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ON GINZBURG'S LAGRANGIAN CONSTRUCTION OF REPRESENTATIONS OF $GL(n)$

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Abstract. In [3] V. Ginzburg observed that one can realize irreducible representations of the group $GL(n,\mathbb{C})$ in the cohomology of certain Springer's fibers for the group GL_d (for all $d \in \mathbb{N}$). However, Ginzburg's construction of the action of $GL(n)$ on this cohomology was a bit artificial (he defined the action of Chevalley generators of the Lie algebra gl_n on the corresponding cohomology by certain explicit correspondences, following the work of A. Beilinson, G. Lusztig and R. MacPherson ([1]), who gave a similar construction of the quantum group $U_q(gl_n)$).

In this note we give a very simple geometric definition of the action of the whole group $GL(n,\mathbb{C})$ on the above cohomology and simplify the results of [3].

1. Introduction

1.1. The combinatorics. Throughout this note we will fix natural numbers n and d. E will be a fixed n-dimensional (complex) vector space E and G will denote the group $GL(E)$. Let, in addition, $\mathfrak{g}_d$ denote the Lie algebra $\mathbf{gl}(d,\mathbb{C})$.

We set

$$\mathcal{P}_{n,d} = \{(n_1,\dots,n_k)|\ \sum_{i=1}^{k} n_i = d; \quad n_i \leq n \text{ for all } i=1,\dots,k \quad \text{and } n_1 \geq n_2 \geq \dots \geq n_k \geq 1\}.$$

With any $\mathbf{P} \in \mathcal{P}_{n,d}$ one usually associates three objects: an irreducible representation $\rho(\mathbf{P})$ of the symmetric group S_d, an irreducible representation $V(\mathbf{P})$ of G and a nilpotent orbit $\mathcal{O}_{\mathbf{P}}$ in $\mathfrak{g}_d$.

First of all, $\mathcal{O}_{\mathbf{P}}$ by definition consists of all nilpotent matrices in $\mathfrak{g}_d$, which have k Jordan blocks of sizes $n_1,\dots,n_k$.

The representation $\rho(\mathbf{P})$ is characterized uniquely by the following property:

Let $S_{\mathbf{P}} = S_{n_1}\times\ldots\times S_{n_k}$ be the corresponding subgroup of S_d. If the partition $\mathbf{P}$ is a (strict) refinement of a partition $\mathbf{P}'$, we shall write $\mathbf{P} < \mathbf{P}'$. We have:

$$\mathrm{Hom}_{S_d}(\rho(\mathbf{P}), \mathrm{Ind}_{S_{\mathbf{P}}}^{S_d}(\mathrm{sign})) \neq 0,\ \mathrm{Hom}_{S_d}(\rho(\mathbf{P}), \mathrm{Ind}_{S_{\mathbf{P}'}}^{S_d}(\mathrm{sign})) = 0,\ \text{if } \mathbf{P} < \mathbf{P}'.$$

Finally, we set $V(\mathbf{P}) = \mathrm{Hom}_{S_d}(\rho(\mathbf{P}), E^{\otimes d})$.

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1.2. The variety $X_{n,d}$. Fix n and d as above. Let $X_{n,d}$ be the complex algebraic variety whose $\mathbb{C}$-points consist of all partial flags (filtrations) $\mathcal{V}$ of the form

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^d. \tag{1.1}$$

The variety $X_{n,d}$ splits into connected components numbered by the set $\mathcal{Q}_{n,d}$ of n-tuples of non-negative integers $\mathbf{d} = (d_1, \dots, d_n)$ with $d_1 + \cdots + d_n = d$:

For $\mathbf{d} \in \mathcal{Q}_{n,d}$ as above, the corresponding connected component (denoted as $X_{n,d}^{\mathbf{d}}$) consists of flags $\mathcal{V}$, for which $\dim(V_i/V_{i-1}) = d_i$. We set $|\mathbf{d}| = \dim X_{n,d}^{\mathbf{d}}$.

Let now

$$\mathcal{N}_{n,d} = \{a \in \mathfrak{g}_d |\ a^n = 0\}. \tag{1.2}$$

$\mathcal{N}_{n,d}$ is a closed subvariety of the nilpotent cone of $\mathfrak{g}_d$.

Let $\widetilde{\mathcal{N}_{n,d}} = T^* X_{n,d}$ denote the cotangent bundle to $X_{n,d}$. It may be identified with the set of all pairs

$$\{\mathcal{V} \in X_{n,d}, a \in \mathfrak{g}_d |\ \text{such that } a(V_i) \subset V_{i-1}\}. \tag{1.3}$$

For $\mathbf{d} \in \mathcal{Q}_{n,d}$ we shall denote by $\widetilde{\mathcal{N}_{n,d}}^{\mathbf{d}}$ the corresponding connected component of $\widetilde{\mathcal{N}_{n,d}}$.

Let $\pi^{n,d} : \widetilde{\mathcal{N}_{n,d}} \to \mathfrak{g}_d$ denote the map, given by $\pi^{n,d}((\mathcal{V}, a)) = a$. Then $\pi^{n,d}$ is actually a map from $\widetilde{\mathcal{N}_{n,d}}$ to $\mathcal{N}_{n,d}$ and it is a semi-small (cf. [2]) resolution of singularities of $\mathcal{N}_{n,d}$.

1.3. The main result. Recall that we have fixed an n-dimensional complex vector space E and we have set $G = GL(E)$.

Fix now a decomposition

$$E = E_1 \oplus \cdots \oplus E_n \quad \text{such that } \dim E_i = 1. \tag{1.4}$$

For $\mathbf{d} \in \mathcal{Q}_{n,d}$ we shall denote by $E^{\mathbf{d}}$ the 1-dimensional vector space $E_1^{\otimes d_1} \otimes \ldots \otimes E_n^{\otimes d_n}$, where $\mathbf{d} = (d_1, \dots, d_n)$.

Let $T \subset G$ be the maximal torus corresponding to the decomposition $E = E_1 \oplus \cdots \oplus E_n$. An element $\mathbf{d} \in \mathcal{Q}_{n,d}$ defines a character of T: this is the character by which T acts on $E^{\mathbf{d}}$.

Let now $\mathcal{L}_E^{n,d}$ denote the perverse sheaf on $\widetilde{\mathcal{N}_{n,d}}$ described as follows:

For $\mathbf{d} \in \mathcal{Q}_{n,d}$, we set the restriction of $\mathcal{L}_E^{n,d}$ to $\widetilde{\mathcal{N}_{n,d}}^{\mathbf{d}}$ to be the 1-dimensional constant sheaf $E^{\mathbf{d}}[2|\mathbf{d}|]$. We endow $\mathcal{L}_E^{n,d}$ with a T-equivariant structure by letting T act on $E^{\mathbf{d}}[2|\mathbf{d}|]$ via the corresponding character.

The following result is essentially due to V. Ginzburg (cf. [3]) in a slightly less invariant form. The main purpose of this paper is to provide a simple proof of this result.

Theorem 1.4. *There is a canonical T-equivariant isomorphism*

$$\pi_*^{n,d}(\mathcal{L}_E^{n,d}) \simeq \bigoplus_{\mathbf{P}\in\mathcal{P}_{n,d}} IC_{\mathbf{P}} \otimes V(\mathbf{P}), \tag{1.5}$$

where $IC_{\mathbf{P}}$ denotes the intersection cohomology sheaf on the closure of $\mathcal{O}_{\mathbf{P}}$ in $\mathcal{N}_{n,d}$.

Remark. Note that the right hand side of (1.5) is independent of the decomposition $E = E_1 \oplus ... \oplus E_n$, though the sheaf $\mathcal{L}_E^{n,d}$ does depend on it.

For any $\mathbf{d} \in \mathcal{Q}_{n,d}$ and $\mathbf{P} \in \mathcal{P}_{n,d}$ let us denote by $V(\mathbf{P})_{\mathbf{d}}$ the corresponding weight space of $V(\mathbf{P})$. It is easy to see that $\mathcal{Q}_{n,d}$ exhaust all the weights that can appear in $V(\mathbf{P})$, i.e.

$$V(\mathbf{P}) = \bigoplus_{\mathbf{d}\in\mathcal{Q}_{n,d}} V(\mathbf{P})_{\mathbf{d}}. \tag{1.6}$$

By decomposing both sides of (1.5) with respect to the characters of T, we obtain:

Corollary 1.5. *There is a canonical T-equivariant isomorphism*

$$\pi_*^{n,d}\left(\mathcal{L}_E^{n,d}|_{\widetilde{\mathcal{N}_{n,d}}^{\mathbf{d}}}\right) \simeq \bigoplus_{\mathbf{P}\in\mathcal{P}_{n,d}} IC_{\mathbf{P}} \otimes V(\mathbf{P})_{\mathbf{d}}. \tag{1.7}$$

By applying Theorem 1.4 to $E = \mathbb{C}^n$ with the standard decomposition into one-dimensional subspaces and the definition of intersection cohomology sheaf we obtain the following result:

For $a \in \mathcal{N}_{n,d}$ consider

(1.8)
$$H^{\mathrm{top}}((\pi^{n,d})^{-1}(a),\mathbb{C}) := \bigoplus_{\mathbf{d}\in\mathcal{Q}_{n,d}} H^{\frac{1}{2}\operatorname{codim}_{\mathcal{N}_{n,d}}(\mathcal{O}_{\mathbf{P}})}((\pi^{n,d})^{-1}(a)\cap X_{n,d}^{\mathbf{d}},\mathbb{C}).$$

Corollary 1.6. *(cf.* [3] *and* [4]*) Let $\mathbf{P} \in \mathcal{P}_{n,d}$ and let $a \in \mathcal{O}_{\mathbf{P}}$. Then the vector space $H^{top}((\pi^{n,d})^{-1}(a),\mathbb{C})$ can be naturally identified with the space of the representation $V(\mathbf{P})$ of $GL(n,\mathbb{C})$. Under this identification, the direct summand*

$$H^{\frac{1}{2}\operatorname{codim}_{\mathcal{N}_{n,d}}(\mathcal{O}_P)}((\pi^{n,d})^{-1}(a)\cap X_{n,d}^{\mathbf{d}},\mathbb{C}) \subset H^{top}((\pi^{n,d})^{-1}(a),\mathbb{C}),$$

goes over to the weight space $V(\mathbf{P})_{\mathbf{d}} \subset V(\mathbf{P})$.

Remark. The fact that $\pi^{n,d}$ is semi-small means that for $a \in \mathcal{O}_{\mathbf{P}}$ the variety $(\pi^{n,d})^{-1}(a)$ has no irreducible components of dimension larger than $\frac{1}{2}\operatorname{codim}_{\mathcal{N}_{n,d}}(\mathcal{O}_{\mathbf{P}})$. Hence, the vector space

$$H^{\frac{1}{2}\operatorname{codim}_{\mathcal{N}_{n,d}}(\mathcal{O}_{\mathbf{P}})}((\pi^{n,d})^{-1}(a)\cap X_{n,d}^{\mathbf{d}},\mathbb{C})$$

is endowed with a natural basis, consisting of the fundamental classes of irreducible components of dimension $\frac{1}{2}\operatorname{codim}_{\mathcal{N}_{n,d}}(\mathcal{O}_{\mathbf{P}})$.

2. Proof of Theorem 1.4

2.1. Digression on the Springer correspondence. Let $\mathfrak{g}$ be any reductive Lie algebra and let W denote the corresponding Weyl group. Let us choose an invariant non-degenerate bilinear form on $\mathfrak{g}$, by means of which we shall identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$.

Recall the construction of the Springer sheaf on $\mathfrak{g}$. Let $\widetilde{\mathfrak{g}}$ denote the variety of all pairs $(\mathfrak{b}, x)$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $x \in \mathfrak{b}$. Let $p: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ denote the natural morphism.

We set $\mathbf{Spr} = p_* \mathbb{C}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}]$.

It is well-known (cf. [2]) that $\mathbf{Spr}$ is a semi-simple perverse sheaf on $\mathfrak{g}$. Moreover, one has a canonical isomorphism

$$\operatorname{End}(\mathbf{Spr}) \simeq \mathbb{C}[W], \tag{2.1}$$

where $\mathbb{C}[W]$ denotes the group algebra of W.

If now ρ is any (not necessarily irreducible) finite-dimensional representation of W, we attach to it a perverse sheaf $\mathcal{S}_\rho$ on $\mathfrak{g}$ by setting

$$\mathcal{S}_\rho = (\mathbf{Spr} \otimes \rho)^W \tag{2.2}$$

It follows easily from (2.1) that $\mathcal{S}_\rho$ is semi-simple and is irreducible if and only if ρ is irreducible.

Let $\mathbf{F}$ denote the Fourier-Deligne transform functor, that maps the category of $\mathbb{C}^*$-equivariant perverse sheaves on $\mathfrak{g}$ to itself (this functor is well-defined, since we have chosen an identification $\mathfrak{g} \simeq \mathfrak{g}^*$).

We shall need the following version of the Springer correspondence for the Lie algebra $\mathfrak{g} = \mathfrak{g}_d$ (cf. [2], [4]):

Theorem 2.2. *Let $\rho = \rho(\mathbf{P})$ for $\mathbf{P} \in \mathcal{P}_{n,d}$. Then $\mathbf{F}(\mathcal{S}_\rho) \simeq IC_{\mathbf{P}}$, where $IC_{\mathbf{P}}$ is viewed as a perverse sheaf on $\mathfrak{g}_d$ via the closed embedding $\mathcal{N}_{n,d} \hookrightarrow \mathfrak{g}_d$.*

2.3. A reformulation of Theorem 1.4. Consider $E^{\otimes d}$ as a representation of the group $S_d \times G$. Since our constructions are functorial, the sheaf $\mathcal{S}_{E^{\otimes d}}$ carries a G-action and hence a T-action.

Using Theorem 2.2, we can reformulate Theorem 1.4 as follows:

Theorem 2.4. *One has a canonical T-equivariant isomorphism*

$$\pi_*^{n,d}(\mathcal{L}_E^{n,d}) \simeq \mathbf{F}(\mathcal{S}_{E^{\otimes d}}) \tag{2.3}$$

(*note that T acts naturally on both sides of* (2.3)).

The rest of this section is occupied by the proof of Theorem 2.4.

Proof. Since $\mathbf{F}$ is an involution on the category of $\mathbb{C}^*$-equivariant sheaves on $\mathfrak{g}$, we must construct an isomorphism between $\mathbf{F}(\pi_*^{n,d}(\mathcal{L}_E^{n,d}))$ and $\mathcal{S}_{E^{\otimes d}}$.

First, we shall give an explicit description of the sheaf $\mathbf{F}(\pi_*^{n,d}(\mathcal{L}_E^{n,d}))$.

Let $\widetilde{\mathfrak{g}}_{n,d}$ denote the variety of all pairs $(\mathcal{V}, x)$ where $\mathcal{V} \in X_{n,d}$ and $x \in \mathfrak{g}_d$ such that $x(V_i) \subset V_i$. We have natural projections $p: \widetilde{\mathfrak{g}}_{n,d} \to \mathfrak{g}_d$ and $q: \widetilde{\mathfrak{g}}_{n,d} \to X_{n,d}$ given by $p((\mathcal{V}, x)) = x$ and $q((\mathcal{V}, x)) = \mathcal{V}$, respectively.

The variety $\widetilde{\mathfrak{g}}_{n,d}$ is a vector bundle over $X_{n,d}$ and for $\mathbf{d} \in \mathcal{Q}_{n,d}$ we shall denote by $\widetilde{\mathfrak{g}}_{n,\mathbf{d}}$ the preimage of the corresponding connected component of $X_{n,d}$.

It is known that p is a small map in the sense of Goresky-MacPherson and that its restriction to any connected component of $\widetilde{\mathfrak{g}}_{n,d}$ is surjective. In particular, any connected component of $\widetilde{\mathfrak{g}}_{n,d}$ has dimension $d^2 = \dim \mathfrak{g}_d$.

We introduce a perverse sheaf $\mathcal{K}_E$ over $\widetilde{\mathfrak{g}}_{n,d}$ as follows:

For $\mathbf{d} \in \mathcal{Q}_{n,d}$ its direct summand (denoted by $\mathcal{K}_E^{\mathbf{d}}$) that lives over the connected component $\widetilde{\mathfrak{g}}_{n,\mathbf{d}}$ is set to be $E^{\mathbf{d}}[d^2]$.

Lemma 2.5. *One has a canonical T-equivariant isomorphism*

$$\mathbf{F}(\pi_*^{n,d}(\mathcal{L}_E^{n,d})) \simeq p_* \mathcal{K}_E. \tag{2.4}$$

Proof. Both $\widetilde{\mathcal{N}_{n,d}}$ and $\widetilde{\mathfrak{g}}_{n,d}$ are vector subbundles of the trivial bundle $X_{n,d} \times \mathfrak{g}_d$ over $X_{n,d}$. Since we have chosen an identification $\mathfrak{g}_d \simeq \mathfrak{g}_d^*$, this trivial bundle is canonically self-dual and

$$\widetilde{\mathcal{N}_{n,d}} = \widetilde{\mathfrak{g}}_{n,d}^{\perp}, \tag{2.5}$$

where $\widetilde{\mathfrak{g}}_{n,d}^{\perp}$ denotes the orthogonal complement (in the trivial bundle $X_{n,d} \times \mathfrak{g}_d$) to the vector subbundle $\widetilde{\mathfrak{g}}_{n,d}$.

Let $\widetilde{\mathbf{F}}$ denote the Fourier-Deligne transform functor that maps the category of $\mathbb{C}^*$-equivariant perverse sheaves on $X_{n,d} \times \mathfrak{g}_d$ to itself. It follows from the standard properties of the functor $\widetilde{\mathbf{F}}$ that one has a canonical T-equivariant isomorphism

$$\widetilde{\mathbf{F}}(\mathcal{L}_E^{n,d}) \simeq \mathcal{K}_E.$$

This observation, together with the fact that Fourier transform commutes with direct images, implies Lemma 2.5. □

2.6. It remains to show that $p_* \mathcal{K}_E$ is isomorphic to $\mathcal{S}_{E^{\otimes d}}$. But the map p is small in the sense of Goresky-MacPherson (cf. [2]) and, therefore, $p_* \mathcal{K}_E$ is equal to the Goresky-MacPherson extension of its restriction on the set $\mathfrak{g}_d^{rs}$ of regular semisimple elements in $\mathfrak{g}_d$. Since $\mathcal{S}_{E^{\otimes d}}$ is also equal to the Goresky-MacPherson extension of its restriction on $\mathfrak{g}_d^{rs}$, it is enough to construct a T-equivariant isomorphism of the restrictions of $\mathcal{S}_{E^{\otimes d}}$ and $p_* K_E$ to $\mathfrak{g}_d^{rs}$.

Let $\mathbb{C}^{(d)}$ denote the d-th symmetric power of $\mathbb{C}$, i.e. the quotient of $\mathbb{C}^d$ by the natural action of the symmetric group S_d. Let ${}^0\mathbb{C}^d$ denote the complement to all the diagonals in $\mathbb{C}^d$, i.e.

$$ {}^0\mathbb{C}^d = \{(z_1, ..., z_d) \in \mathbb{C}^d | \ z_i \neq z_j \text{ for } i \neq j\}. \tag{2.6}$$

Set ${}^0\mathbb{C}^{(d)}$ to be the image of ${}^0\mathbb{C}^d$ in $\mathbb{C}^{(d)}$ and we denote by σ_d the natural map ${}^0\mathbb{C}^d \to {}^0\mathbb{C}^{(d)}$.

For $\mathbf{d} \in \mathcal{Q}_{n,d}$ let ${}^0\mathbb{C}^{(\mathbf{d})}$ denote the quotient of ${}^0\mathbb{C}^d$ by $S_{\mathbf{d}}$. Let $\sigma_{\mathbf{d}} : {}^0\mathbb{C}^{(\mathbf{d})} \to {}^0\mathbb{C}^{(d)}$ be the natural map.

For $\mathbf{d} \in \mathcal{Q}_{n,d}$ let $\widetilde{\mathfrak{g}}^{rs}_{n,\mathbf{d}}$ denote the open subset of $\widetilde{\mathfrak{g}}_{n,\mathbf{d}}$, consisting of all pairs $(\mathcal{V}, x) \in \widetilde{\mathfrak{g}}_{n,\mathbf{d}}$ such that $x \in \mathfrak{g}^{rs}_d$.

One has a Cartesian square:

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}^{rs}_{n,\mathbf{d}} & \xrightarrow{e_{\mathbf{d}}} & {}^0\mathbb{C}^{(\mathbf{d})} \\ {\scriptstyle p}\downarrow & & {\scriptstyle \sigma_{\mathbf{d}}}\downarrow \\ \mathfrak{g}^{rs}_d & \xrightarrow{e_d} & {}^0\mathbb{C}^{(d)} \end{array} \tag{2.7}$$

Here e_d is the map which associates to any $x \in \mathfrak{g}^{rs}_d$ the collection of its eigenvalues and $e_{\mathbf{d}}$ is the map which associates to every pair $(\mathcal{V}, x) \in \widetilde{\mathfrak{g}}_{n,\mathbf{d}}$ the eigenvalues of x in the spaces $V_1, V_2/V_1, ..., V_n/V_{n-1}$ respectively.

Let S_E denote the d-th symmetric power of the constant local system with fiber E on $\mathbb{C}$. In other words, S_E is a locally constant sheaf on ${}^0\mathbb{C}^{(d)}$ defined as

$$\mathrm{S}_E := (E^{\otimes d} \otimes (\sigma_d)_* \mathbb{C})^{S_d}.$$

We have a T-equivariant identification:

$$e_d^* \, \mathrm{S}_E[d^2] \simeq \mathcal{S}_{E^{\otimes d}}.$$

Analogously, for $\mathbf{d} \in \mathcal{Q}_{n,d}$, let $\mathrm{K}^{\mathbf{d}}_E$ be the constant sheaf over ${}^0\mathbb{C}^{(\mathbf{d})}$ with fiber $E^{\mathbf{d}}$. We have:

$$e_{\mathbf{d}}^* \, \mathrm{K}^{\mathbf{d}}_E[d^2] \simeq \mathcal{K}_E$$

(in a T-equivariant way).

The assertion of Theorem 2.4 now follows from the following lemma.

Lemma 2.7. *There exists canonical T-equivariant isomorphism*

$$\bigoplus_{\mathbf{d} \in \mathcal{Q}_{n,d}} (\sigma_{\mathbf{d}})_* \, \mathrm{K}^{\mathbf{d}}_E \simeq \mathrm{S}_E \, . \tag{2.8}$$

Proof. The lemma follows easily from the following linear algebra observation: there exists canonical $T \times S_d$-equivariant isomorphism

$$\bigoplus_{\mathbf{d} \in \mathcal{Q}_{n,d}} E^{\mathbf{d}} \simeq E^{\otimes d}. \tag{2.9}$$

The proof is clear. □ □

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