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CANONICAL BASES FOR HECKE ALGEBRA QUOTIENTS

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Dedicated to Professor R.W. Carter on the occasion of his 65th birthday

Abstract. We establish the existence of an IC basis for the generalized Temperley–Lieb algebra associated to a Coxeter system of arbitrary type. We determine this basis explicitly in the case where the Coxeter system is simply laced and the algebra is finite dimensional.

Introduction

An important construction in the theory of quantum groups and quantum algebras is that of canonical bases. The original example of this construction is the Kazhdan–Lusztig basis of the Hecke algebra associated to a Coxeter system, which first appeared in [11]. Another well known example is the canonical basis for U^+, the "plus part" of the quantized enveloping algebra associated to a semisimple Lie algebra over $\mathbb{C}$; this was discovered independently by Kashiwara [10] and Lusztig [12]. In each of these examples, the basis which arises has many deep and beautiful properties, some of which have geometric interpretations.

The general theory of such bases is defined for an $\mathcal{A}$-module (where $\mathcal{A}$ is the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$) equipped with an involutive $\mathbb{Z}$-linear map that sends v to v^{-1}. This theory was developed by Du in [2], where the bases arising are called IC bases. The letters "IC" stand for "intersection cohomology"; the name alludes to the fact that many of the natural examples have interpretations in terms of perverse sheaves. However, the existence and uniqueness of such a basis is not guaranteed.

The Temperley–Lieb algebra, a finite dimensional algebra arising in statistical mechanics [13] and knot theory [9], may be defined as a certain quotient of the Hecke algebra of a Coxeter system of type A. It is possible to generalize this construction to an arbitrary Coxeter system, obtaining the so-called "generalized Temperley–Lieb algebras" as quotients. In this paper, we show that IC bases exist for these Hecke algebra quotients associated to a Coxeter system of arbitrary type. We also determine the bases explicitly in the case of Coxeter

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systems of types A, D and E; it turns out that in each of these cases the basis coincides with a previously familiar basis for the corresponding generalized Temperley–Lieb algebra. The situation for non-simply-laced Coxeter systems turns out to be more complicated and surprising, as we will discuss.

It is tempting to think that IC bases for generalized Temperley–Lieb algebras may be obtained from the well-known Kazhdan–Lusztig bases for the Hecke algebra by projection to the quotient, but this is in fact far from clear. In general, the kernel of the canonical map from the Hecke algebra to the generalized Temperley–Lieb algebra is not spanned by the Kazhdan–Lusztig basis elements which it contains, even for Coxeter systems of low rank such as type D_4, and this causes complications. However, the situation in type A is relatively simple, as C.K. Fan and the first author have shown [4].

Our results give rise to some interesting problems concerning the general properties of IC bases for generalized Temperley–Lieb algebras; we mention a few of these in our remarks.

1. Generalized Temperley–Lieb algebras

Let X be a Coxeter graph, of arbitrary type, and let $W(X)$ be the associated Coxeter group with distinguished set of generating involutions $S(X)$. Denote by $\mathcal{H}(X)$ the Hecke algebra associated to $W(X)$. (The reader is referred to [8, §7] for the basic theory of Hecke algebras arising from Coxeter systems.) The $\mathcal{A}$-algebra $\mathcal{H}(X)$ has a basis consisting of elements T_w, with w ranging over $W(X)$, that satisfy

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where ℓ is the length function on the Coxeter group $W(X)$, $w \in W(X)$, and $s \in S(X)$. The parameter q is equal to v^2, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

We define $J(X)$ to be the ideal of $\mathcal{H}(X)$ generated by all elements

$$\sum_{w \in \langle s_i, s_j \rangle} T_w,$$

where (s_i, s_j) runs over all pairs of elements of $S(X)$ that correspond to adjacent nodes in the Coxeter graph. (If the nodes corresponding to (s_i, s_j) are connected by a bond of infinite strength, we omit the corresponding relation.)

Definition 1.1. The generalized Temperley–Lieb algebra, $TL(X)$, is defined to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(X)/J(X)$.

The algebra $TL(X)$ has a basis which arises naturally from the T-basis of $\mathcal{H}(X)$. To define it, we introduce some standard combinatoric notions.

Definition 1.2. Let s_i and s_j be elements of $S(X)$, the set of generating involutions, such that s_i and s_j correspond to adjacent nodes in the Coxeter graph. Let w_{ij} be the longest element in $\langle s_i, s_j \rangle$.

We call an element $w \in W(X)$ *complex* if it can be written as $x_1 w_{ij} x_2$, where $x_1, x_2 \in W(X)$ and $\ell(x_1 w_{ij} x_2) = \ell(x_1) + \ell(w_{ij}) + \ell(x_2)$.

Denote by $W_c(X)$ the set of all elements of $W(X)$ which are not complex.

Let t_w denote the image of the basis element $T_w \in \mathcal{H}(X)$ in the quotient $TL(X)$.

Theorem 1.3 (Graham). *The set $\{t_w : w \in W_c\}$ is an $\mathcal{A}$-basis for the algebra $TL(X)$.*

Proof. This is [5, Theorem 6.2]. □

We will call the basis of Theorem 1.3 the "t-basis" of the algebra $TL(X)$. It plays an important rôle in the sequel, since the canonical basis will be defined in terms of it.

Lemma 1.4. *The algebra $TL(X)$ has a $\mathbb{Z}$-linear automorphism of order 2 which sends v to v^{-1} and t_w to $t_{w^{-1}}^{-1}$.*

Note. The statement of the lemma makes sense: it is well known that the elements $T_w \in \mathcal{H}(W)$ are invertible, from which it follows that the elements t_w are invertible also.

Proof. It is known from [11] that the Hecke algebra $\mathcal{H}(X)$ over $\mathbb{Z}[v, v^{-1}]$ has a $\mathbb{Z}$-linear automorphism of order 2 which exchanges v and v^{-1} and sends T_w to $T_{w^{-1}}^{-1}$. It is therefore enough to show that this automorphism of $\mathcal{H}(X)$ fixes the ideal $J(X)$.

Given a finite Coxeter group W' with longest element w_0, it is a standard result that, for any $w_1 \in W'$, there exists an element $w_2 \in W'$ such that $w_0 = w_1 w_2$ and $\ell(w_0) = \ell(w_1) + \ell(w_2)$. It follows from this fact that

$$\left(\sum_{w \in \langle s_i, s_j \rangle} T_w \right) T_{w_{ij}}^{-1} = \left(\sum_{w \in \langle s_i, s_j \rangle} T_{w^{-1}}^{-1} \right),$$

so that the right hand side of the above equation lies in the ideal $J(X)$. We deduce that the automorphism of $\mathcal{H}(X)$ given above fixes the ideal $J(X)$ setwise, and thus induces an automorphism of $TL(X)$ which has the required properties. □

Lemma 1.5. *Let $w \in W$ (not necessarily in W_c). Then*

$$t_w = \sum_{\substack{x \in W_c \\ x \leq w}} D_{x,w} t_x,$$

where $D_{x,w} \in \mathbb{Z}[q]$*. Furthermore,* $D_{w,w} = 1$ *if* $w \in W_c$*.*

Proof. We proceed by induction on $\ell(w)$. The proposition is trivial if $w \in W_c$, which covers the cases where $\ell(w) \leq 1$, as well as the last assertion in the statement.

Now consider the case where $w = su$, $\ell(s) = 1$ and $\ell(u) = \ell(w) - 1$. Then, by induction,

$$t_w = t_s t_u = \sum_{\substack{x \in W_c \\ x \leq u}} D_{x,u} t_s t_x.$$

It follows from standard properties of $\mathcal{H}(X)$ that

$$t_s t_x = \begin{cases} t_{sx} & \text{if } \ell(sx) > \ell(x), \\ q t_{sx} + (q-1) t_x & \text{if } \ell(sx) < \ell(x). \end{cases}$$

It is easy to see that $sx \leq w$ and $x \leq w$ for each basis element t_x appearing in the sum. If $sx < w$ then, by induction, t_{sx} is a $\mathbb{Z}[q]$-linear combination of basis elements t_z $(z \in W_c)$ where $z \leq sx \leq w$.

The only remaining case is when $sx = w$, $w \notin W_c$. This forces $x = u$ and thus $u \in W_c$, so there is only one term in the sum. Since w is complex, it can be written as $x_1 w_{ij} x_2$ as in Definition 1.2. We now use the fact that in $TL(X)$ we have

$$t_{w_{ij}} = -\sum_{y < w_{ij}} t_y.$$

This enables $t_w = t_{x_1} t_{w_{ij}} t_{x_2}$ to be expressed as a $\mathbb{Z}[q]$-linear combination of elements t_y, where $y < w$. By induction, these elements t_y are $\mathbb{Z}[q]$-linear combinations of basis elements t_z $(z \in W_c)$ where $z \leq y \leq w$.

The lemma follows. □

2. IC bases

We now recall the basic properties of IC bases from [2, §1].

Let $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate, and let $\mathcal{A}^- = \mathbb{Z}[v^{-1}]$ and $\mathcal{A}^+ = \mathbb{Z}[v]$. Let $^-$ be the involution on the ring $\mathcal{A}$ which satisfies $\bar{v} = v^{-1}$.

Let M be a free $\mathcal{A}$-module with basis $\{m_i\}_{i \in I}$ and an involutive $\mathbb{Z}$-linear map $^-\colon M \longrightarrow M$ such that $\overline{am} = \bar{a}\bar{m}$ for any $m \in M$ and $a \in \mathcal{A}$. For each $i \in I$, let r_i be an integer.

Let $\mathcal{L}$ be the free $\mathcal{A}^-$-submodule with basis $\{m'_i\}_{i \in I}$, where $m'_i := v^{r_i} m_i$. Let $\pi : \mathcal{L} \longrightarrow \mathcal{L}/v^{-1}\mathcal{L}$ be the canonical projection.

Definition 2.1. If there exists a unique basis $\{c_i\}_{i \in I}$ for $\mathcal{L}$ such that $\overline{c_i} = c_i$ and $\pi(c_i) = \pi(m'_i)$, then the basis $\{c_i\}_{i \in I}$ is called an IC basis of M with respect to the triple $(\{m_i\}_{i \in I}, {}^-, \mathcal{L})$.

Note that the existence of an IC basis for M depends only on the original basis $\{m_i\}_{i \in I}$, the map $^-$ and the integers r_i.

Theorem 2.2 (Du). *Let $(I, \leq)$ be a poset such that the sets $\{i : i \in I, i \leq j\}$ are finite for all $j \in I$. Suppose that*

$$\overline{m'_j} = \sum_{\substack{i \in I \\ i \leq j}} a_{ij} m'_i$$

with $a_{ij} \in \mathcal{A}$ such that $a_{ii} = 1$ for all $i \in I$. Then an IC basis of M with respect to $(\{m_i\}, {}^-, \mathcal{L})$ exists.

Proof. This comes from [2, Theorem 1.2, Remark 1.2.1 (1)]. □

The importance of this result for us is that the t-basis of the algebra $TL(X)$ fits naturally into this setup, as we now explain.

Fix a Coxeter graph X. Let $I = I(X)$ be the set $\{w : w \in W_c\}$. We make I into a poset $(I, \leq)$ by restricting the Bruhat–Chevalley order on the Coxeter group $W = W(X)$ to the subset I. (Standard properties of this order imply that the poset $(I, \leq)$ has the finiteness property required by Theorem 2.2.) We take $r_w = -\ell(w)$ and $m_w = t_w$, so that $m'_w = v^{-\ell(w)} t_w$, and we take ${}^-$ to be the automorphism of $TL(X)$ defined in Lemma 1.4.

Maintaining this notation, we have the following central result.

Theorem 2.3. *Let X be an arbitrary Coxeter graph. There exists an IC basis for the algebra $TL(X)$ with respect to $(\{m_i\}, {}^-, \mathcal{L})$.*

Proof. It is enough to show that the formula for $\overline{m'_w}$ (where $w \in W_c$) given in Theorem 2.2 can be satisfied for suitable elements $a_{ij} \in \mathcal{A}$. We proceed by induction on $\ell(w)$, the case $\ell(w) = 0$ being trivial ($w = e$ and $\overline{m_e} = m_e$).

To deal with the inductive step, we suppose that $w = su$, where $\ell(s) = 1$ and $\ell(u) = \ell(w) - 1$. It is clear from the definition of W_c that $u \in W_c$.

Using the fact that $m'_s = v^{-1} t_s$, it is easily verified that

$$\overline{m'_s} = m'_s - (v - v^{-1}) m'_e,$$

where m'_e is the identity in $TL(X)$. Thus

$$\overline{m'_w} = \overline{m'_s m'_u} = (m'_s - (v - v^{-1}) m'_e) \overline{m'_u}.$$

By induction,

$$\overline{m'_u} = \sum_{\substack{z \in W_c \\ z \leq u}} a_{zu} m'_z.$$

For each z appearing in the sum, we have $z \leq u \leq w$ and also $sz \leq w$ by standard properties of the Bruhat–Chevalley order. It therefore follows from Lemma 1.5 that

$$\overline{m'_w} = \sum_{\substack{y \in W_c \\ y \leq w}} a_{yw} m'_y$$

for suitable $a_{yw} \in \mathcal{A}$.

It remains to show that $a_{ww} = 1$. Considering lengths and using Lemma 1.5, we find that the only term in the expression for $\overline{m'_s m'_u}$ which can contribute to the coefficient of m'_w arises from the product $m'_s \times a_{uu} m'_u$. It is clear that $m'_s m'_u = m'_w$, and we have $a_{uu} = 1$ by induction, completing the proof. □

Remark 2.4.

(1) The structure constants associated to the IC basis of $TL(X)$ lie in $\mathbb{N}[v, v^{-1}]$ if X is of type A, D or E_n. (This will follow from Theorem 3.6.) We do not know an example where this positivity property fails for the IC basis, and the property certainly holds for some non-simply-laced types, such as $TL(H_n)$. In [7, §4.1], the first author constructed a basis of $TL(H_n)$ with structure constants in $\mathbb{N}[v, v^{-1}]$ using morphisms in the category of decorated tangles. (This category was introduced in [6].) It turns out that the basis of $TL(H_n)$ in [7] is precisely the IC basis of $TL(H_n)$ arising from Theorem 2.3. This means that the results of this paper give an elementary characterisation of the basis in [7]. We do not supply the details here, but we hope to do so in a forthcoming paper.

(2) An interesting problem to consider is that of identifying the Coxeter graphs X for which the IC basis of $TL(X)$ is equal to the image of the set $\mathcal{C}$ of Kazhdan–Lusztig basis elements $C'_w \in \mathcal{H}(X)$ indexed by $w \in W_c$. Coxeter graphs X of type A have this property, by [4, Theorem 3.8.2] and our Theorem 3.6. It seems likely that the set $\mathcal{C}$ projects to the IC basis in type D, as well. This problem is closely related to the question of whether an element $v^{-\ell(w)} T_w \in \mathcal{H}(X)$ $(w \notin W_c)$ necessarily lies in the lattice $\mathcal{L}$ after passing to $TL(X)$. One can also express the problem in terms of a degree bound on the polynomials $D_{x,w}$ of Lemma 1.5. A significant partial result would be to know that $v^{-\ell(w)} T_w \in \mathcal{H}(X)$ projects into $\mathcal{L}$ when $w = su$, $s \in S(X)$, $u \in W_c$ and $w \notin W_c$. This would allow an inductive construction of the IC basis.

3. The ADE case

In §3, we restrict ourselves to the case where the Coxeter graph is of type A, D or E. However, we allow the graphs of type E to be of arbitrary rank. This means that a graph of type E_n $(n \geq 6)$ may consist of a straight line of nodes numbered $1, 2, \dots, n-1$ together with a node numbered 0 which is connected only to node 3. (Contrast this with the more familiar definition of type E which is the same but requires $n \leq 8$.)

It is known [5, Theorem 7.1] that the algebras $TL(E_n)$ are finite dimensional for all values of n. In fact, for simply laced X, the algebra $TL(X)$ is finite dimensional if and only if X is of type A, type D or type E_n for some n. If X satisfies these hypotheses, we say it is of type ADE.

Definition 3.1. If $s \in S(X)$, we define b_s to be the element $v^{-1}t_s + v^{-1}t_e$.

If $w \in W_c$ and $w = s_1 s_2 \cdots s_r$ (reduced), we define

$$b_w := b_{s_1} b_{s_2} \cdots b_{s_r}.$$

This definition may appear to depend on the reduced expression chosen for w, but in fact it does not, because b_s and $b_{s'}$ commute whenever s and s' commute, and every reduced expression for w can be obtained from any other by a sequence of commutation moves (see [3]).

It is well known (and follows easily from Theorem 1.3 and Lemma 1.5) that the set $\{b_w : w \in W_c\}$ is a basis for $TL(X)$. We call this the *monomial basis*.

We are particularly interested in the case where the Coxeter graph X is connected and simply laced (all bonds have strength 2 or 3).

Lemma 3.2. *If X is connected and simply laced then the algebra $TL(X)$ is generated as an associative, unital algebra by the elements b_s (one for each node of X) and defining relations*

$$\begin{aligned} b_s^2 &= q_c b_s, & & \\ b_s b_t &= b_t b_s, & & \textit{if } s \textit{ and } t \textit{ are not connected,} \\ b_s b_t b_s &= b_s, & & \textit{if } s \textit{ and } t \textit{ are connected,} \end{aligned}$$

where $q_c := v + v^{-1}$.

Proof. This is a standard result from [3]. □

Our aim in §3 is to prove that in types A, D and E, the monomial basis is the IC basis. The following is a key ingredient of the proof that we will give.

Lemma 3.3. *Let X be a Coxeter graph of type ADE. Let $w \in W_c(X)$ and let $s_1 s_2 \cdots s_r$ be a reduced expression for w. Then for any $1 \leq i_1 < i_2 < \cdots < i_k \leq r$ $(k < r)$, we have $b_{s_{i_1}} b_{s_{i_2}} \cdots b_{s_{i_k}} = q_c^m b_x$ for some $x \in W_c(X)$, where $m \leq r - k - 1$ and $q_c := v + v^{-1}$.*

Proof. This result is a special case of [5, Lemma 9.13]. Let b be an arbitrary monomial in the generators b_s. The relations of $TL(X)$ ensure that b is equal to $q_c^m b_x$ for some nonnegative integer m and some $x \in W_c$. Then [5, Lemma 9.13] shows that the removal of one generator from the monomial b results in an expression equal to $q_c^{m'} b_{x'}$, where $x' \in W_c$ and $m' \leq m + 1$. Furthermore, if $b = b_w$ for some $w \in W_c$, we have $m' = m$. This means that the maximum exponent of q_c which could occur after $r - k$ generators have been removed from b_w, as in the statement, is $r - k - 1$. □

Remark 3.4. It is possible to prove Lemma 3.3 without appealing to the results of [5] by using a combinatoric argument based on tom Dieck's graphical calculus

[1], and in fact the latter approach establishes the lemma for a slightly wider class of Coxeter systems, including type $\widetilde{E}_7$. However, we do not pursue this here.

In the following lemma we use the standard notation $\varepsilon_x := (-1)^{\ell(x)}$.

Lemma 3.5. *Let $w \in W_c$. Then*

$$v^{-\ell(w)} t_w = \varepsilon_w \sum_{\substack{x \in W_c \\ x \leq w}} \varepsilon_x \widetilde{Q}_{x,w} b_x,$$

where $\widetilde{Q}_{w,w} = 1$ and $\widetilde{Q}_{x,w} \in v^{-1}\mathcal{A}^-$ if $x < w$.

Proof. Fix $w \in W_c$ and fix a reduced expression $w = s_1 s_2 \cdots s_r$. We have

$$\begin{aligned} v^{-\ell(w)} t_w &= (v^{-1} t_{s_1})(v^{-1} t_{s_2}) \cdots (v^{-1} t_{s_r}) \\ &= (b_{s_1} - v^{-1})(b_{s_2} - v^{-1}) \cdots (b_{s_r} - v^{-1}). \end{aligned}$$

By expanding the last product and using Lemma 3.2 and the subexpression characterisation of the Bruhat–Chevalley order, we can see that all the b_x occurring in the sum satisfy $x \in W_c$ and $x \leq w$.

More precisely, the product expands to a sum of terms

$$v^{k-\ell(w)} b_{s_{i_1}} b_{s_{i_2}} \cdots b_{s_{i_k}}.$$

If $k < \ell(w)$, Lemma 3.3 shows that this is equal to

$$v^{k-\ell(w)} q_c^m b_x,$$

where $x \in W_c$ and $m \leq \ell(w)-k-1$. It follows that the coefficient of b_x, and hence $\widetilde{Q}_{x,w}$, lies in $v^{-1}\mathcal{A}^-$ in this case. The other possibility is that $k = \ell(w)$, which produces the basis element b_w with coefficient 1 and no other basis elements. It follows that $\widetilde{Q}_{w,w} = 1$, as required. $\square$

We are ready to prove the main result of §3.

Theorem 3.6. *Let X be a Coxeter graph of type ADE. Then the IC basis for $TL(X)$ defined in Theorem 2.3 is precisely the monomial basis $\{b_w : w \in W_c\}$.*

Proof. It is easily checked that the generators b_s are fixed by the involution $^-$, from which it follows that any monomial in these generators also has this property, so that $\overline{b_w} = b_w$.

Lemma 3.5 shows that the transfer matrix $(\varepsilon_w \varepsilon_x \widetilde{Q}_{x,w})$ from the monomial basis to the m_i'-basis is upper unitriangular with respect to a total refinement of the partial order $\leq$, and all the entries above the diagonal lie in $v^{-1}\mathcal{A}^-$.

Elementary linear algebra shows that the inverse of this matrix, $(\widetilde{P}_{x,w})$, is also upper unitriangular with all the entries above the diagonal in $v^{-1}\mathcal{A}^-$. This shows that

$$b_w = \sum_{\substack{x \in W_c \\ x \leq w}} \widetilde{P}_{x,w}(v^{-\ell(x)} t_x),$$

and therefore that $\pi(b_w) = \pi(v^{-\ell(w)} t_w)$.

We have now shown that the monomial basis is the IC basis with respect to the triple $(\{m_i\}_{i \in I}, {}^-, \mathcal{L})$, as claimed. $\square$

Remark 3.7.

(1) It is not true that the monomial basis of $TL(X)$ equals the IC basis for all simply laced X. For example, take X to be the Coxeter graph of type $\widetilde{A}_3$ consisting of four nodes connected by four edges in the shape of a square. If we number the nodes $1, 2, 3, 4$ around the square, the element $b_1 b_3 b_2 b_4 b_1 b_3$ is not an IC basis element. Similar remarks hold for X of type $\widetilde{A}_l$, where $l > 3$ is odd.

If X is non-simply-laced, the monomial and IC bases do not agree: any element in W_c of the form $w = ss's$ (where $s, s' \in S(X)$) has the property that $b_s b_{s'} b_s$ is not an IC basis element.

(2) Another problem to consider is that of determining the precise relationship between the elements $\widetilde{P}_{x,w}$ in the proof of Theorem 3.6 and the Kazhdan–Lusztig polynomials $P_{x,w}$ of [11]. It is natural to be curious about this relationship, because our elements $\widetilde{P}_{x,w}$ play a rôle analogous to that of $v^{\ell(x)-\ell(w)} P_{x,w}$ in [11]. Similarly, the elements $\widetilde{Q}_{x,w}$ are analogous to the inverse Kazhdan–Lusztig polynomials.

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