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VOLUME, CHEEGER AND GROMOV

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Abstract. It is shown that a manifold of bounded local geometry with Cheeger constant bigger than h and Gromov hyperbolicity constant smaller than δ has either infinite volume or it's volume is bounded by a function depending only on h and δ and the bounded geometry parameters.

1. Introduction

In this note we will restrict the discussion to the set of complete Riemannian manifolds M of some fixed dimension n, with all sectional curvatures bounded from below by say -1, and injectivity radius bigger than $r_0 > 0$. Our goal is to prove the following.

Theorem 1. *Given $h, \delta > 0$, assume the Cheeger constant of M is bigger than h and the Gromov hyperbolicity constant of M is smaller than δ, then either M has infinite volume or it's diameter is bounded by $f(h, \delta) < \infty$, a function which depends only on δ and h and the bounded geometry parameters.*

Since we assume bounded geometry with fixed bounds, a bound on the diameter implys a bound on the volume.

The theorem was inspired by a related result from Benjamini (1998).

We start with definitions.

Definition (Cheeger constant).

$$h(M) = \inf \frac{\text{Area}(\partial A)}{\min(\text{Vol}(A), \text{Vol}(A^c))},$$

where A runs over open subsets of M with finite volume. A^c is the compliment of A, ∂A is the boundary of A, Area(∂A) denotes the $(n-1)$-dimensional volume of ∂A, and Vol denotes n-dimensional volume.

Definition (δ-hyperbolic). Let M be a manifold. Given three points $a, b, c \in M$, pick geodesics between any two to get a geodesic triangle. Denote the geodesics by $[a, b], [a, c], [b, c]$. Say the triangle is δ-thin if for any $p \in [a, b]$

$$\min(d(p, [a, c]), d(p, [b, c])) \leq \delta,$$

and the same for $p \in [a, c]$ or $[b, c]$. M is said to be δ-hyperbolic if all geodesic triangles in M are δ-thin. Let

$$\delta(M) = \inf\{\delta | M \text{ is } \delta\text{-thin}\}.$$

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Note that the real hyperbolic spaces $\mathbb{H}^n$ have infinite volume, strictly positive Cheeger constant and finite hyperbolicity constant.

2. Proof

Proof. Given M let $h = h(M)$, $\delta = \delta(M)$ and assume that d is chosen so that for any ball $B(a, r) \subset M$ with radius $r > 1$, $\mathrm{Vol}(B(a, r)) < d^r$. Such a d exists and depends only on the dimension and the bounded geometry conditions we assumed at the start. See for instance Chavel (1993).

From now on we will assume M has finite volume. Since M has bounded geometry it is compact. Set $R = \log_d \mathrm{Vol}(M)$ and pick $C > 0$ depending only on h and d such that

$$(h/2)\mathrm{Vol}B(a, (1/2 - 2C)R) > d^{2CR},$$

for any ball centered at any $a \in M$.

Let a, b be two points that realize the diameter of M, and γ a geodesic between a and b. Let m be the midpoint of γ. Pick a ball B of radius r' $CR \leq r' \leq 2CR$ around m, for which $\mathrm{Area}(\partial B) < d^{2CR}$. Such an r' exists because of the volume upper bound in terms of d and the fact that $\mathrm{Vol}(B) = \int_0^{r'} \mathrm{Area}(\partial B(m, r))dr$. The distance from a to b is at least R. Hence the distance from a to B is at least $(1/2 - 2C)R$. The same is true for the distance from b to B. Let $M \setminus B$ be the manifold with boundary obtained from M by cutting B off. Denote by $d_{M\setminus B}$ the distance function on $M \setminus B$. By Gromov (1987) 7.1.A there is $c_\delta > 1$ such that

$$d_{M\setminus B}(a, b) \geq c_\delta^{CR},$$

(the actual bound in Gromov (1987) is $\delta(2^{CR/\delta} - 2)$). Now assume

$$\mathrm{Vol}_{M\setminus B}(B(a, d_{G\setminus B}(a, b)/2)) \leq \mathrm{Vol}_{M\setminus B}(B(b, d_{G\setminus B}(a, b)/2)).$$

That is, the Volume of the ball in $M \setminus B$ centred at a of half the distance in $M \setminus B$, from a to b, is not bigger than the volume of the similar ball centered at b. Now let $A(n) = B_{M\setminus B}(a, n) \setminus B_{M\setminus B}(a, n-1)$. Thus, for any $n < d_{M\setminus B}(a, b)/2$, by integrating the areas of $\partial(B_{M\setminus B}(a, r))$, $n - 1 \leq r \leq n$,

$$\mathrm{Vol}(A(n)) \geq h\mathrm{Vol}(B_{M\setminus B}(a, n-1)) - \mathrm{Area}(\partial B).$$

Yet C was chosen so that for $r \geq (1/2 - 2C)R$,

$$\mathrm{Area}(\partial B) \leq (h/2)\mathrm{Vol}B(a, (1/2 - 2C)R) \leq (h/2)\mathrm{Vol}(B_{G\setminus B}(a, r)).$$

(For $r < (1/2 - 2C)R$), $B_{M\setminus B}(a, r)$ is disjoint from B). So for $n \leq d_{M\setminus B}(a, b)/2$,

$$\mathrm{Vol}(B_{M\setminus B}(a, n)) > (1 + h/2)\mathrm{Vol}(B_{M\setminus B}(a, n-1)).$$

We get then,

$$\begin{aligned} d^R \geq \mathrm{Vol}(M) &\geq (1+h/2)^{d_{M\setminus B}(a,b)/2} \\ &\geq (1+h/2)^{c_\delta^{CR}/2}, \end{aligned}$$

which forces an upper bound on the diameter R in terms of d,h and δ. □

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References

[1] I. Benjamini, *Expanders are not hyperbolic*, Israel J. Math., to appear.

[2] I. Chavel, *Riemannian geometry, a modern introduction*, *Cambridge Tracts in Mathematics, 108*, Cambridge University Press, Cambridge, 1993.

[3] M. Gromov, *Hyperbolic groups*, *Essays in group theory*, pp. 75–263, *Math. Sci. Res. Inst. Publ.* **8**, Springer, 1987.

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