Mathematical Research Letters **6**, 203–211 (1999)

SOME LOWER BOUNDS ON THE NUMBER OF RESONANCES IN EUCLIDEAN SCATTERING

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The purpose of this note is to give some new lower bounds on the number of resonances, or scattering poles, for non-trivial, real-valued, smooth, compactly supported potentials in dimension $n \geq 3$, odd. Let $N(r)$ be the number of resonances, counted with multiplicity, with norm less than r. We prove that

$$\limsup_{r\to\infty} \frac{N(r)}{r(\log r)^{-p}} = \infty,$$

for any $p > 1$; this is the first quantitative lower bound to hold in this generality. We give some similar results for scattering by non-trapping metric perturbations and for scattering by certain obstacles with fractal boundaries. The non-zero resonances are defined, equivalently, as the poles of the meromorphic continuation of the resolvent or of the scattering matrix ([6]). For the situations we consider, except, possibly, for a finite number of points, the poles of the meromorphic continuation of the resolvent correspond, with multiplicity, to the poles of the determinant of the scattering matrix ([5, 22]).

Rather little is known about the poles for general potentials. Let $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ be real-valued, $n \geq 3$ odd, and let $R(\lambda) = (\Delta + V - \lambda^2)^{-1}$ be the resolvent of $\Delta + V$. We take the convention that, aside from a finite number of λ, $R(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^n)$ when $\operatorname{Im}\lambda < 0$. It has a meromorphic continuation to $\mathbb{C}$ as an operator from $L^2_{\mathrm{comp}}(\mathbb{R}^n)$ to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Let

$$N(r) = \#\{\lambda_j : \lambda_j \text{ is a pole of } R(\lambda), \text{ listed with multiplicity, and } |\lambda_j| < r\}.$$

In [21], for odd $n \geq 3$, Zworski showed that for $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, $N(r) = \mathcal{O}(r^n)$, and this bound is optimal in that there are potentials for which $N(r) \geq cr^n$, $c > 0$; in [20] he obtained asymptotics for $N(r)$ when $n = 1$. Only relatively recently has it been shown that if V is a smooth, super-exponentially decaying potential, $V \not\equiv 0$, then there are infinitely many poles. This was done for $n = 3$ ([12]), for a combination of a potential and metric perturbation when $n = 3$ ([16]), and then for potential scattering in all odd dimensions ([17]); see [17] for a brief history of the problem.

Let $S(\lambda)$ be the scattering matrix, $s(\lambda) = \det S(\lambda)$, and, for λ real, $\sigma(\lambda) = (2\pi i)^{-1}\int_0^\lambda s^{-1}(\tau)s'(\tau)d\tau + s_{0\pm}$ be the scattering phase, where $s_{0\pm} = 1$ if 0 is a resonance and is 0 otherwise. We use knowledge of $s(\lambda)$ and $\sigma(\lambda)$ near 0 and infinity, both of which are rather well understood in the potential case, and a

Received December 9, 1998. Revised February 11, 1999.

representation of $s(\lambda)$ from [22] to obtain some information about $N(r)$. Our methods are in the spirit of what has been done for surfaces with cusps (e.g. [13, 14]). However, since the results of [13, 14] make extensive use of the fact that most of the resonances lie near the real axis, such precise results are not possible here.

From [22] we know that $s(\lambda) = \pm e^{ig(\lambda)} P(-\lambda)/P(\lambda)$, where

$$P(\lambda) = \prod_{\lambda_j \text{ pole, } \lambda_j \neq 0} E(\lambda/\lambda_j, n), \quad E(z,p) = (1-z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}),$$

$\{\lambda_j\}$ are all the non-zero poles of the resolvent, including those corresponding to eigenvalues, and g is a polynomial of order at most n. We take the positive sign when 0 is not a resonance and the negative sign otherwise ([8]). Moreover, using the fact that for real λ, $|s(\lambda)| = 1$, $\sigma'(\lambda)$ is even, and $\sigma(\lambda)$ is real, we obtain that $g(\lambda)$ is real when λ is real, odd, and has $g(0) = 0$.

In the spirit of this note, we give a different proof of the following lemma (see [17, 15], and references). Effectively, we are using the behaviour of the scattering phase near 0 rather than the asymptotics of the heat kernel, as was noted as a possibility by Müller (See [22, Section 2], and also [23, Section 3].). Note that it was observed by Melrose ([12]) that the existence of at least one pole is enough to show that there are infinitely many.

Lemma. *The resolvent of* $\Delta + V$*, where* $V \in C_c^\infty(\mathbb{R}^n)$ *is real-valued,* $n \geq 3$ *odd, has at least one nonzero pole if* $V \not\equiv 0$*.*

Proof. Suppose there are no non-zero poles. Then $s(\lambda) = \pm e^{ig(\lambda)}$ and $\sigma(\lambda) = (2\pi)^{-1} g(\lambda) + s_{0\pm}$. However, from results on the behaviour of the scattering matrix near 0 ([7, 8]), under these assumptions $g(\lambda) = \alpha_{n-2}\lambda^{n-2} + \alpha_n \lambda^n$ where α_{n-2}, α_n are constants. Since the coefficient of λ^{n-4} in the expansion at infinity of the scattering phase is a nonzero multiple of $\int V^2(x)dx$ ([1, 3, 4]), it follows that $V \equiv 0$. □

We shall use the following notation: For $f, g \geq 0$, we say that $f(r) = \Omega(g(r))$ as $r \to \infty$ if for any $C > 0$, $R > 0$, there is an $r_1 > R$ with $f(r_1) > Cg(r_1)$.

We note that if $\sum_{m=1}^\infty |a_m|^{-p-1} < \infty$, then $\prod_1^\infty E(z/a_m, p)$ converges; we make use of this in the theorems and proposition below.

Theorem 1. *If* $V \in C_c^\infty(\mathbb{R}^n)$ *is real-valued,* $V \not\equiv 0$*,* $n \geq 3$ *odd, then as* $r \to \infty$*,* $N(r) = \Omega(r(\log r)^{-p})$ *for any* $p > 1$*.*

Proof. Suppose $N(r) = \mathcal{O}(r(\log r)^{-p})$ for some $p > 1$. Then we may write

$$s(\lambda) = \pm e^{ig(\lambda)} \prod \frac{\lambda_j + \lambda}{\lambda_j - \lambda},$$

where this g is perhaps different from the previous one, but has the same properties.

Using the fact that if λ_j is a resonance with non-zero real part, then $-\overline{\lambda_j}$ is also a resonance, we have for real λ

$$\begin{aligned} 2\pi\sigma(\lambda) &= g(\lambda) - \sum \int_0^\lambda \frac{2\operatorname{Im}\lambda_j}{(\tau - \operatorname{Re}\lambda_j)^2 + (\operatorname{Im}\lambda_j)^2} d\tau \\ &= g(\lambda) - 2\sum \left(\operatorname{Arc}\tan\left(\frac{\lambda - \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right) - \operatorname{Arc}\tan\left(\frac{-\operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right)\right). \end{aligned}$$

Again using the symmetry of the poles, we have

$$\begin{aligned} 2\pi\sigma(\lambda) = g(\lambda) &- 2 \sum_{\operatorname{Re}\lambda_j = 0} \operatorname{Arc}\tan\left(\frac{\lambda}{\operatorname{Im}\lambda_j}\right) \\ &- 2 \sum_{\operatorname{Re}\lambda_j > 0} \left(\operatorname{Arc}\tan\left(\frac{\lambda - \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right) + \operatorname{Arc}\tan\left(\frac{\lambda + \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right)\right). \end{aligned}$$

This could also be obtained directly. The term involving the arctangent is the sum of the arguments of $(\lambda_j+\lambda)(\lambda_j-\lambda)^{-1}$ and $(-\overline{\lambda_j}+\lambda)(-\overline{\lambda_j}-\lambda)^{-1}$ if $\operatorname{Re}\lambda_j > 0$, minus the value of the arguments when $\lambda = 0$.

Using the fact that for $x > 0$,

$$\operatorname{Arc}\tan x + \operatorname{Arc}\tan y = \operatorname{Arc}\tan \frac{x+y}{1-xy} + \begin{cases} 0 & \text{if } xy < 1 \\ \pi & \text{if } xy > 1 \end{cases}$$

we obtain, when $|\lambda| \neq |\lambda_j|$, $\operatorname{Re}\lambda_j > 0$,

$$\begin{aligned} (1) \quad \operatorname{Arc}\tan\left(\frac{\lambda - \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right) &+ \operatorname{Arc}\tan\left(\frac{\lambda + \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right) \\ &= \operatorname{Arc}\tan\left(\frac{2\lambda\operatorname{Im}\lambda_j}{|\lambda_j|^2 - \lambda^2}\right) + \begin{cases} 0 & \text{if } |\lambda_j|^2 > \lambda^2 \\ \pi & \text{if } |\lambda_j|^2 < \lambda^2. \end{cases} \end{aligned}$$

We wish to bound

$$\begin{aligned} (2) \quad h(\lambda) = 2 &\sum_{\operatorname{Re}\lambda_j = 0} \operatorname{Arc}\tan\left(\frac{\lambda}{\operatorname{Im}\lambda_j}\right) \\ &+ 2 \sum_{\operatorname{Re}\lambda_j > 0} \left(\operatorname{Arc}\tan\left(\frac{\lambda - \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right) + \operatorname{Arc}\tan\left(\frac{\lambda + \operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right)\right), \end{aligned}$$

from above and below. From our bounds, we will then derive a contradiction to the known asymptotics of the scattering phase.

For the bound from above, note that

$$(3)\quad \sum_{\substack{\operatorname{Re}\lambda_j=0\\|\lambda_j|<2\lambda}} \operatorname{Arc}\tan\left(\frac{\lambda}{\operatorname{Im}\lambda_j}\right) + \sum_{\substack{\operatorname{Re}\lambda_j>0\\|\lambda_j|<2\lambda}}\left(\operatorname{Arc}\tan\left(\frac{\lambda-\operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right)+\operatorname{Arc}\tan\left(\frac{\lambda+\operatorname{Re}\lambda_j}{\operatorname{Im}\lambda_j}\right)\right) \leq \frac{\pi}{2}N(2\lambda) \leq C(1+\lambda(\log\lambda)^{-p}).$$

Using the fact that $|\operatorname{Arc}\tan x| \leq |x|$, we can, for large λ, bound

$$(4)\quad 2\sum_{\substack{\operatorname{Re}\lambda_j=0\\|\lambda_j|>2\lambda}} \operatorname{Arc}\tan\left(\frac{\lambda}{\operatorname{Im}\lambda_j}\right) \leq 2\sum_{\substack{\operatorname{Re}\lambda_j=0\\|\lambda_j|>2\lambda}}\left|\frac{\lambda}{\operatorname{Im}\lambda_j}\right| \leq 2\lambda\int_{2\lambda}^{\infty} r^{-1}\frac{d}{dr}N(r)dr \leq C\lambda(\log\lambda)^{-p} + C\int_{2\lambda}^{\infty} r^{-1}(\log r)^{-p}dr \leq C\lambda(\log\lambda)^{-\epsilon},$$

for some $\epsilon > 0$, where C is a positive constant whose value may change from line to line. A similar bound can be made for the second sum in (2), showing that $h(\lambda) = \mathcal{O}(\lambda(\log\lambda)^{-\epsilon})$.

For the lower bound, we use the fact that if $|\lambda_j| < \lambda/2$, $\operatorname{Im}\lambda_j > 0$, and $\operatorname{Re}\lambda_j = 0$, then $\operatorname{Arc}\tan(\lambda/\operatorname{Im}\lambda_j) > \pi/4$. Since

$$\pi + \operatorname{Arc}\tan\frac{2\lambda\operatorname{Im}\lambda_j}{|\lambda_j|^2-\lambda^2} > \frac{\pi}{2},$$

using (1) and the fact that the contribution to h of the poles with positive imaginary part is positive, we obtain

$$\frac{\pi}{2}N(\frac{\lambda}{2}) - 2\pi n_e \leq h(\lambda),$$

where n_e is the sum of the number of eigenvalues and zero-resonances, counted with multiplicity. The non-zero eigenvalues correspond to the poles of the resolvent with negative imaginary part.

Summarizing, we have for large λ

$$\frac{\pi}{2}N(\frac{\lambda}{2}) - 2\pi n_e \leq h(\lambda) \leq C\lambda(\log\lambda)^{-\epsilon}.$$

Therefore, h is unbounded but grows more slowly than linearly, and since $2\pi\sigma(\lambda) = g(\lambda) - h(\lambda)$, this contradicts the known asymptotics of the scattering phase at infinity ([1, 3, 4]). $\square$

Set

$$N_I(r) = \#\{\lambda_j \text{ poles of } R(\lambda), \text{ counted with multiplicity} : |\lambda_j| < r, \operatorname{Re}\lambda_j = 0\}.$$

The following result may be compared to [18, Example 5.3], which gives sufficient conditions for the Dirichlet or Neumann Laplacian on an exterior domain to have

$$N(r) - N_I(r) \geq c_\epsilon r^{n-1-\epsilon},$$

for any $\epsilon > 0$.

Theorem 2. *Let $V \in C_c^\infty(\mathbb{R}^n)$ be real-valued, $n \geq 3$ odd, and suppose $N_I(r) \geq cr^{n-1}$ for some $c > 0$. Then either $N(r) - N_I(r) = \Omega(r^{n-1}(\log r)^{-p})$ for any $p > 1$ or $N_I(r) = \Omega(r^n(\log r)^{-p})$, any $p > 1$.*

Note that the results of [9, 19] show that the class of potentials satisfying the hypotheses of the theorem includes certain potentials of fixed sign.

Proof. Suppose that $N(r) = \mathcal{O}(r^n(\log r)^{-p})$, some $p > 1$. Then we may write $s(\lambda) = \pm e^{ig(\lambda)}P(-\lambda)/P(\lambda)$ with

$$P(\lambda) = \prod_{\lambda_j \text{pole},\ \lambda_j \neq 0} E(\lambda/\lambda_j, n-1).$$

Considering only the contribution of the purely imaginary poles to the scattering phase, we will show below that for large λ,

$$(5)\quad C_1\lambda^{n-1} \leq \left| \int_0^\lambda \prod_{\substack{\lambda_j \text{ pole} \\ \lambda_j \neq 0,\ \operatorname{Re}\lambda_j = 0}} \frac{E(\tau/\lambda_j, n-1)}{E(-\tau/\lambda_j, n-1)} \frac{\partial}{\partial\tau} \left(\prod_{\substack{\lambda_j \text{ pole} \\ \lambda_j \neq 0, \operatorname{Re}\lambda_j = 0}} \frac{E(-\tau/\lambda_j, n-1)}{E(\tau/\lambda_j, n-1)} \right) d\tau \right|$$
$$= \left| \sum_{\substack{\lambda_j \text{ pole},\ \lambda_j \neq 0 \\ \operatorname{Re}\lambda_j = 0}} \int_0^\lambda \frac{2i\tau^{n-1}(-1)^{(n-3)/2}}{(\operatorname{Im}\lambda_j)^{n-2}(\tau^2 + (\operatorname{Im}\lambda_j)^2)} d\tau \right| \leq C_2\lambda^n(\log\lambda)^{-\epsilon},$$

for some $C_1, C_2, \epsilon > 0$. If we have $N(r) - N_I(r) = \mathcal{O}(r^{n-1}(\log r)^{-p})$ for some $p > 1$, then we have, when λ is large,

$$\left| \int_0^\lambda \prod_{\substack{\lambda_j \text{ pole} \\ \operatorname{Re}\lambda_j \neq 0}} \frac{E(\tau/\lambda_j, n-1)}{E(-\tau/\lambda_j, n-1)} \frac{\partial}{\partial\tau} \left(\prod_{\substack{\lambda_j \text{ pole} \\ \operatorname{Re}\lambda_j \neq 0}} \frac{E(-\tau/\lambda_j, n-1)}{E(\tau/\lambda_j, n-1)} \right) d\tau \right|$$
$$\leq C\lambda^{n-1}(\log\lambda)^{-\epsilon},$$

for some $C, \epsilon > 0$. The proof of this estimate is very similar to that of (2) and (5), using, of course, the bound $\mathcal{O}(r^{n-1}(\log r)^{-p})$ on the number of poles with nonzero real part. This estimate, combined with (5) and the fact that g is an odd polynomial, contradicts the known asymptotics of the scattering phase ([1, 3, 4]), which has highest order term λ^{n-2}. Therefore, at least one of our assumptions is incorrect.

We have now finished the proof, except for showing (5). To prove the lower bound, note that, using $N_I(r) \geq cr^{n-1}$,

(6)
$$\begin{aligned}
&\sum_{\substack{\lambda_j \,\mathrm{pole}, \mathrm{Re}\,\lambda_j = 0 \\ \mathrm{Im}\,\lambda_j > 0, |\lambda_j| < \lambda}} \int_0^\lambda \frac{2\tau^{n-1}}{(\mathrm{Im}\,\lambda_j)^{n-2}(\tau^2 + (\mathrm{Im}\,\lambda_j)^2)} d\tau \\
&\qquad \geq \int_0^\lambda \int_0^\lambda \frac{2\tau^{n-1}}{r^{n-2}(\tau^2 + r^2)} d\tau \frac{d}{dr} N_I(r) dr - C \\
&\qquad \geq \int_0^\lambda \frac{2c\lambda\tau^{n-1}}{\tau^2 + \lambda^2} d\tau - \int_0^\lambda \int_0^\lambda \frac{d}{dr} \frac{2\tau^{n-1}}{r^{n-2}(\tau^2 + r^2)} d\tau N_I(r) dr - C \\
&\qquad \geq 2c\lambda \int_0^\lambda \frac{\tau^{n-1}}{\tau^2 + \lambda^2} d\tau - C \\
&\qquad \geq C'\lambda^{n-1} - C,
\end{aligned}$$

where $C' > 0$, and here and below C is a constant which may change from line to line.

For the upper bound in (5), note that much as in (4) and (6), using the upper bound on $N(r)$ it suffices to bound from above

$$\int_2^\infty \int_0^\lambda \frac{\tau^{n-1}}{\tau^2 + r^2} r (\log r)^{-p} d\tau dr. \tag{7}$$

If we write this as the sum of two integrals, where in the first we integrate over $r \in (2, \lambda)$ and the second $r \in (\lambda, \infty)$, we get for the first

(8)
$$\begin{aligned}
&\int_2^\lambda \int_0^\lambda \frac{\tau^{n-1}}{\tau^2 + r^2} r (\log r)^{-p} d\tau dr \\
&\qquad \leq \int_2^\lambda \int_0^\lambda \tau^{n-3} r (\log r)^{-p} d\tau dr \\
&\qquad \leq C\lambda^{n-2} \int_2^\lambda r (\log r)^{-p} dr \leq C\lambda^n (\log \lambda)^{-p},
\end{aligned}$$

for large λ. For the part of the integral (7) with $\lambda \leq r < \infty$, we obtain, after a change of variables,

$$\int_1^\infty \int_0^1 \frac{\lambda^n \tau^{n-1} r}{r^2+\tau^2} (\log(r\lambda))^{-p} d\tau dr \leq C\lambda^n (\log \lambda)^{-\epsilon},$$

for some $\epsilon > 0$. □

We include several results, for scattering by metric perturbations and obstacles, whose proofs are much the same. We continue to use $N(r)$ to denote the counting function for the poles, although we consider different kinds of perturbations of the Laplacian.

Proposition. *If the resolvent of the Laplacian associated to a smooth, compactly supported, non-trapping metric perturbation of* $\mathbb{R}^n$*,* $n \geq 3$ *odd, has any poles, then* $N(r) = \Omega(r(\log r)^{-p})$*, any* $p > 1$*.*

Proof. This proof follows just as the proof of Theorem 1, using the results of [11] for the asymptotics of the scattering phase. The proof works because of the existence of a complete asymptotic expansion of the scattering phase. □

We note that [15, 17] have shown that if $n = 3$ or $n = 5$, then there are infinitely many resonances for a non-flat, compactly supported metric perturbation of $\mathbb{R}^n$. For general odd n, [15] reduced the question of the existence of infinitely many resonances to the question of whether the Minakshisundaram coefficients d_j, $j \geq 2$, are 0.

Examples. In [10], Levitin and Vassiliev constructed examples of bounded domains $\tilde{\Omega} \subset \mathbb{R}^n$ with interior Minkowski dimension $\tilde{D}$, $n-1 < \tilde{D} < n$, such that

$$N_{ev}(\mu) = c_n \operatorname{Vol} \tilde{\Omega} \mu^n + \tilde{c}_{\tilde{\Omega},n} \mu^{\tilde{D}} + o(\mu^{\tilde{D}})$$

([10, Theorem C]), with

$$N_{ev}(\mu) = \#\{\lambda_j^2 \leq \mu^2 : \lambda_j^2 \text{ is an eigenvalue of the Laplacian on } \tilde{\Omega} \text{ with Dirichlet boundary conditions, counted with multiplicity}\},$$

and $\tilde{c}_{\tilde{\Omega},n} \neq 0$. (See [10] for a definition of interior and exterior Minkowski dimension.) Using their examples and a result of [2], it is relatively easy to construct families of obstacles Ω in $\mathbb{R}^n$ with exterior Minkowski dimension $\hat{D}$, $n-1 < \hat{D} < n$, such that the Laplacian on $\mathbb{R}^n \setminus \Omega$ with Dirichlet boundary conditions has scattering phase with the asymptotics

$$\sigma(\lambda) = c_n \lambda^n \operatorname{Vol}(\Omega) \lambda^n + \hat{c}_{\Omega,n} \lambda^{\hat{D}} + o(\lambda^{\hat{D}}),$$

with $\hat{c}_{\Omega,n} \neq 0$ (see [2]). For such obstacles, when $n \geq 3$, odd, we have $N(r) = \Omega(r^{\hat{D}}(\log r)^{-p})$ for any $p > 1$. Again, the proof is by contradiction. If $N(r) =$

$\mathcal{O}(r^{\hat{D}}(\log r)^{-p})$ for some $p > 1$, then we can write $s(\lambda) = \pm e^{ig(\lambda)}P(-\lambda)/P(\lambda)$, with

$$P(\lambda) = \prod_{0 \neq \lambda_j,\ \text{pole}} E(\lambda/\lambda_j, n-1).$$

Again using the bound $N(r) = \mathcal{O}(r^{\hat{D}}(\log r)^{-p})$, we obtain, as $\lambda \to \infty$, $s(\lambda) = \alpha_n \lambda^n + \mathcal{O}(\lambda^{\hat{D}}(\log \lambda)^{-\epsilon})$ for some $\epsilon > 0$, a contradiction.

Acknowledgments. It is a pleasure to thank R.B. Melrose, M. Zworski, and the referee for their comments on earlier versions of this note, and M. Zworski for helpful discussions.

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