesic is a projection in R^n_x of a generalized null bicharacteristic of the Hamiltonian $g(x,\xi) - 1$ as defined by Melrose and Sjöstrand [3].

Clearly, (1.2) is fulfilled when $\mathcal{O}_1 = \emptyset$ and $g(x,\xi) = c^2|\xi|^2, c = Const$. Our main result is the following

Theorem 1.1. *Let Γ be both g- and g^0-strictly concave with respect to $\mathcal{O}$ and let the assumption (1.2) be fulfilled. Suppose that*

$$r(x', \xi') > r_0(x', \xi') \qquad \forall (x', \xi') \in T^*\Gamma \setminus 0. \tag{1.3}$$

Then there exist positive constants C_1 and C_2 so that

$$\|\lambda R_\chi(\lambda)\|_{\mathcal{L}(H)} \le C_1 \qquad \text{for } \lambda \in R,\ |\lambda| \ge C_2. \tag{1.4}$$

Note that the same estimate of the cutoff resolvent on the real axis holds for nontrapping perturbations of the Laplacian. Here *nontrapping* means that every generalized geodesic leaves any compact in a finite time (see (A.3)). Clearly, the operator P is not a nontrapping perturbation of the Laplacian in the sense of this definition.

As an immediate consequence of this theorem (e.g., see [9]) we get the following

Corollary 1.2. *Under the same assumptions as in Theorem* 1.1*, there exists a constant* $\gamma > 0$ *so that there are no resonances of* P *in the strip* $0 \leq Im\lambda \leq \gamma$*.*

Note that for nontrapping perturbations there is a larger free of resonances region of the form $\operatorname{Im}\lambda \leq N\log|\lambda| - C_N, \forall N > 0$. Moreover, in the case of scattering by strictly convex obstacles (with Dirichlet boundary conditions) there is a free region of the form $\operatorname{Im}\lambda \leq C_1|\lambda|^{1/3} - C_2$, $C_1, C_2 > 0$ (see [2]). It is easy to check, however, that when $\mathcal{O}_1 = \emptyset$, $\mathcal{O}_2$ is a ball, and $g(x,\xi) = c^2|\xi|^2, c = Const > 1$, there exist infinitely many resonances $\{\lambda_j\}$ of P such that $\operatorname{Im}\lambda_j \to \gamma_0 > 0$. This example shows that one can not expect to obtain for P a free of resonances region near the real axis larger than a strip.

It is proved in [9] that (1.4) implies an uniform decay of the local energy of the solutions of the corresponding wave equation. More precisely, denote by $u(t)$ the solution of the equation

$$\begin{cases} (\partial_t^2 - P)u(t) = 0, \\ u(0) = f_1, \partial_t u(0) = f_2. \end{cases}$$

Given any compact $K \subset \overline{\Omega}_1$, set

$$p_0(t) = \sup\left\{ \frac{\|\nabla_x u\|_{L^2(K)} + \|\partial_t u\|_{L^2(K)}}{\|\nabla_x f_1\|_{L^2(K)} + \|f_2\|_{L^2(K)}}, (0,0) \neq (f_1, f_2) \in [C^\infty(\overline{\mathcal{O}}) \oplus C^\infty(\overline{\Omega})]^2, \operatorname{supp} f_j \subset K \right\}.$$

According to the result of [9], the above theorem implies the following

Corollary 1.3. *Under the same assumptions as in Theorem* 1.1*, there exist constants* $C, \beta > 0$ *so that*

$$p_0(t) \leq \begin{cases} Ce^{-\beta t}, & n \text{ odd}, \\ Ct^{-n}, & n \text{ even}. \end{cases}$$

In other words, we have the same uniform decay of the local energy as in the case of nontrapping perturbations.

To prove (1.4) we first obtain in Section 2 a precise à priori estimate of the solutions of boundary value problems in $\mathcal{O}$ under the assumption (1.2). This assumption allows to extend the Hamiltonian g to a nontrapping Hamiltonian $\widetilde{g}$ in $T^*\Omega_1$ (see the appendix), which in turn allows to make use of the results of Melrose-Sjöstrand [3], [4] on the propagation of the singularities. Thus we localize our à priori estimate near the boundary Γ, where we make use in an essential way of the fact that Γ is g-strictly concave. In Section 3 we obtain à priori estimates of the solutions of boundary value problems in Ω using the parametrix of the solutions of the Helmholtz equation in the exterior of a strictly convex obstacle studied in the appendices of [1], [7]. Finally, in Section 4 we combine these à priori estimates together with (1.3) to get (1.4).

2. A priori estimates of the solutions of interior boundary value problems

Let $\mathcal{O} \subset R^n$ be as in the introduction and let $u \in H^2(\mathcal{O})$ satisfy the equation

$$\begin{cases} (\Delta_g + \lambda^2)u = \lambda v \quad \text{in} \quad \mathcal{O}, \\ Bu|_{\Gamma_1} = 0, \\ u|_\Gamma = f, \quad \partial_{\nu'} u|_\Gamma = \lambda h. \end{cases} \tag{2.1}$$

Throughout this paper $H^1(K)$ will denote the Sobolev space equipped with the norm

$$\|w\|_{H^1(K)} := \sum_{0\le|\alpha|\le 1} \|(\lambda^{-1}\partial_x)^\alpha w\|_{L^2(K)}.$$

Also, given a symbol $\chi \in C^\infty(T^*\Gamma)$, $\mathrm{Op}_\lambda(\chi)$ will denote the $\lambda - \Psi DO$ defined by

$$[\mathrm{Op}_\lambda(\chi)f](x) := \left(\frac{\lambda}{2\pi}\right)^{n-1} \int\int e^{i\lambda\langle x-y,\xi\rangle}\chi(x,\xi)f(y)d\xi dy.$$

Set

$$M = \|v\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\Gamma)} + \|h\|_{L^2(\Gamma)}.$$

The main result of this section is the following:

Theorem 2.1. *Let Γ be both g- and g_0- strictly concave with respect to $\mathcal{O}$, and let the assumption* (1.2) *be fulfilled. Then there exist constants $C, \lambda_0 > 0$ so that for real $\lambda \ge \lambda_0$ we have*

$$\|u\|_{H^1(\mathcal{O})} \le C\,M. \tag{2.2}$$

Proof. Choose a real-valued function $\varphi(t) \in C_0^\infty(R), 0 \le \varphi \le 1, \varphi = 1$ for $|t| \le \delta/2, \varphi = 0$ for $|t| \ge \delta, d\varphi(t)/dt \le 0$ for $t \ge 0$, where $0 < \delta \ll 1$. Given a $x \in \mathcal{O}$, denote by $d(x)$ the distance between x and Γ along the geodesics of g. Since Γ is g-strictly concave with respect to $\mathcal{O}$, it is well known that $d(x) \in C^\infty(\mathcal{O})$ if $d(x) \le \delta$. Hence $\psi(x) := \varphi(d(x)) \in C^\infty(\mathcal{O})$, $\psi = 1$ near Γ. We will derive (2.2) from the following:

Proposition 2.2. *Let Γ be g-strictly concave with respect to $\mathcal{O}$. Then there exist constants $C, \delta_0 > 0$ so that for every $\delta \in (0, \delta_0]$ we have*

$$\|\psi u\|_{H^1(\mathcal{O})} \leq C\, M + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})}. \tag{2.3}$$

Let $\widetilde{\varphi}(t) \in C_0^\infty(R), \widetilde{\varphi} = 1$ for $|t| \leq \delta/4, \widetilde{\varphi} = 0$ for $|t| \geq \delta/3$, and set $\widetilde{\psi}(x) = \widetilde{\varphi}(d(x))$. According to Proposition A.1, under the assumption (1.2) there exists a nontrapping Hamiltonian $\widetilde{g}(x,\xi)$ on $T^*\Omega_1$ of the form

$$\widetilde{g}(x,\xi) = \widetilde{c}(x)^2 \sum_{i,j=1}^{n} \widetilde{g}_{ij}(x)\xi_i\xi_j,$$

where $\widetilde{c}(x) = c(x), \widetilde{g}_{ij}(x) = g_{ij}(x)$ in $\mathcal{O}$, $\widetilde{c}(x) = 1, \widetilde{g}_{ij}(x) = \delta_{ij}$ outside some compact. Denote by $\Delta_{\widetilde{g}}$ the selfadjoint realisation of the Laplace-Beltrami operator corresponding to $\widetilde{g}$ with boundary condition B on Γ_1. Define the resolvent

$$\widetilde{R}(\lambda) = (\Delta_{\widetilde{g}} + \lambda^2)^{-1} : L^2(\Omega_1; \widetilde{c}(x)^{-2}dx) \to L^2(\Omega_1; \widetilde{c}(x)^{-2}dx) \qquad \text{for } \operatorname{Im}\lambda < 0.$$

Since $\widetilde{g}$ is nontrapping, it follows from the results in [3], [4] on the propagation of the singularities combined with the results in [8] that for real $\lambda \gg 1$ we have

$$\widetilde{R}_\chi(\lambda) := \chi\widetilde{R}(\lambda)\chi = O(\lambda^{-1}) : H^{s_1}(\Omega_1) \to H^{s_2}(\Omega_1), \quad s_j = 0, 1.$$

If we extend u as zero in Ω, we can write

$$\begin{cases} (\Delta_{\widetilde{g}} + \lambda^2)(1-\widetilde{\psi})u &= [\widetilde{\psi}, \Delta_{\widetilde{g}}]u + (1-\widetilde{\psi})\lambda v \qquad \text{in} \quad \Omega_1, \\ B(1-\widetilde{\psi})u|_{\Gamma_1} &= 0, \end{cases}$$

and hence

$$(1-\widetilde{\psi})u = \widetilde{R}_\chi(\lambda)\left([\widetilde{\psi}, \Delta_{\widetilde{g}}]u + (1-\widetilde{\psi})\lambda v\right).$$

Thus we get

$$\begin{aligned} &\|(1-\widetilde{\psi})u\|_{H^1(\mathcal{O})} \leq \\ &\quad \|\widetilde{R}_\chi(\lambda)[\widetilde{\psi}, \Delta_{\widetilde{g}}]\|_{\mathcal{L}(H^1(\Omega_1))}\|\psi u\|_{H^1(\mathcal{O})} + \lambda\|\widetilde{R}_\chi(\lambda)\|_{\mathcal{L}(L^2(\Omega_1), H^1(\Omega_1))}\|v\|_{L^2(\mathcal{O})} \\ &\qquad\qquad \leq C_1\|\psi u\|_{H^1(\mathcal{O})} + C_1\|v\|_{L^2(\mathcal{O})}. \end{aligned}$$

Therefore,

$$\|u\|_{H^1(\mathcal{O})} \leq \|\widetilde{\psi}u\|_{H^1(\mathcal{O})} + \|(1-\widetilde{\psi})u\|_{H^1(\mathcal{O})} \leq C_2\|\psi u\|_{H^1(\mathcal{O})} + C_2\|v\|_{L^2(\mathcal{O})}. \tag{2.4}$$

It is easy to see that (2.2) follows from (2.3) and (2.4). □

Proof of Proposition 2.2. Set $P = -\lambda^{-2}\Delta_g - 1$. We are going to write P in local coordinates near the boundary. Fix a $\rho_0 \in \Gamma$ and let $V \subset R^n$ be a small neighbourhood of ρ_0. Choose local coordinates (x', x_n) in V so that $\rho_0 = (0,0), x_n > 0$ is the distance from a point $\rho \in V \cap \mathcal{O}$ to Γ, $V \cap \Gamma$ is given by $x_n = 0$, while $x' \in R^{n-1}$ are coordinates on $V \cap \Gamma$. In these coordinates P is written in the form

$$P = \mathcal{D}_n^2 - Q$$

where $\mathcal{D}_j := \mathcal{D}_{x_j} := (i\lambda)^{-1}\partial_{x_j}$, Q is a second-order differential operator of the form $Q = Q_0 + Q_1$, where $Q_0 = q(x', x_n, \mathcal{D}_{x'})$ is symmetric with respect to the scalar product in $L^2(R^{n-1}), \forall x_n \geq 0$ small enough, and Q_1 is a first-order differential operator of the form $Q_1 = \lambda^{-1} q_1(x, \mathcal{D}_x)$. More precisely, the principal symbol, q_0, of Q_0 is of the form

$$q_0(x', x_n, \xi') = 1 - r(x', \xi') - x_n \widetilde{q}_0(x', \xi') + O(x_n^2),$$

where r is just the Hamiltonian induced on $T^*\Gamma$,

$$\widetilde{q}_0(x', \xi') = \{g, \{g, x_n\}\}(x', 0, \xi', 0).$$

In view of (1.1), we have

$$C_1|\xi'|^2 \leq r(x', \xi') \leq C_2|\xi'|^2, \quad C_1|\xi'|^2 \leq \widetilde{q}_0(x', \xi') \leq C_2|\xi'|^2,$$

for some constants $C_1, C_2 > 0$. Let $\widetilde{U} \subset U$ be small neighbourhoods of 0 on the hyperplane $x_n = 0$, and let $\eta \in C_0^\infty(U), \eta = 1$ on $\widetilde{U}$. Set $w = \eta u$. Throughout this section $\|\cdot\|_0, \|\cdot\|_1, \|\cdot\|_{0,+}$ and $\|\cdot\|_{1,+}$ will denote the norms in the spaces $L^2(R^{n-1}), H^1(R^{n-1}), L^2(R^{n-1} \times R^+)$ and $H^1(R^{n-1} \times R^+)$, respectively, while $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_+$ will denote the scalar products in $L^2(R^{n-1})$ and $L^2(R^{n-1} \times R^+)$, respectively. We have

$$Pw = \eta P u + [\eta, Q]u,$$

and hence

$$\|Pw(\cdot, x_n)\|_0 \leq C\|Pu(\cdot, x_n)\|_{L^2(U)} + C\lambda^{-1}\|u(\cdot, x_n)\|_{H^1(U)}. \tag{2.5}$$

Let φ be as above with $\delta > 0$ small enough. We have

$$P(\varphi w) = \varphi P w - \lambda^{-2}(\varphi'' w + 2\varphi' \partial_{x_n} w) - [Q_1, \varphi]w.$$

Here φ' and φ'' denote the first and the second derivative, respectively. Integrating by parts gives

$$\int_0^\infty \mathcal{D}_n^2(\varphi w) \cdot \overline{\varphi w} dx_n = \int_0^\infty |\mathcal{D}_n(\varphi w)|^2 dx_n + i\lambda^{-1} \overline{w}|_{x_n=0} \cdot \mathcal{D}_n w|_{x_n=0}. \tag{2.6}$$

This implies

$$-\int_0^\infty Q_0(\varphi w) \cdot \overline{\varphi w} dx_n + \int_0^\infty |\mathcal{D}_n(\varphi w)|^2 dx_n \leq O(\lambda^{-1})(|w|_{x_n=0}|^2 + |\mathcal{D}_n w|_{x_n=0}|^2)$$
$$+ O(\lambda^{-1}) \int_0^\delta \sum_{0 \leq |\alpha| \leq 1} |\mathcal{D}_x^\alpha w|^2 dx_n + O(\lambda) \int_0^\delta |Pw|^2 dx_n.$$

Integrating this inequality with respect to x' leads to

$$-\langle Q_0(\varphi w), \varphi w \rangle_+ + \|\mathcal{D}_n(\varphi w)\|_{0,+}^2 \leq O(\lambda^{-1})(M^2 + \|u\|_{H^1(\mathcal{O})}^2). \tag{2.7}$$

For $x_n \geq 0$ set

$$E(x_n) = \langle Q_0(\varphi w(\cdot, x_n)), \varphi w(\cdot, x_n)\rangle + \|\mathcal{D}_n(\varphi w(\cdot, x_n))\|_0^2,$$

and

$$F(x_n) = \langle [\partial_{x_n}, Q_0](\varphi w(\cdot, x_n)), \varphi w(\cdot, x_n)\rangle.$$

We have

$$\begin{aligned} E'(x_n) &:= \frac{dE(x_n)}{dx_n} \\ &= F(x_n) + 2\mathrm{Re}\langle Q_0(\varphi w(\cdot, x_n)), \partial_{x_n}(\varphi w)(\cdot, x_n)\rangle \\ &\qquad\qquad - 2\mathrm{Re}\langle \mathcal{D}_n^2(\varphi w(\cdot, x_n)), \partial_{x_n}(\varphi w)(\cdot, x_n)\rangle \\ &= F(x_n) - 2\mathrm{Re}\langle Q_1(\varphi w(\cdot, x_n)), \partial_{x_n}(\varphi w)(\cdot, x_n)\rangle \\ &\qquad\qquad - 2\mathrm{Re}\langle P(\varphi w(\cdot, x_n)), \partial_{x_n}(\varphi w)(\cdot, x_n)\rangle \\ &\leq F(x_n) + 2|\langle \lambda Q_1(\varphi w(\cdot, x_n)), \mathcal{D}_n(\varphi w)(\cdot, x_n)\rangle| \\ &\quad + 2\lambda|\langle \varphi P w(\cdot, x_n)), \mathcal{D}_n(\varphi w)(\cdot, x_n)\rangle| \\ &\quad + 2|\langle \lambda[Q_1, \varphi] w(\cdot, x_n)), \mathcal{D}_n(\varphi w)(\cdot, x_n)\rangle| \\ &\quad + 4\varphi\varphi'\|\mathcal{D}_n w(\cdot, x_n)\|_0^2 + 2\lambda^{-1}|\varphi''\varphi + 2\varphi'^2||\langle w(\cdot, x_n), \mathcal{D}_n w(\cdot, x_n)\rangle| \\ &\quad + 2\lambda^{-2}|\varphi'\varphi''|\|w(\cdot, x_n)\|_0^2. \end{aligned}$$

Since $\varphi\varphi' \leq 0$, we get,$\forall \beta > 0$,

$$\begin{aligned} E'(x_n) &\leq F(x_n) + O(\beta)\|\varphi w(\cdot, x_n)\|_1^2 + \\ &\quad O_\delta(\lambda^{-1})\|w(\cdot, x_n)\|_0^2 + O_\delta(\lambda^{-1})\|\mathcal{D}_n w(\cdot, x_n)\|_0^2 \\ &\quad + \beta^{-1}\|\mathcal{D}_n(\varphi w)(\cdot, x_n)\|_0^2 + O(\lambda^2\beta)\|\varphi P w(\cdot, x_n)\|_0^2. \end{aligned} \tag{2.8}$$

Taking $\beta = 1$, for $0 \leq x_n \leq \delta$, we get

$$\begin{aligned} E(x_n) &= \int_0^{x_n} E'(t)dt + E(0) \\ &\leq O(1)M^2 + O(1)\|\eta f\|_{H^1(\Gamma)}^2 + O_\delta(\lambda^{-1})\|u\|_{H^1(\mathcal{O})}^2 + \\ &\qquad O(1)\|\varphi w\|_{1,+}^2 + O(\lambda^2)\|\varphi P w\|_{0,+}^2. \end{aligned}$$

Integrating this inequality gives

$$\begin{aligned} &\langle Q_0(\varphi w), \varphi w\rangle_+ + \|\mathcal{D}_n(\varphi w)\|_{0,+}^2 \\ &\quad \leq O(\delta)M^2 + O(\delta)\|\eta f\|_{H^1(\Gamma)}^2 + O_\delta(\lambda^{-1})\|u\|_{H^1(\mathcal{O})}^2 \\ &\qquad + O(\delta)\|\varphi w\|_{1,+}^2 + O(\delta\lambda^2)\|\varphi P w\|_{0,+}^2. \end{aligned} \tag{2.9}$$

Combining (2.7) and (2.9) leads to

$$\begin{aligned} \|\mathcal{D}_n(\varphi w)\|_{0,+} &\leq O(\delta^{1/2})M + O(\delta^{1/2})\|\eta f\|_{H^1(\Gamma)} \\ &\quad + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\delta^{1/2})\|\varphi w\|_{1,+} + O(\delta^{1/2}\lambda)\|\varphi P w\|_{0,+}. \end{aligned} \tag{2.10}$$

Furthermore, taking $\beta = \delta^{1/2}$ in (2.8) and combining with (2.10) gives

(2.11)
$$-\int_0^\infty F(x_n)dx_n \leq O(\delta^{1/2})M^2 + O(\delta^{1/2})\|\eta f\|^2_{H^1(\Gamma)} + O_\delta(\lambda^{-1})\|u\|^2_{H^1(\mathcal{O})} + O(\delta^{-1/2})\|\mathcal{D}_n(\varphi w)(\cdot, x_n)\|_0^2 + O(\delta^{1/2})\|\varphi w\|^2_{1,+} + O(\delta^{1/2}\lambda^2)\|\varphi P w\|^2_{0,+}$$
$$\leq O(\delta^{1/2})M^2 + O(\delta^{1/2})\|\eta f\|^2_{H^1(\Gamma)} + O_\delta(\lambda^{-1})\|u\|^2_{H^1(\mathcal{O})} + O(\delta^{1/2})\|\varphi w\|^2_{1,+} + O(\delta^{1/2}\lambda^2)\|\varphi P w\|^2_{0,+}.$$

Let $\chi_1 \in C^\infty(T^*\Gamma), \chi_1 = 1$ for $r(x', \xi') \leq 1 - 2\varepsilon_0, \chi_1 = 0$ for $r(x', \xi') \geq 1 - \varepsilon_0, 0 < \varepsilon_0 \ll 1$. Then, if $\delta > 0$ is small enough, we clearly have

$$\langle Q_0 \mathrm{Op}_\lambda(\chi_1)\varphi w, \mathrm{Op}_\lambda(\chi_1)\varphi w\rangle_+ \geq C_1\|\mathrm{Op}_\lambda(\chi_1)\varphi w\|^2_{0,+}, \qquad C_1 > 0. \tag{2.12}$$

It is easy to see that (2.9) holds with φw replaced by $\mathrm{Op}_\lambda(\chi_1)\varphi w$ (and hence with ηf replaced by $\mathrm{Op}_\lambda(\chi_1)\eta f$ in the RHS). Therefore, using that

$$\|\mathrm{Op}_\lambda(\chi_1)\eta f\|_{H^1(\Gamma)} \leq C\|f\|_{L^2(\Gamma)},$$

by (2.9) and (2.12) we conclude

$$\|\mathrm{Op}_\lambda(\chi_1)\varphi w\|_{0,+} \leq O(\delta^{1/2})M + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\delta^{1/2})\|\varphi w\|_{1,+} + O(\delta^{1/2}\lambda)\|\varphi P w\|_{0,+}. \tag{2.13}$$

Let $\chi_2 \in C^\infty(T^*\Gamma), \chi_2 = 1$ for $1 - \varepsilon_0 \leq r(x', \xi') \leq 1 + \varepsilon_0, \chi_2 = 0$ for $r(x', \xi') \leq 1 - 2\varepsilon_0$ and $r(x', \xi') \geq 1 + 2\varepsilon_0$. Since the principal symbol of $[\partial_{x_n}, Q_0]$ is equal to $\partial_{x_n} q_0 = -\widetilde{q}_0(x', \xi') + O(x_n) \leq -C < 0$ on $\mathrm{supp}\chi_2$ provided $0 \leq x_n \leq \delta$, $\delta > 0$ small enough, we have

$$-\langle [\partial_{x_n}, Q_0]\mathrm{Op}_\lambda(\chi_2)\varphi w, \mathrm{Op}_\lambda(\chi_2)\varphi w\rangle_+ \geq C_2\|\mathrm{Op}_\lambda(\chi_2)\varphi w\|^2_{0,+}, \quad C_2 > 0. \tag{2.14}$$

It is easy to see that (2.11) holds with φw replaced by $\mathrm{Op}_\lambda(\chi_2)\varphi w$. As above, we get by (2.11) and (2.14),

$$\|\mathrm{Op}_\lambda(\chi_2)\varphi w\|_{0,+} \leq O(\delta^{1/4})M + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\delta^{1/4})\|\varphi w\|_{1,+} + O(\delta^{1/4}\lambda)\|\varphi P w\|_{0,+}. \tag{2.15}$$

Let $\chi_3 \in C^\infty(T^*\Gamma), \chi_3 = 1$ for $r(x', \xi') \geq 1 + 2\varepsilon_0, \chi_3 = 0$ for $r(x', \xi') \leq 1 + \varepsilon_0$. Since $-q_0 \geq C(1 + |\xi'|^2), C > 0$, on $\mathrm{supp}(\varphi\chi_3)$, we have

$$-\langle Q_0 \mathrm{Op}_\lambda(\chi_3)\varphi w, \mathrm{Op}_\lambda(\chi_3)\varphi w\rangle_+ \geq C_3\int_0^\infty \|\mathrm{Op}_\lambda(\chi_3)\varphi w(\cdot, x_n)\|_1^2 dx_n, \quad C_3 > 0. \tag{2.16}$$

It is easy to see that (2.7) holds with φw replaced by $\mathrm{Op}_\lambda(\chi_3)\varphi w$. Therefore, by (2.7) and (2.16),

$$\|\mathrm{Op}_\lambda(\chi_3)\varphi w\|_{1,+} \leq O(\lambda^{-1/2})M + O(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})}. \tag{2.17}$$

Summing up (2.10), (2.13), (2.15) and (2.17), and taking $\delta > 0$ small enough lead to the following estimate

$$\|\varphi w\|_{1,+} \leq O(\delta^{1/4})M + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\delta^{1/4})\|\varphi w\|_{1,+} + O(\delta^{1/4}\lambda)\|\varphi P w\|_{0,+}. \tag{2.18}$$

It follows from (2.18) and (2.5) that

$$\|\varphi u\|_{H^1(\widetilde{U}\times R^+)} \leq O(\delta^{1/4})M + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\delta^{1/4})\|\varphi u\|_{H^1(U\times R^+)}.$$

Thus, by a partition of the unity on Γ we conclude

$$\|\psi u\|_{H^1(\mathcal{O})} \leq O(\delta^{1/4})M + O_\delta(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\delta^{1/4})\|\psi u\|_{H^1(\mathcal{O})},$$

which, after taking $\delta > 0$ small enough, independent of λ, implies (2.3). □

Consider now the problem

$$\begin{cases} (\Delta_g + \lambda^2)u = \lambda v \quad \text{in} \quad \mathcal{O}, \\ u|_\Gamma = f, \\ \lambda^{-1}\partial_{\nu'} u|_\Gamma + A(\lambda)f = h, \end{cases} \tag{2.19}$$

where

$$A(\lambda) = O(1) : H^1(\Gamma) \to L^2(\Gamma). \tag{2.20}$$

Suppose that

$$\operatorname{Re}\langle A(\lambda)f, f\rangle_{L^2(\Gamma)} \leq o(1)\|f\|^2_{L^2(\Gamma)}, \qquad \forall f \in H^1(\Gamma). \tag{2.21}$$

We also suppose that for any $\chi \in C^\infty(T^*\Gamma)$ which is equal either to zero or to $(1 + |\xi'|)^s, s = 0, 1$, outside some compact, we have

$$\|[\mathrm{Op}_\lambda(\chi), A(\lambda)]\|_{\mathcal{L}(H^s(\Gamma))} = o(1), \quad s = 0, 1. \tag{2.22}$$

Choose now a function $\chi \in C^\infty(T^*\Gamma), \chi = 1$ for $r(x', \xi') \leq 1 + \varepsilon_0, \chi = 0$ for $r(x', \xi') \geq 1 + 2\varepsilon_0, 0 < \varepsilon_0 \ll 1$.

Proposition 2.3. *Under the assumptions* (2.20)–(2.22), *we have*

$$\| Op_\lambda(1-\chi)f\|_{H^1(\Gamma)} \leq O(1)\|h\|_{L^2(\Gamma)} + o(1)\|v\|_{L^2(\mathcal{O})} + o(1)\|f\|_{L^2(\Gamma)} + o(1)\|u\|_{L^2(\mathcal{O})}. \tag{2.23}$$

Proof. We will use the same notations as in the proof of Proposition 2.2. Using (2.6), we obtain

$$\begin{aligned} &- \langle Q_0\mathrm{Op}_\lambda(\chi_3)\varphi w, \mathrm{Op}_\lambda(\chi_3)\varphi w\rangle_+ + \|\mathcal{D}_n\mathrm{Op}_\lambda(\chi_3)\varphi w\|^2_{0,+} \\ &\quad \leq \lambda^{-1}\mathrm{Re}\,\langle A(\lambda)\mathrm{Op}_\lambda(\chi_3)\eta f, \mathrm{Op}_\lambda(\chi_3)\eta f\rangle + O_\gamma(\lambda^{-1})\|h\|^2_{L^2(\Gamma)} \\ &\qquad + O(\gamma\lambda^{-1})\|\mathrm{Op}_\lambda(\chi_3)\eta f\|^2_0 + O(\gamma)\|\mathrm{Op}_\lambda(\chi_3)\varphi w\|^2_{1,+} \\ &\qquad + \lambda^{-1}o(1)\|f\|^2_{L^2(\Gamma)} + O_\gamma(\lambda^{-2})\|u\|^2_{H^1(\mathcal{O})} + O_\gamma(\lambda^{-2})\|v\|^2_{L^2(\mathcal{O})}, \end{aligned}$$

for any $\gamma > 0$. Since the principal symbol of $-Q_0$ is $\geq C(1+|\xi'|^2), C > 0$, on $\operatorname{supp}\chi_3$, $0 \leq x_n \leq \delta$, $\delta > 0$ small enough, in view of (2.21) we get, taking $\gamma > 0$ small enough,

$$\text{(2.24)}\quad \lambda\|\mathrm{Op}_\lambda(\chi_3)\varphi w\|_{1,+}^2 \leq O_\gamma(1)\|h\|_{L^2(\Gamma)}^2 + O(\gamma)\|\mathrm{Op}_\lambda(\chi_3)\eta f\|_0^2 + o(1)\|f\|_{L^2(\Gamma)}^2 + O_\gamma(\lambda^{-1})\|u\|_{H^1(\mathcal{O})}^2 + O_\gamma(\lambda^{-1})\|v\|_{L^2(\mathcal{O})}^2.$$

On the other hand, we have
(2.25)

$$\begin{aligned}\|\mathrm{Op}_\lambda(\chi_3)\eta f\|_0^2 &= -\int_0^\infty \frac{d}{dx_n}\|\mathrm{Op}_\lambda(\chi_3)\varphi w(\cdot,x_n)\|_0^2 dx_n \\ &= -2\int_0^\infty \mathrm{Re}\,\langle \mathrm{Op}_\lambda(\chi_3)\varphi w(\cdot,x_n), \partial_{x_n}\mathrm{Op}_\lambda(\chi_3)\varphi w(\cdot,x_n)\rangle dx_n \\ &\leq O(\lambda)\left(\|\mathrm{Op}_\lambda(\chi_3)\varphi w\|_{0,+}^2 + \|\mathcal{D}_n\mathrm{Op}_\lambda(\chi_3)\varphi w\|_{0,+}^2\right).\end{aligned}$$

By (2.24) and (2.25), taking $\gamma > 0$ small enough, we get

$$\text{(2.26)}\quad \|\mathrm{Op}_\lambda(\chi_3)\eta f\|_0 \leq O(1)\|h\|_{L^2(\Gamma)} + o(1)\|f\|_{L^2(\Gamma)} + O(\lambda^{-1/2})\|u\|_{H^1(\mathcal{O})} + O(\lambda^{-1/2})\|v\|_{L^2(\mathcal{O})}.$$

It is easy to see that (2.26) still holds with χ_3 replaced by $(1+|\xi'|)\chi_3$ in the LHS, and $\|f\|_{L^2(\Gamma)}$ replaced by $\|f\|_{H^1(\Gamma)}$ in the RHS. Therefore, making a suitable partition of the unity on $\operatorname{supp}(1-\chi)$ and taking into account the estimate

$$\|f\|_{H^1(\Gamma)} \leq C\|f\|_{L^2(\Gamma)} + C\|\mathrm{Op}_\lambda(1-\chi)f\|_{H^1(\Gamma)},$$

we conclude

$$\text{(2.27)}\quad \|\mathrm{Op}_\lambda(1-\chi)f\|_{H^1(\Gamma)} \leq O(1)\|h\|_{L^2(\Gamma)} + o(1)\|v\|_{L^2(\mathcal{O})} + o(1)\|f\|_{L^2(\Gamma)} + o(1)\|u\|_{H^1(\mathcal{O})}.$$

On the other hand, by Green's formula we have

$$\lambda^{-2}\int_{\mathcal{O}} g(x,\nabla_x u(x))c(x)^{-2}dx = \|c(x)^{-1}u\|_{L^2(\mathcal{O})}^2 - \lambda^{-1}\mathrm{Re}\,\langle f, h - A(\lambda)f\rangle_{L^2(\Gamma)} - \lambda^{-1}\mathrm{Re}\,\langle c^{-2}u, v\rangle_{L^2(\mathcal{O})}.$$

Hence,

$$\text{(2.28)}\quad \begin{aligned}\|u\|_{H^1(\mathcal{O})} &\leq O(1)(\|u\|_{L^2(\mathcal{O})} + \|v\|_{L^2(\mathcal{O})} + \|h\|_{L^2(\Gamma)} + \|f\|_{H^1(\Gamma)}) \\ &\leq O(1)(\|u\|_{L^2(\mathcal{O})} + \|v\|_{L^2(\mathcal{O})} + \|h\|_{L^2(\Gamma)} + \|f\|_{L^2(\Gamma)} \\ &\quad + \|\mathrm{Op}_\lambda(1-\chi)f\|_{H^1(\Gamma)}).\end{aligned}$$

Now (2.23) follows from (2.27) and (2.28). □

3. A priori estimates of the solutions of exterior boundary value problems

Throughout this section $\Omega \subset R^n$ will be the exterior of a strictly convex domain with C^∞ boundary Γ. Let $u \in H^2_{loc}(\Omega)$ satisfy the equation

$$\left\{ \begin{array}{c} (\Delta + \lambda^2)u = \lambda v \quad \text{in} \quad \Omega, \\ u|_\Gamma = f,\ \lambda^{-1}\partial_\nu u|_\Gamma = h, \\ u\ -\ \lambda - \text{outgoing}, \end{array} \right. \tag{3.1}$$

where $\operatorname{supp} v \subset \Omega_a := \{x \in \Omega : |x| \leq a\}, a \gg 1$. One of the purposes of this section is to prove the following

Theorem 3.1. *There exist constants $C, \lambda_0 > 0$ so that for real $\lambda \geq \lambda_0$ we have*

$$\|u\|_{H^1(\Omega_a)} + \|h\|_{L^2(\Gamma)} \leq C\|v\|_{L^2(\Omega_a)} + C\|f\|_{H^1(\Gamma)}. \tag{3.2}$$

Proof. Observe first that we have

$$\|h\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega_a)} + C\|v\|_{L^2(\Omega_a)} + C\|f\|_{H^1(\Gamma)}, \tag{3.3}$$

with a constant $C > 0$ independent of λ. With the same notations as in the proof of Proposition 2.2 we have

$$\begin{aligned} \|\eta h\|_0^2 \leq E(0) + C\|f\|^2_{H^1(\Gamma)} = -\int_0^\infty E'(x_n)dx_n + C\|f\|^2_{H^1(\Gamma)} \\ \leq C\|u\|^2_{H^1(\Omega')} + C\|v\|^2_{L^2(\Omega')} + C\|f\|^2_{H^1(\Gamma)}, \end{aligned} \tag{3.4}$$

where $\Omega' \subset \Omega$ is some small neighbourhood of Γ. Clearly, (3.3) follows from (3.4) by a partition of the unity on Γ.

To estimate the norm of u observe that

$$u = G(\lambda)v + K(\lambda)f,$$

where $G(\lambda)v$ solves the problem

$$\left\{ \begin{array}{rcl} (\Delta + \lambda^2)G(\lambda)v & = & v \quad \text{in} \quad \Omega, \\ G(\lambda)v|_\Gamma & = & 0, \\ G(\lambda)v & - & \lambda - \text{outgoing}, \end{array} \right.$$

and $K(\lambda)f$ solves the problem

$$\left\{ \begin{array}{rcl} (\Delta + \lambda^2)K(\lambda)f & = & 0 \quad \text{in} \quad \Omega, \\ K(\lambda)f|_\Gamma & = & f, \\ K(\lambda)f & - & \lambda - \text{outgoing}. \end{array} \right.$$

Clearly, (3.2) follows from (3.3) and the following

Proposition 3.2. *For real $\lambda \gg 1$, we have*

$$G(\lambda) = O(\lambda^{-1}) : H^{s_1}(\Omega_a) \to H^{s_2}(\Omega_a), \quad s_j = 0, 1, \tag{3.5}$$

and

$$K(\lambda) = O(1) : H^1(\Gamma) \to H^1(\Omega_a). \tag{3.6}$$

Proof. As in the proof of Theorem 2.1, (3.5) follows from the fact that the strictly convex obstacles are nontrapping. In what follows we will derive (3.6) from (3.5). This would follow if there exist a small neighbourhood $\Omega' \subset \Omega$ of Γ and operators

$$\mathcal{H}(\lambda) = O(1) : H^1(\Gamma) \to H^1(\Omega'), \quad \mathcal{K}(\lambda) = O(\lambda^{-\infty}) : H^1(\Gamma) \to L^2(\Omega'), \tag{3.7}$$

such that for any $f \in H^1(\Gamma)$ we have

$$\begin{cases} (\Delta + \lambda^2)\mathcal{H}(\lambda)f &= \mathcal{K}(\lambda)f \quad \text{in} \quad \Omega', \\ \mathcal{H}(\lambda)f|_\Gamma &= f. \end{cases}$$

Indeed, let $\psi \in C^\infty(\overline{\Omega})$, $\text{supp}\psi \subset \overline{\Omega}'$, $\psi = 1$ near Γ. We have

$$\begin{cases} (\Delta + \lambda^2)\psi\mathcal{H}(\lambda)f &= [\Delta, \psi]\mathcal{H}(\lambda)f + \psi\mathcal{K}(\lambda)f \quad \text{in} \quad \Omega, \\ \psi\mathcal{H}(\lambda)f|_\Gamma &= f. \end{cases}$$

Therefore, we can write

$$\begin{cases} (\Delta + \lambda^2)(K(\lambda)f - \psi\mathcal{H}(\lambda)f) &= -[\Delta, \psi]\mathcal{H}(\lambda)f - \psi\mathcal{K}(\lambda)f \quad \text{in} \quad \Omega, \\ (K(\lambda)f - \psi\mathcal{H}(\lambda)f)|_\Gamma &= 0, \end{cases}$$

and hence

$$K(\lambda)f = \psi\mathcal{H}(\lambda)f - G(\lambda)([\Delta, \psi]\mathcal{H}(\lambda)f + \psi\mathcal{K}(\lambda)f). \tag{3.8}$$

Clearly, (3.6) follows from (3.5), (3.7) and (3.8).

Operators $\mathcal{H}(\lambda)$ and $\mathcal{K}(\lambda)$ with the above properties are constructed explicitly in the appendix of [1] (see also the appendix of [7]). In what follows we will recall this construction. Let $\chi_j \in C^\infty(T^*\Gamma), \chi_j \geq 0$, $j = 1, 2, 3$, be real valued functions such that $\chi_1 + \chi_2 + \chi_3 = 1$ and $\text{supp}\,\chi_1 \subset \{\zeta \in T^*\Gamma : \|\zeta\| \leq 1 - \varepsilon_0\}$, $\text{supp}\,\chi_2 \subset \{\zeta \in T^*\Gamma : 1 - 2\varepsilon_0 \leq \|\zeta\| \leq 1 + 2\varepsilon_0\}$ and $\text{supp}\,\chi_3 \subset \{\zeta \in T^*\Gamma : \|\zeta\| \geq 1 + \varepsilon_0\}$, $0 < \varepsilon_0 \ll 1$. The operator $\mathcal{H}(\lambda)$ is of the form

$$\mathcal{H}(\lambda) = \mathcal{H}_1(\lambda) + \mathcal{H}_2(\lambda) + \mathcal{H}_3(\lambda),$$

where $\mathcal{H}_1(\lambda)$ is a finite sum of operators with kernels which can be written in local coordinates $x = (x', x_n) \in \Omega', 0 \leq x_n \ll 1, x' \in \Gamma, z' \in \Gamma$, in the form

$$\mathcal{A}_1(x, z') = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int e^{i\lambda(\theta_1(x,\xi') - \langle z', \xi' \rangle)} a(x, \xi', \lambda) d\xi',$$

where the amplitude a is obtained by solving the transport equations, the support of a with respect to x' and ξ' being contained in $\operatorname{supp}\chi_1$. The phase θ_1 is real valued, satisfies the eikonal equation $|\nabla_x \theta_1|^2 = 1$, $\theta_1|_{x_n=0} = \langle x', \xi' \rangle$, and $-\partial_{x_n}\theta_1|_{x_n=0} \geq C > 0$.

Furthermore, $\mathcal{H}_2(\lambda)$ is a finite sum of operators of the form $\mathcal{A}_2 J^{-1}$, where J is an elliptic $\lambda - FIO$ of class $L^{0,0}_{0,0}(\Gamma)$ and $\mathcal{A}_2$ has a kernel of the form

$$\mathcal{A}_2(x, z') = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int e^{i\lambda(\theta_2(x,\xi') - \langle z', \xi' \rangle)} \left(b_1(x, \xi', \lambda) \frac{Ai_-(\lambda^{2/3}\rho(x, \xi'))}{Ai_-(\lambda^{2/3}\alpha)} \right.$$
$$\left. + i\lambda^{-1/3} b_2(x, \xi', \lambda) \frac{Ai'_-(\lambda^{2/3}\rho(x, \xi'))}{Ai_-(\lambda^{2/3}\alpha)} \right) d\xi',$$

where $\alpha = (r_0(x', \xi') - 1)/\partial_{x_n} g^0(x', 0, \xi', 0)$, $Ai_-(s) := Ai(e^{-2\pi i/3}s)$, $Ai(s)$ being the Airy function. The amplitudes b_1 and b_2 satisfy the corresponding transport equations and their supports with respect to x' and ξ' are contained in $\operatorname{supp}\chi_2$, while the phases θ_2 and ρ satisfy the corresponding eikonal equation. Note that the kernel of the operator J above is equal to $\mathcal{A}_2(x', 0, \xi')$.

Finally, $\mathcal{H}_3(\lambda)$ is a finite sum of operators with kernels of the form

$$\mathcal{A}_3(x, z') = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int e^{i\lambda(\theta_3(x,\xi') - \langle z', \xi' \rangle)} c(x, \xi', \lambda) d\xi',$$

where the amplitude c is obtained by solving the corresponding transport equations, the support of c with respect to x' and ξ' being contained in $\operatorname{supp}\chi_3$. The phase θ_3 satisfies the eikonal equation $\operatorname{mod} O(x_n^\infty)$, $\theta_3|_{x_n=0} = \langle x', \xi' \rangle$, and $\operatorname{Im}\partial_{x_n}\theta_3|_{x_n=0} \geq C > 0$.

The operator $\mathcal{K}$ is a finite sum of operators with kernels as above but with amplitudes which are $O(\lambda^{-\infty})$. It is easy to see that $\mathcal{H}(\lambda)$ and $\mathcal{K}(\lambda)$ have the properties required above. □

Introduce the Neumann operator

$$N(\lambda)f = \lambda^{-1}\partial_\nu K(\lambda) f|_\Gamma.$$

Applying Theorem 3.1 with $v \equiv 0$ leads to the following

Corollary 3.3. *For real* $\lambda \gg 1$, *we have*

$$N(\lambda) = O(1) : H^1(\Gamma) \to L^2(\Gamma). \tag{3.9}$$

In what follows in this section we will prove the following

Proposition 3.4. *For real* $\lambda \gg 1$, *we have*

$$Re\,\langle N(\lambda)f, f\rangle_{L^2(\Gamma)} \leq C\lambda^{-1/3}\|f\|^2_{L^2(\Gamma)}, \qquad \forall f \in H^1(\Gamma), \tag{3.10}$$

with some constant $C > 0$ *independent of* λ.

Proof. Set

$$\mathcal{N}(\lambda)f = \lambda^{-1}\partial_\nu \mathcal{H}(\lambda)f|_\Gamma.$$

It is proved in [1] that

$$\|N(\lambda) - \mathcal{N}(\lambda)\|_{\mathcal{L}(L^2(\Gamma))} = O(\lambda^{-\infty}). \tag{3.11}$$

Therefore, it suffices to prove (3.10) for the operator $\mathcal{N}(\lambda)$. Set

$$\mathcal{N}_j(\lambda)f = \lambda^{-1}\partial_\nu \mathcal{H}_j(\lambda)f|_\Gamma,$$

$j = 1, 2, 3$. We will make use of the analysis of $\mathcal{N}_j(\lambda)$ carried out in the appendix of [1] (see also the appendix of [7]). We will keep the same notations. We have that $\mathcal{N}_1(\lambda)$ is a $\lambda - \Psi DO$ of class $L^{0,0}_{0,0}$, $\mathcal{N}_3(\lambda)$ is a $\lambda - \Psi DO$ of class $L^{1,0}_{0,0}$, while $\mathcal{N}_2(\lambda)$ is a $\lambda - FIO$ of class $L^{0,0}_{2/3,0}$. Clearly, (3.10) would follow from the following

Lemma 3.5. *There exist selfadjoint operators on $L^2(\Gamma)$, T_1 and T_2 with domain of definition $H^1(\Gamma)$, $T_1 \le 0$, so that $\mathcal{N}(\lambda) - T_1 - iT_2$ is a sum of $\lambda - \Psi DO$ and $\lambda - FIO$ of class $L^{0,-1/3}_{2/3,0}$.*

Proof. Since the symbol of $\mathcal{N}_1(\lambda)$ is $-i\chi_1\sqrt{1-\|\zeta\|^2} \mod S^{0,-1}_{0,0}$, we have

$$\mathcal{N}_1(\lambda) = iB_1 \mod L^{0,-1}_{0,0}, \tag{3.12}$$

B_1 being selfadjoint on $L^2(\Gamma)$. Furthermore, since the symbol of $\mathcal{N}_3(\lambda)$ is $-\chi_3\sqrt{\|\zeta\|^2-1} \mod S^{0,-1}_{0,0}$, we have

$$\mathcal{N}_3(\lambda) = -B_2^2 \mod L^{0,-1}_{0,0}, \tag{3.13}$$

B_2 being selfadjoint on $L^2(\Gamma)$. Choose real valued functions $\eta_j \in S^{0,0}_{2/3,0}, \eta_j \ge 0$, $j = 1, 2, 3$, such that $\eta_1 + \eta_2 + \eta_3 = \chi_2$ and $\operatorname{supp}\eta_1 \subset \{\zeta \in T^*\Gamma : 1 - \varepsilon_0 \le \|\zeta\| \le 1 - \lambda^{-2/3}\}$, $\operatorname{supp}\eta_2 \subset \{\zeta \in T^*\Gamma : 1 - \lambda^{-2/3} \le \|\zeta\| \le 1 + \lambda^{-2/3}\}$ and $\operatorname{supp}\eta_3 \subset \{\zeta \in T^*\Gamma : 1 + \lambda^{-2/3} \le \|\zeta\| \le 1 + \varepsilon_0\}$. It follows from the analysis in the appendix of [1] that

$$\mathcal{N}_2(\lambda)\mathrm{Op}_\lambda(\eta_1) = \mathrm{Op}_\lambda(-i\eta_1\sqrt{1-\|\zeta\|^2}) \mod L^{0,-1/3}_{2/3,0},$$

$$\mathcal{N}_2(\lambda)\mathrm{Op}_\lambda(\eta_2) \in L^{0,-1/3}_{2/3,0},$$

$$\mathcal{N}_2(\lambda)\mathrm{Op}_\lambda(\eta_3) = \mathrm{Op}_\lambda(-\eta_3\sqrt{\|\zeta\|^2-1}) \mod L^{0,-1/3}_{2/3,0}.$$

Hence

$$\mathcal{N}_2(\lambda) = -B_3^2 + iB_4 \mod L^{0,-1/3}_{2/3,0}, \tag{3.14}$$

B_3 and B_4 being selfadjoint on $L^2(\Gamma)$. Now the lemma follows from (3.12)–(3.14). □

Remark. It follows from (3.9) and (3.10) that the conditions (2.20)–(2.22) are satisfied with $A(\lambda) = \alpha N(\lambda)$.

4. Estimates of the cutoff resolvent on the real axis

Consider the problem

$$
\left\{
\begin{array}{rcll}
(\Delta_g + \lambda^2)u_1 & = & \lambda v_1 & \text{in} \quad \mathcal{O}, \\
(\Delta + \lambda^2)u_2 & = & \lambda v_2 & \text{in} \quad \Omega, \\
u_1|_\Gamma = u_2|_\Gamma & = & f, & \\
Bu_1|_{\Gamma_1} & = & 0, & \\
\partial_{\nu'} u_1|_\Gamma + \alpha \partial_\nu u_2|_\Gamma & = & 0, & \\
u_2 & - & \lambda - \text{outgoing}, &
\end{array}
\right.
\tag{4.1}
$$

where $u_1 \in H^2(\mathcal{O}), u_2 \in H^2_{loc}(\Omega)$, $\operatorname{supp} v_2 \subset \Omega_a, a \gg 1$. In what follows C will denote a positive constant independent of λ. Clearly, (1.4) is equivalent to the estimate

$$\|u_1\|_{L^2(\mathcal{O})} + \|u_2\|_{L^2(\Omega_a)} \le C\|v_1\|_{L^2(\mathcal{O})} + C\|v_2\|_{L^2(\Omega_a)}. \tag{4.2}$$

By Theorems 2.1 and 3.1 we have

$$\|u_2\|_{L^2(\Omega_a)} \le C\|v_2\|_{L^2(\Omega_a)} + C\|f\|_{H^1(\Gamma)}, \tag{4.3}$$

and

$$\|u_1\|_{L^2(\Omega_a)} \le C\|v_1\|_{L^2(\mathcal{O})} + C\|v_2\|_{L^2(\Omega_a)} + C\|f\|_{H^1(\Gamma)}. \tag{4.4}$$

Hence, to prove (4.2) it suffices to show

$$\|f\|_{H^1(\Gamma)} \le C\|v_1\|_{L^2(\mathcal{O})} + C\|v_2\|_{L^2(\Omega_a)}. \tag{4.5}$$

To this end, observe that $u := u_1, v := v_1$ satisfy the equation

$$
\left\{
\begin{array}{rcll}
(\Delta_g + \lambda^2)u & = & \lambda v & \text{in} \quad \mathcal{O}, \\
u|_\Gamma & = & f, & \\
Bu|_{\Gamma_1} & = & 0, & \\
\lambda^{-1}\partial_{\nu'} u|_\Gamma + \alpha N(\lambda) f & = & h, &
\end{array}
\right.
\tag{4.6}
$$

where $h = -\lambda^{-1}\partial_\nu G(\lambda)v_2|_\Gamma$, in view of Theorem 3.1, satisfies the estimate

$$\|h\|_{L^2(\Gamma)} \le C\|v_2\|_{L^2(\Omega_a)}. \tag{4.7}$$

By Green's formula we have

$$
\begin{aligned}
-\alpha\lambda \mathrm{Im}\,\langle N(\lambda)f, f\rangle_{L^2(\Gamma)} + \lambda \mathrm{Im}\,\langle h, f\rangle_{L^2(\Gamma)} &= \mathrm{Im}\,\langle \partial_{\nu'} u|_\Gamma, u|_\Gamma\rangle_{L^2(\Gamma)} \\
&= -\lambda \mathrm{Im}\,\langle c^{-2}u, v\rangle_{L^2(\mathcal{O})}.
\end{aligned}
$$

Hence, $\forall \varepsilon > 0$, we have

$$
\begin{aligned}
-\mathrm{Im}\,\langle N(\lambda)f, f\rangle_{L^2(\Gamma)} \le\; & O(\varepsilon^2)\|u\|^2_{L^2(\mathcal{O})} + O_\varepsilon(1)\|v\|^2_{L^2(\mathcal{O})} \\
& + O(\varepsilon^2)\|f\|^2_{L^2(\Gamma)} + O_\varepsilon(1)\|h\|^2_{L^2(\Gamma)}.
\end{aligned}
\tag{4.8}
$$

Let $\chi \in C^\infty(T^*\Gamma), \chi = 1$ in $\{(x',\xi') \in T^*\Gamma : r(x',\xi') \leq 1+\varepsilon_0\}, \chi = 0$ in $\{(x',\xi') \in T^*\Gamma : r(x',\xi') \geq 1+2\varepsilon_0\}, 0 < \varepsilon_0 \ll 1$. By (4.8),

$$\begin{aligned} -\operatorname{Im}\langle N(\lambda)\mathrm{Op}_\lambda(\chi)f, \mathrm{Op}_\lambda(\chi)f\rangle_{L^2(\Gamma)} &\leq O(\varepsilon^2)\|u\|^2_{L^2(\mathcal{O})} + O_\varepsilon(1)\|v\|^2_{L^2(\mathcal{O})} \\ &+ O(\varepsilon^2)\|f\|^2_{L^2(\Gamma)} + O_\varepsilon(1)\|h\|^2_{L^2(\Gamma)} + O(1)\|\mathrm{Op}_\lambda(1-\chi)f\|^2_{H^1(\Gamma)}. \end{aligned} \tag{4.9}$$

In view of (1.3) we have $\operatorname{supp}\chi \subset \{\zeta \in T^*\Gamma : \|\zeta\| < 1\}$, and hence $N(\lambda)\mathrm{Op}_\lambda(\chi)$ is a $\lambda - \Psi DO$ with principal symbol $-i\chi\sqrt{1-\|\zeta\|^2}$. Therefore

(4.10)

$$-\operatorname{Im}\langle N(\lambda)\mathrm{Op}_\lambda(\chi)f, \mathrm{Op}_\lambda(\chi)f\rangle_{L^2(\Gamma)} \geq C\|\mathrm{Op}_\lambda(\chi)f\|^2_{L^2(\Gamma)} - o(1)\|f\|^2_{L^2(\Gamma)}, \quad C > 0.$$

By (4.9) and (4.10),

$$\begin{aligned} \|\mathrm{Op}_\lambda(\chi)f\|_{L^2(\Gamma)} &\leq O(\varepsilon)\|u\|_{L^2(\mathcal{O})} + O_\varepsilon(1)\|v\|_{L^2(\mathcal{O})} \\ &+ O(\varepsilon)\|f\|_{L^2(\Gamma)} + O_\varepsilon(1)\|h\|_{L^2(\Gamma)} + O(1)\|\mathrm{Op}_\lambda(1-\chi)f\|_{H^1(\Gamma)}. \end{aligned}$$

Hence, taking $\varepsilon > 0$ small enough, we obtain

$$\begin{aligned} \|f\|_{H^1(\Gamma)} &\leq C\|\mathrm{Op}_\lambda(\chi)f\|_{L^2(\Gamma)} + C\|\mathrm{Op}_\lambda(1-\chi)f\|_{H^1(\Gamma)} \\ &\leq O(\varepsilon)\|u\|_{L^2(\mathcal{O})} + O_\varepsilon(1)\|v\|_{L^2(\mathcal{O})} \\ &+ O(\varepsilon)\|f\|_{L^2(\Gamma)} + O_\varepsilon(1)\|h\|_{L^2(\Gamma)} + O(1)\|\mathrm{Op}_\lambda(1-\chi)f\|_{H^1(\Gamma)}, \end{aligned}$$

which combined with (2.23), (4.4) and (4.7) implies (4.5).

Appendix

We keep the same notations as in the introduction. Given $\varrho > 0$ set $B(\varrho) = \{x \in R^n : |x| < \varrho\}$. Let $\widetilde{g}(x,\xi)$ be a smooth Hamiltonian in $T^*\Omega_1$ of the form

$$\widetilde{g}(x,\xi) = \widetilde{c}(x)^2 \sum_{i,j=1}^n \widetilde{g}_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \quad C > 0. \tag{A.1}$$

Suppose that

$$\overline{\mathcal{O}}_1 \subset B(\varrho) \text{ and } \widetilde{g}(x,\xi) = |\xi|^2, \ \forall (x,\xi) \in T^*(\Omega_1 \setminus B(\varrho)), \tag{A.2}$$

for some $\varrho > 0$. Then $\widetilde{g}(x,\xi)$ is said to be nontrapping in Ω_1 if the following condition holds:

(A.3) There exists $T > 0$ such that for any generalized $\widetilde{g}$-geodesics $\gamma(t)$ with $\gamma(0) \in B(\varrho) \cap \overline{\Omega}_1$ we have $\gamma(T) \notin B(\varrho) \cap \overline{\Omega}_1$.

Proposition A.1. *Let Γ be both g- and g^0- strictly concave with respect to $\mathcal{O}$. Suppose that g satisfies* (1.2). *Then there exits a smooth extension $\widetilde{g}(x,\xi)$ of g in $T^*\Omega_1$ satisfying* (A.1), (A.2) *and* (A.3).

Proof. The idea of the proof is to find a deformation $\widetilde{g}$ of g satisfying (A.1) and (A.2) in $T^*\Omega_1$ and a smooth family Ω^s, $s \geq 0$, of domains in R^n with smooth boundaries Γ^s such that:

- $\Omega^0 = \Omega$, $\cap_{s\geq 0}\Omega^s = \emptyset$, and $\overline{\Omega}^s \subset \Omega^t$ for $s > t$,

- For each $s \geq 0$, Γ^s is both g- and g^0- strictly convex with respect to Ω^s.

Let f be a smooth function in R^n such that $f > 0$ and $df \neq 0$ in Ω, and $\Gamma^s = \{f = s\}$. Then given $(x,\xi) \in T^*R^n$ such that $x \in \Gamma$ and $g(\nabla f(x), \xi) \geq 0$, we obtain that $f(\exp(tH_{\widetilde{g}})(x,\xi))$ is a strictly increasing function in $t > 0$. Therefore, in view of (1.2) we have that $\widetilde{g}$ is nontrapping in Ω_1.

Since $\mathcal{O}_2$ is a bounded strictly convex domain in R^n with a smooth boundary Γ, the map

$$\Gamma \times R_+ \ni (x', x_n) \longrightarrow x' + x_n \nu(x') \in \Omega$$

is a diffeomorphism, where $\nu(x')$ is the unit normal to Γ at x' pointing inside Ω. We set $f = x_n$. The Hamiltonian g^0 becomes $g^0(x,\xi) = \xi_n^2 + r^0(x,\xi')$ in $T^*(\Gamma \times (-\varepsilon, +\infty))$ for some $\varepsilon > 0$, where $r^0(x', x_n, \cdot)$ is a positive definite quadratic form on $T^*_{x'}\Gamma$ for each $x_n \geq -\varepsilon$. Moreover,

$$g(x,\xi) = (a(x)\xi_n - b(x,\xi'))^2 + r(x,\xi'),\ (x',\xi') \in T^*\Gamma,\ |x_n| \leq \varepsilon,$$

where $b(x,\xi')$ is linear with respect to $\xi' \in T^*_{x'}\Gamma$, and $r^0(x,\cdot)$ is a positive definite quadratic form on $T^*_{x'}\Gamma$ for $|x_n| \leq \varepsilon$. The inequality (A.1) implies $a(x',0) \neq 0$ and we can suppose that a is positive in $\Gamma \times [-\varepsilon, \varepsilon]$. Set $r_0(x',\xi') = r(x',0,\xi')$. Let $|x_n| \leq \varepsilon$ and $\{g, x_n\}(x,\xi) = 2a(x)(a(x)\xi_n - b(x,\xi')) = 0$. Then we obtain

$$\begin{aligned} \{g,\{g,x_n\}\}(x,\xi) &= 2a(x)\{r, a\xi_n - b\}(x,\xi') \\ &= -2a^2(x)\frac{\partial r}{\partial x_n}(x,\xi') + 2b(x,\xi')\{r,a\}(x,\xi') \\ &\qquad - 2a(x)\{r,b\}(x,\xi'). \end{aligned} \tag{A.4}$$

Moreover, in view of (1.1) we can suppose that for $|x_n| \leq \varepsilon$ and $\xi_n = b(x,\xi')/a(x)$ we have

$$\{g,\{g,x_n\}\}(x,\xi) \geq C r_0(x',\xi'),\ (x',\xi') \in T^*\Gamma.$$

Let $\chi \in C^\infty(R)$ be such that $\chi \geq 0$, $\chi'' \geq 0$, and $\chi = 0$ for $t \leq 1/3$ and $\chi = t - 1/2$ for $t \geq 2/3$. Then $0 \leq \chi(t) \leq \chi'(t)$ for $0 \leq t \leq 1$. Set $G = (a\xi_n - b)^2 + R$, where

$$R(x,\xi') = R_\varepsilon(x,\xi') = r(x,\xi') - \chi(x_n/\varepsilon) r_0(x',\xi').$$

For $0 \leq x_n \leq \varepsilon$ we have $R(x,\xi') \geq 2^{-1} r_0(x',\xi')(1 - O(\varepsilon))$. Moreover, if $\{G, x_n\} = 2a(a\xi_n - b) = 0$, we obtain from (A.4)

$$\begin{aligned} \{G,\{G,x_n\}\}(x,\xi) = \{g,\{g,x_n\}\}(x,\xi) + \frac{2a^2(x)}{\varepsilon}\chi'(x_n/\varepsilon) r_0(x',\xi')(1 + O(\varepsilon)) \\ \geq C r_0(x',\xi') \end{aligned}$$

for some $C > 0$ and any $0 \leq x_n \leq \varepsilon$. On the other hand, for $2\varepsilon/3 \leq x_n \leq \varepsilon$ we have

$$-\frac{\partial R}{\partial x_n}(x,\xi') = -\frac{\partial r}{\partial x_n}(x,\xi') + \frac{1}{\varepsilon} r_0(x',\xi') \geq \frac{1}{\varepsilon} r_0(x',\xi'),$$

and

$$\{G,\{G,x_n\}\}(x,\xi) \geq \frac{C}{\varepsilon} r_0(x',\xi'), \quad C > 0,$$

if $\{G, x_n\} = 0$. Now we can deform a to 1 and b to 0.

Choose $\psi \in C^\infty(R)$ such that $0 \le \psi \le 1$, $\psi = 1$ for $t \le 3/4$ and $\psi = 0$ in a neighbourhood of $t = 1$. Set $F = (u\xi_n - v)^2 + R$, where $u(x) = u_\varepsilon(x) = (a(x) - 1)\psi(x_n/\varepsilon) + 1$ and $v(x, \xi') = v_\varepsilon(x, \xi') = b(x, \xi')\psi(x_n/\varepsilon)$. Then $u > 0$ on $\Gamma \times [0, \varepsilon]$ and $\{F, x_n\} = 0$ implies $\xi_n = v/u$. Moreover, in view of (A.4), for $\xi_n = v/u$ and $2\varepsilon/3 \le x_n \le \varepsilon$ we have

$$\begin{aligned}\{F, \{F, x_n\}\}(x, \xi) &= -2u_\varepsilon(x)^2 \frac{\partial R_\varepsilon}{\partial x_n}(x, \xi') + 2\psi(x_n/\varepsilon)v_\varepsilon(x, \xi')\{R_\varepsilon(x, \xi'), a(x)\} \\ &\qquad - 2\psi(x_n/\varepsilon)\{R_\varepsilon(x, \xi'), b(x, \xi')\} \\ &\ge \frac{C}{\varepsilon}(1 - O(\varepsilon))r_0(x, \xi'),\end{aligned}$$

for some $C > 0$ as $\varepsilon \searrow 0$. On the other hand, $F = G$ for $x_n \le 2\varepsilon/3$ and $F(x, \xi) = \xi_n^2 + R(x, \xi')$ for $x_n \in I = [\varepsilon_1, \varepsilon]$ for some $\varepsilon_1 < \varepsilon$. We have $R(x, \xi') \ge C_0 r^0(x, \xi')$ in $T^*\Gamma \times I$, for some $0 < C_0 < 1$. Let $\phi \in C^\infty(R)$ be such that $0 \le \phi \le 1$, $\phi' \le 0$ and $\phi = 1$ in a neighbourhood of $[0, \varepsilon_1]$ and $\phi = 0$ in a neighbourhood of $t = \varepsilon$. Set $F_1(x, \xi) = \xi_n^2 + R_1(x, \xi')$, where $R_1(x, \xi') = \phi(x_n)R(x, \xi') + C_0(1 - \phi(x_n))r^0(x, \xi')$. Then $R_1(x, \xi') \ge C_0 r^0(x, \xi')$ in $T^*\Gamma \times I$ and

$$\{F_1, \{F_1, x_n\}\} = -\phi \frac{\partial R}{\partial x_n} - C_0(1 - \phi)\frac{\partial r^0}{\partial x_n} - \phi'(R - C_0 r^0) \ge C_1 r^0$$

in $T^*\Gamma \times I$ for some $C_1 > 0$. Hence, we can extend F_1 for $x_n \ge \varepsilon$ setting $F_1(x, \xi) = \xi_n^2 + C_0 r^0(x, \xi')$. Fix $\gamma > 0$ and choose $\varphi \in C^\infty(R)$ such that $0 \le \varphi \le 1$, $-\gamma \le \varphi' \le 0$, $\phi = 1$ in a neighbourhood of $t = \varepsilon$ and $\phi = 0$ for $t \ge C_\gamma$, where C_γ is sufficiently large. Setting $\widetilde{g}(x, \xi) = F_1(x, \xi)$ for $x_n \le \varepsilon$ and $\widetilde{g}(x, \xi) = \xi_n^2 + (1 + (C_0 - 1)\varphi(x_n))r^0(x, \xi')$ for $x_n \ge \varepsilon$, and choosing $\gamma > 0$ sufficiently small, we obtain the desired extension of g. □

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