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MIXED MOTIVES AND ALGEBRAIC CYCLES III

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In this paper we will define a category which is a candidate for the abelian category of mixed motives over a field k. The category is a full subcategory of the triangulated category of mixed motives $\mathcal{D}(k)$ constructed in [Ha 2] (referred to as Part II in the sequel).

According to Grothendieck, the category of pure homological motives over a field k is a semi-simple abelian $\mathbb{Q}$-category $\mathcal{M}(k)$ satisfying the following properties: there is a functor h : (Smooth Projective Varieties/k)$^{opp} \to \mathcal{M}(k)$; for any Weil cohomology H^*, one has a commutative diagram,

$$\begin{tikzcd}
(\text{Smooth Proj}/k)^{opp} \arrow[r, "h"] \arrow[dr, "H^*"'] & \mathcal{M}(k) \arrow[d] \\
 & \text{Vect}
\end{tikzcd}$$

where Vect is the category of finite dimensional vector spaces over the coefficient field of H^* and $\mathcal{M}(k) \to$ Vect is an exact faithful functor. The existence of such $\mathcal{M}(k)$ is a consequence of his standard conjectures on cycle classes in $H^*(X)$ for smooth projective varieties X, see [Kl 1], [Kl 2].

Extending this one can question the existence of the abelian category of mixed motives $M\mathcal{M}(k)$ that fits in the following commutative diagram.

$$\begin{tikzcd}
(Q-\text{Proj}/k)^{opp} \arrow[r, "h"] \arrow[dr, "H^*"'] & M\mathcal{M}(k) \arrow[d] \\
 & \text{Vect}
\end{tikzcd}$$

where $(Q-\text{Proj}/k)$ is the category of quasi-projective varieties and $M\mathcal{M}(k) \to$ Vect is again an exact faithful functor. The category $M\mathcal{M}(k)$ is no more semi-simple.

The triangulated category of mixed motives $\mathcal{D}(k)$ was constructed using algebraic cycles and does not directly fits the above picture. In this part III, we discuss the conjectural properties of the category $\mathcal{D}(k)$. Specifically, we show

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the conjunction of the conjectures of Grothendieck, Murre and Soulé-Beilinson is responsible for the existence of the appropriate t-structure on $\mathcal{D}(k)$. These extended "standard conjectures" concern the algebraic part of the cohomology and the (higher) Chow groups of smooth projective varieties. By definition the category $M\mathcal{M}(k)$ is the heart of this t-structure.

Besides the pure theory, the works of Beilinson, Bloch, Deligne, Jannsen and Murre have much influenced this paper. Shuji Saito has independently found similar lines of arguments regarding the existence of the t-structure.

In §1 we recall the properties of the category $\mathcal{D}(k)$. In §2 we discuss the "standard conjectures" and their consequences about the category of pure motives. We derive the existence of the t-structure from the conjectures in §3.

§1. The triangulated category of mixed motives $\mathcal{D}(k)$.

For an equi-dimensional smooth projective variety X over k, let $C^r(X)$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-algebraic cycles of codimension r on X modulo an adequate equivalence relation. In particular, for rational equivalence $C^r(X) = CH^r(X) \otimes \mathbb{Q}$. Throughout the paper we write $CH^r(X)$ for $CH^r(X) \otimes \mathbb{Q}$.

The additive category of *finite C-symbols* C Symb(k) has objects are of the form

$$\bigoplus_{\alpha \in A} (X_\alpha, r_\alpha),$$

where A is a finite set, X_α is an *irreducible* smooth projective variety over k and $r_\alpha \in \mathbb{Z}$. The homomorphism group is the $\mathbb{Q}$-vector space

$$\mathrm{Hom}_{\mathrm{C\ Symb}(k)}(\oplus(X_\alpha, r_\alpha), \oplus(Y_\beta, s_\beta)) = \bigoplus_{\alpha,\beta} C^{\dim X_\alpha + s_\beta - r_\alpha}(X_\alpha \times Y_\beta)\ .$$

The composition of morphisms is induced by the composition of correspondences. In the case C is the rational Chow group, we denote the corresponding category simply by $\mathrm{Symb}_{\mathrm{finite}}(k)$. There is a natural contravariant functor

$$h : (\text{Smooth Proj}/k) \to \mathrm{C\ Symb}(k)\ .$$

The category $C\mathcal{M}(k)$ of *C-motives* is the pseudo-abelianization of the additive category C Symb(k). Explicitly, $C\mathcal{M}(k)$ has objects

$$\left(\bigoplus_{\alpha \in A} (X_\alpha, r_\alpha), P \right),$$

where A is a finite set, X_α is irreducible smooth projective, $r_\alpha \in \mathbb{Z}$, and

$$P = (P_{\alpha\beta}) \in \bigoplus_{\alpha\,\beta} C^{\dim X_\alpha + r_\beta - r_\alpha}(X_\alpha \times X_\beta) \text{ such that } P \circ P = P\ .$$

The homomorphism groups are

$$\begin{aligned}\mathrm{Hom}\,(\,(\oplus(X_\alpha, r_\alpha), P), (\oplus(Y_\beta, s_\beta), Q)\,)\\ = Q\circ\left(\bigoplus_{\alpha,\beta} C^{\dim X_\alpha + s_\beta - r_\alpha}(X_\alpha \times Y_\beta)\right)\circ P\ ;\end{aligned}$$

the composition of maps are given by composition of correspondences.

When C^* is the rational Chow group (resp. algebraic cycles modulo numerical equivalence), the resulting category we denote by $CH\mathcal{M}(k)$ and call the category of *Chow motives* (resp. $\mathcal{M}(k)$, the category of *Grothendieck motives*). There is a canonical functor cano : $CH\mathcal{M}(k) \to \mathcal{M}(k)$.

Let H^* : $(\mathrm{Smooth\,Proj}/k)^{opp} \to (\mathrm{Vect})$ be a Weil cohomology. If numerical equivalence coincides with homological equivalence for H^*, then one has a faithful functor

$$H^* : \mathcal{M}(k) \to (\mathrm{Vect})$$

where

$$H^*(X,p,r) = \oplus H^i(X,p,r), \quad H^i(X,p,r) = p_* H^{i+2r}(X)\ .$$

(p gives rise to $p_* \in \mathrm{End}(H^*(X))$.)

We recall the properties of the triangulated category of mixed motives from Part II, particularly §4. There the category $\mathcal{D}(k)$ was defined so that its objects are “diagrams” of smooth projective varieties and correspondences among them.

Let k be an arbitrary field. There is a triangulated $\mathbb{Q}$-category $\mathcal{D}(k)$ with the following properties:

(1) $\mathcal{D}(k)$ has dual, tensor product, internal Hom, the unit object $\mathbb{Q}$, and the Tate objects $\mathbb{Q}(r)$;

(2) There is a contravariant functor h : $(\mathrm{Smooth\,Proj}/k) \to \mathcal{D}(k)$. If X is smooth and projective, one has

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}, h(X)\otimes\mathbb{Q}(r)[2r-m]) = K_m(X)^{(r)}_{\mathbb{Q}}\ .$$

Here the right side is an Adams-graded piece of the K-group of X.

(3) If k'/k is an extension of fields, there is the base extension functor

$$\mathcal{D}(k) \to \mathcal{D}(k'), \quad K \mapsto K\otimes_k k'\ .$$

(4) Let H^* be one of the etale, Betti or algebraic de Rham cohomology: for X smooth projective over k,

$$H^*(X) = H^*(X\otimes_k \bar{k}, \mathbb{Q}_\ell), \quad H^*(X(\mathbb{C}),\mathbb{Q}), \quad H^*_{DR}(X/k),$$

and $\Lambda = \mathbb{Q}_\ell, \mathbb{Q}$ or k be the coefficient field. There is the corresponding realization functor

$$\mathbb{R}\Gamma : \mathcal{D}(k) \to D^b_f(\Lambda - \mathrm{Vect}),$$

(the target the derived category of complexes of Λ-vector spaces with bounded and finite dimensional cohomology) and the induced $H^* : \mathcal{D}(k) \to (\Lambda - \mathrm{Vect})$.

If k is a finite field, the etale cohomology functor Γ factors through

$$\mathbb{R}\Gamma : \mathcal{D}(k) \to D^b_c(\mathrm{Spec}\ k, \mathbb{Q}_\ell)$$

(cf. [BBD],[De 3] for the definition of the category $D^b_c(\mathrm{Spec}\ k, \mathbb{Q}_\ell)$ for X over a finite field). For this see the proof of Proposition (3.6).

(5) There is a triangulated subcategory $\mathcal{D}_{\mathrm{finite}}(k) \subset \mathcal{D}(k)$ of "finite diagrams" satisfying all the properties (1)-(4). $\mathcal{D}(k)$ is the pseudo-abelianization of $\mathcal{D}_{\mathrm{finite}}(k)$. There is a commutative diagram with arrows full embeddings

$$\begin{array}{ccc} \mathrm{Symb}(k) & \to & \mathcal{D}_{\mathrm{finite}}(k) \\ \downarrow & & \downarrow \\ CH\mathcal{M}(k) & \hookrightarrow & \mathcal{D}(k)\ . \end{array}$$

(6) The category $\mathcal{D}_{\mathrm{finite}}(k)$ has the following structure. To an interval $[a, b] \subset [-\infty, +\infty]$ where $a, b \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and an object K of $\mathcal{D}_{\mathrm{finite}}(k)$, there is associated an object $V_{[a,b]}K$ in $\mathcal{D}_{\mathrm{finite}}(k)$. The following properties are to be satisfied:

(a) For $a \le a'$ and $b \le b'$ there are morphisms

$$V_{[a,b]}K \to V_{[a',b']}K,$$

satisfying the transitivity, namely they give a functor from the category of ordered pairs.

One has $V_{[a,a]}K = K^{-a}[a]$ where K^{-a} is an object of $\mathrm{Symb}(k)$. One has $V_{[-\infty,b]}K \xrightarrow{\sim} V_{[a,b]}K$ for $a << 0$, $V_{[a,b]}K \xrightarrow{\sim} V_{[a,\infty]}K$ for $b >> 0$, and $V_{[-\infty,\infty]}K = K$.

(b) There are distinguished triangles

$$V_{[a,b]}K \to V_{[a,c]}K \to V_{[b+1,c]}K \xrightarrow{[1]}\ .$$

The diagram

$$\begin{array}{ccc} V_{[b+1,c]}K & \to & V_{[a,b]}K[1] \\ \downarrow & & \downarrow \\ V_{[b'+1,c']}K & \to & V_{[a',b']}K[1] \end{array}$$

commutes where $a \le a'$, $b \le b'$ and $c \le c'$.

(c) If $u : K \to L$ is a morphism one can choose morphisms $V_{[a,b]}u : V_{[a,b]}K \to V_{[a,b]}L$ so that they commute with morphisms in (a) and (b). (The choice is not unique so $K \mapsto V_{[a,b]}K$ is not a functor.

We refer to this structure as the *V-truncation*. Denote $V_aK = V_{[-\infty,a]}K$ (an increasing "filtration") and $Gr_a^V K$ for $V_{[a,a]}K$. There are distinguished triangles

$$V_{[-\infty,a-1]}K \to V_{[-\infty,a]}K \to Gr_a^V K \xrightarrow{[1]} ,$$

and

$$Gr_a^V K \to V_{[a,+\infty]}K \to V_{[a+1,+\infty]}K \xrightarrow{[1]} .$$

Using the notation in [Part II, §4], the objects $V_{[a,b]}K$ are defined as follows.

$$V_{[a,b]}(K^m; f^{m,n}) = \big((V_{[a,b]}K)^m; \quad (V_{[a,b]}f)^{m,n}\big),$$

where

$$(V_{[a,b]}K)^m = \begin{cases} K^m & \text{if} \quad m \in [-b,-a] \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(V_{[a,b]}f)^{m,n} = \begin{cases} f^{m,n} & \text{if} \quad [m,n] \subset [-b,-a] \\ 0 & \text{otherwise.} \end{cases}$$

The association $K \mapsto V_{[a,b]}K$ is *not* a functor $\mathcal{D}(k) \to \mathcal{D}(k)$. (Consider, for example the object $K := [X \xrightarrow{id} X]$ placed in degrees zero and one. It is the zero object in $\mathcal{D}(k)$, yet $Gr_a^V K$ are not zero for $a = 0, -1$.) Given a morphism $u : K \to L$, take a representative $(u^{m,n})$, see the paragraph before [Part II, (4.4)]. Then the representative induces a morphism $V_{[a,b]}K \to V_{[a,b]}L$. However, equivalent representatives can induce different morphisms.

§2. "Standard" conjectures and consequences.

We consider the three "standard" conjectures on the Chow groups and K-groups of smooth projective varieties. The statements of the conjectures are followed by the consequences. Then the implications of the conjectures on the properties of the category of pure motives $\mathcal{M}(k)$ and $CH\mathcal{M}(k)$ are discussed.

Grothendieck's standard conjectures [Kl 1]:

These are five conjectures altogether, as stated below. We refer to these as **Conjecture (Gro)** in the sequel.

Let $X \mapsto H^*(X)$ be a Weil cohomology, [Kl §1], which satisfies the hard Lefschetz "theorem". Let X be a smooth projective variety of dimension d

and $L \in H^2(X)$ the class of a hyperplane section of X; the hard Lefschetz theorem reads: $L^{d-i} : H^i(X) \to H^{2d-i}(X)$ is an isomorphism for $i \le d$. As a consequence, one has the primitive decomposition: For $i \le d$, $H^i_{prim}(X) := \mathrm{Ker}\left(L^{d-i+1} : H^i(X) \to H^{2d-i+2}(X)\right)$, the primitive part. Then

$$H^i(X) = \bigoplus_{j \ge max(0,i-d)} L^j H^{i-2j}_{prim}(X) \ .$$

For each $p \ge 0$, denote by $A^p(X) \subset H^{2p}(X)$ be the $\mathbb{Q}$-vector space generated by the classes of algebraic cycles. We call an element of $H^{2p}(X)$ algebraic if it is in $A^p(X)$.

Conjecture (A). For a smooth projective variety X and a hyperplane section L, and $p \le d = \dim X$, the map

$$L^{d-p} : A^p(X) \to A^{d-p}(X),$$

is an isomorphism.

Conjecture (B). Define $\Lambda \in H^{2d-2}(X \times X)$ so that it induces the map $H^*(X) \to H^{*-2}(X)$ given as follows: if $a \in H^i(X)$, and $a = \sum_j L^j a_j$, $a_j \in H^{i-2j}_{prim}(X)$ according to the primitive decomposition, then

$$\Lambda a = \sum_{j \ge max\,(1,i-d)} L^{j-1} a_j \ .$$

Then Λ is the class of an algebraic cycle.

Conjecture (C). Let $\pi^i \in H^{2d-i}(X) \otimes H^i(X)$, $0 \le i \le 2d$, be the Kunneth components of the class of the diagonal Δ_X in $X \times X$. Then π^i are algebraic.

Conjecture (D). Homological equivalence (with respect to the Weil cohomology) coincides with numerical equivalence for algebraic cycles.

Conjecture (I). For a non-zero element $a \in A^p(X) \cap H^{2p}_{prim}(X)$, the rational number $(-1)^p L^{d-p} \cdot a \cdot a$ is positive.

Theorem 2.1 [Kl 1,2].

(1) *Conjecture (B) implies both Conjecture (A) and Conjecture (C). In this case, if $u \in H^*(X \times Y)$ is algebraic, and then its Kunneth components are also algebraic. The Kunneth components π^i are independent of the cohomology theory in the sense there are algebraic cycles $\tilde{\pi}^i$, independent of cohomology theories, whose cycles classes are π^i.*

(2) *Conjecture (A) and Conjecture (I) jointly imply Conjecture (D). In this case, $A^p(X)$ is independent of the cohomology theory and equals C^p_{num}(=*

codim p cycles modulo numerical equivalence). The dimension $H^i(X)$ over the coefficient field is independent of the cohomology theory.

(3) *Provided all the conjecture are true, the category $\mathcal{M}(k)$ of Grothendieck motives is semi-simple and abelian. The functor $H^* : \mathcal{M}(k) \to (\Lambda - Vect)$ is faithful and exact.*

The filtration conjecture by Murre [Mu] (Referred to as **Conjecture (Mu)**):

Let X be an irreducible smooth projective variety over a field k, and π^i $(i = 0, 1, \cdots 2\dim X)$ be the Kunneth components of the diagonal (for a Weil cohomology).

(A) The $\{\pi^i\}$ (in Conjecture (Gro-C)) lifts to an orthogonal set of projectors $\{\Pi^i\}$ in $CH^{\dim X}(X \times X)$ such that $\sum \Pi^i = \Delta_X$.

(B) The correspondences $\Pi^0, \cdots, \Pi^{r-1}$ and $\Pi^{2r+1}, \cdots, \Pi^{2\dim X}$ act as zero on $CH^r(X)$.

(C) For each $\nu \geq 0$, let $F^\nu CH^r(X) = \operatorname{Ker} \Pi_*^{2r} \cap \operatorname{Ker} \Pi_*^{2r-1} \cap \cdots \cap \operatorname{Ker} \Pi_*^{2r-\nu+1}$. Then $F^\nu CH^r(X)$ is independent of the lifting Π^i.

(D) $F^1 CH^r(X) = CH^r(X)_{hom}$, where the latter is the subspace of classes of cycles homologically equivalent to zero (for a Weil cohomology).

Remark. It is proved in [Ja, Theorem 5.2] that Conjecture (Mu) implies the following (part of a conjecture of Beilinson's on Chow groups):

(1) The filtration on $CH^r(X)$ is compatible with the product: one has $F^a CH^r(X) \cdot F^b CH^s(X) \subset F^{a+b} CH^{r+s}(X)$.

(2) The filtration is respected by f^* and f_* for maps $f : X \to Y$.

(3) $F^{r+1} CH^r(X) = 0$.

Vanishing conjecture (**Conjecture (Van)**):

Let (X, P) be an object on $CH\mathcal{M}(k)$ whose realization is of cohomological degrees $\geq 2r - n$ if $n > 0$ and $> 2r$ if $n = 0$. Then one has

$$P_* CH^r(X, n) = 0 \ .$$

We may distinguish the two cases as Conjecture (Van- $(n > 0)$) and Conjecture (Van-$(n = 0)$).

Remark. In the case $P = \Delta_X$ the conjecture is precisely the vanishing conjecture of Soulé-Beilinson: $CH^r(X, n) = 0$ if $2r - n \leq 0$ and $n > 0$ (the case $n = 0$ is vacuous). So Conjecture (Van) is thought of as a generalization of Soulé-Beilinson conjecture to Chow motives.

Conjecture (Van-$(n = 0)$) and Conjectures (Mu- B, C) are related; see Proposition (2.4).

Reformulation of Conjecture (Van). In terms of extension groups in $\mathcal{D}(k)$, one may state the above as follows. Let P^{i+2r} (resp. Q^{j+2s}) be a projector of cohomological degree $i+2r$ (resp. $j+2s$) in $CH^{\dim X}(X\times X)$ (resp. $CH^{\dim Y}(Y\times Y)$); then

$$\mathrm{Hom}_{\mathcal{D}(k)}((X,P^{i+2r},r),(Y,Q^{j+2s},s)[-n])=0$$

if

$$\begin{cases} -i+j\geq -n \quad \text{and} \quad n>0, \quad \text{or} \\ -i+j>0 \quad \text{and} \quad n=0, \quad \text{or} \\ n<0 \end{cases}$$

(Since $Q\circ u\circ P=({}^tP\times Q)_*u$, ${}^tP=$ the transpose of P, the group in question equals

$$({}^tP^{i+2r}\times Q^{j+2s})_*CH^{\dim X-r+s}(X\times Y,n)$$

and $({}^tP^{i+2r}\times Q^{j+2s})$ is a projector of $X\times Y$ of cohomological degree $-i+j+(2\dim X-2r+2s)$.)

For another reformulation, see Proposition (2.9).

Lemma 2.2. (Assume Conjecture (Mu).) *Let $\{p_j\}$ be an orthogonal set of projectors in $H^{2\dim X}(X\times X)$. It can be lifted to an orthogonal set of projectors $\{P_j\}$ in $CH^{\dim X}(X\times X)$. Any other lifting of $\{p_j\}$ is of the form $\{(1+\eta)^{-1}P_j(1+\eta)\}$ where $\eta\in CH^{\dim X}(X\times X)_{\mathrm{hom}}$. If $P\in CH^{\dim X}(X\times X)$ is a projector lifting $\sum p_j$, then $\{P_j\}$ can be so chosen that $\sum P_j=P$. If $\sum p_j=[\Delta_X]_{\mathrm{hom}}$, then necessarily $\sum P_j=[\Delta_X]_{\mathrm{rat}}$.*

Proof. The map $CH^{\dim X}(X\times X)\to A^{2\dim X}(X\times X)$ is a surjective ring homomorphism with kernel $CH^{\dim X}(X\times X)_{\mathrm{hom}}$. By (Mu) (and Remark to it) it is the first step of the finite separated filtration $F^\bullet$ on $CH^{\dim X}(X\times X)$ which is compatible with the product, so it is nilpotent. The claim follows from the following lemma, see [Mu-2, 7.3], [Ja, Lemma 5.4]; (2) and (3) follows from (1).

Lemma. *Let $\phi: A\to B$ is a surjective ring homomorphism of non-commutative rings with nilpotent kernel. Then*

1. *Any orthogonal set of idempotents $\{p_1,\cdots p_m\}$ of B (i.e. $p_ip_j=\delta_{i,j}p_i$) can be lifted to an orthogonal set of idempotents $\{P_1,\cdots,P_m\}$ of A. Moreover, any other lifting is of the form $\{(1+\eta)^{-1}P_j(1+\eta)\}$ where $\eta\in\mathrm{Ker}\,\phi$.*
2. *If $\{p_1,\cdots,p_m\}$ is as above and P is an idempotent of A lifting $\sum p_j$, then $\{P_j\}$ can be so taken that $\sum P_j=P$.*
3. *If $\sum p_j=1_B$, then $\sum P_j=1_A$.*

Proposition 2.3. (Assume Conjecture (Mu).) *If (X,P) is of pure cohomological degree $2r$, then the cycle class map*

$$P_*CH^r(X) \to H^{2r}(X),$$

is injective.

Proof. Take a set $\{\Pi^i\}$ as in Conjecture (Mu-A). If p is the cycle class of P, then $\{p, \pi^{2r}-p\}$ is an orthogonal set of homological projectors. By applying Lemma (2.2) it can be lifted to an orthogonal set of projectors $\{P', \Pi^{2r}-P'\}$ with the prescribed sum Π^{2r}. Since both P and P' lift p, by (2.2) again, there is $\eta \in CH^{\dim X}(X\times X)_{hom}$ such that $P=(1+\eta)^{-1}P'(1+\eta)$.

By (Mu-D), $\operatorname{Ker}\Pi^{2r}_* = CH^r(X)_{hom}$, so $\Pi^{2r}_*CH^r(X)\to H^{2r}(X)$ is injective. We have

$$P_*CH^r(X) = P'_*CH^r(X) \subset \Pi^{2r}_*CH^r(X) \hookrightarrow H^{2r}(X)\ .$$

Proposition 2.4. *Conjectures (Gro), (Mu-A, D) and (Van- ($n=0$)) imply Conjecture (Mu-B, C).*

Proof. First we show (Mu-C). If $\{\Pi^i\}$ is as in (Mu-A)

$$h(X)=\bigoplus_i (X,\Pi^i), \quad \text{and} \quad F^\nu CH^r(X) = CH^r(\bigoplus_{i\le 2r-\nu}(X,\Pi^i))\ .$$

If follows from the reformulation of (Van-($n=0$)), that the subobject $\oplus_{i\le 2r-\nu}(X,\Pi^i)\subset h(X)$ is independent of the of the choice of $\{\Pi^i\}$. Thus, $F^\nu CH^r(X)$ is also independent.

To derive (Mu-B), take Π^i with $i>2r$; then $(X,\Pi^i,0)$ is an object of $CH\mathcal{M}(k)$ of cohomological degree $>2r$; by (Van- ($n=0$)), its Chow group $\Pi^i_*CH^r(X)=0$. [Ja, Theorem 5.2] showed under these hypotheses, namely (Mu-A, C, D and half of B), the following holds true: if $u\in CH^*(X\times Y)$ then $u_*: CH^*(X)\to CH^*(Y)$ respects the filtrations and $u_*: Gr_FCH^*(X)\to Gr_FCH^*(Y)$ depends only on the homology class of u.

The following argument is taken from [Ja, §2]. Assume $i<r$ and consider the projection $\pi^i: H^*(X)\to H^i(X)$, the hard Lefschetz isomorphism L^{d-i} and its inverse λ:

$$\begin{array}{ccc} H^*(X) & & \\ \downarrow \pi^i & \overset{\alpha}{\searrow} & \\ H^i(X) & \underset{\lambda}{\overset{L^{d-i}}{\rightleftarrows}} & H^{2d-i}(X) \end{array}$$

By (Gro), λ is algebraic; let $\alpha = L^{d-i} \circ \pi^i$. Then $\lambda \circ \alpha = \pi^i$. Take liftings $A \in CH^{2d-i}(X \times X)$, $\Lambda \in CH^i(X \times X)$ of α, λ, respectively. Then $\Lambda \circ A$ and Π^i are homologically equivalent, so their actions on $Gr_F CH^r(X)$ are the same. One has a commutative diagram

$$\begin{tikzcd}
Gr_F CH^r(X) \arrow[d, "\Pi^i"'] \arrow[dr, "A"] & \\
Gr_F CH^r(X) & Gr_F CH^{r+d-i}(X) \arrow[l, "\Lambda"']
\end{tikzcd}$$

Since $Gr_F CH^{r+d-i}(X) = 0$ evidently, we have $\Pi^i_* = 0$ on $Gr_F CH^r(X)$ and on $CH^r(X)$.

Recall that there is a full embedding of categories

$$in_{CH\mathcal{M}} : CH\mathcal{M}(k) \to \mathcal{D}(k) .$$

Let $\mathcal{M}(k)^{\deg i}$ (resp. $CH\mathcal{M}(k)^{\deg i}$) be the full subcategory of $\mathcal{M}(k)$ (resp. $CH\mathcal{M}(k)$ consisting of objects K of pure cohomological degree i, namely those objects with $H^j(K) = 0$ if $j \neq i$.

(2.5). (We assume Conjecture (Gro).) Let X be an equi-dimensional smooth projective variety and $p \in A^{\dim X}(X \times X)$ be a projector. Let $p = \sum p^i$, $p^i \in H^{2\dim X - i}(X) \otimes H^i(X)$, be the Kunneth decomposition; p^i are algebraic by Conjecture (Gro) and (2.1) (1). For an object (X, p, r) in $\mathcal{M}(k)$, one sets

$$\underline{H}^i\big((X, p, r)\big) := (X, p^{i+2r}, r) .$$

Note $H^*(\underline{H}^i(X, p, r)) = p_* H^{i+2r}(X) = H^i(X, p, r)$. By linearity this association gives rise to a functor $\underline{H}^i : \mathcal{M}(k) \to \mathcal{M}(k)$. We have the properties:

(1) For an object K of $\mathcal{M}(k)$, one has $H^* \underline{H}^i(K) = H^i(K)$ and $\underline{H}^i(K(r)) \cong \big(\underline{H}^{i+2r}(K)\big)(r)$.

(2) For a smooth variety X, $(X, p, r) = \bigoplus_i \underline{H}^i(X, p, r)$ in $\mathcal{M}(k)$. More generally,
$$K = \oplus \underline{H}^i(K)$$
in $\mathcal{M}(k)$.

(3) If $p^{i+2r} \in H^{2\dim X}(X \times X)$ (resp. $q^{j+2s} \in H^{2\dim Y}(Y \times Y)$) is a projector of pure cohomological degree $i + 2r$ (resp. $j + 2s$), then
$$\mathrm{Hom}_{\mathcal{M}(k)}((X, p^{i+2r}, r), (Y, q^{j+2s}, s)) = 0 \quad \text{for} \quad i \neq j.$$

The proposition below is a consequence of (1)–(3).

Proposition 2.6. (Assume Conjecture (Gro).) *The canonical functor*

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{M}(k)^{\deg i} \to \mathcal{M}(k)$$

is an equivalence of categories.

(2.7). (Assume Conjectures (Mu) and (Van).) Let (X, P, r) be an object of $CH\mathcal{M}(k)$. One has $P = \sum P^i$ where $\{P^i\}$ is an orthogonal set of projectors with P^i of cohomological degree i. Set

$$h^i(X, P, r) = (X, P^{i+2r}, r), \qquad \tau_{\leq i}(X, P, r) := \bigoplus_{j\leq i}(X, P^{j+2r}, r),$$

$$\text{and} \quad \tau_{\geq i}(X, P, r) := \bigoplus_{j\geq i}(X, P^{j+2r}, r)\ .$$

It follows from Conjecture (Van) — see its reformulation, case $n = 0$ — that the subobject $\tau_{\leq i}(X, P, r)$ of (X, P, r) is independent of the choice of the $\{P^i\}$; similarly for the quotient $\tau_{\geq i}(X, P, r)$. Hence

$$h^i(X, P, r) = \tau_{\leq i}(X, P, r)/\tau_{\leq i-1}(X, P, r),$$

is also independent. Moreover $\tau_{\leq i}$, $\tau_{\geq i}$, and h^i are functorial for symbols.

By linearity one has functors

$$h^i, \quad \tau_{\leq i}, \quad \tau_{\geq i} : CH\mathcal{M}(k) \to CH\mathcal{M}(k)\ .$$

There is a non-canonical decomposition $(X, P, r) = \bigoplus h^i(X, P, r)$ and more generally

$$K = \bigoplus_i h^i(K)$$

for K in $CH\mathcal{M}(k)$.

(If Conjectures (Gro),(Mu) and (Van) are assumed) the functors h^i and $\underline{H}^i$ are compatible with the canonical functor cano : $CH\mathcal{M}(k) \to \mathcal{M}(k)$, namely

$$\mathrm{cano} \circ h^i = \underline{H}^i \circ \mathrm{cano}\,.$$

The decomposition $K = \oplus h^i(K)$ and the analogous one for $\mathrm{cano}(K)$ in $\mathcal{M}(k)$ are compatible with cano.

Proposition 2.8. (Assume Conjectures (Gro) and (Mu).) *The canonical functor*

$$CH\mathcal{M}(k)^{\deg i} \to \mathcal{M}(k)^{\deg i}$$

is an equivalence of categories.

Proof. Take (X,P,r) and (Y,Q,s), objects in $CH\mathcal{M}(k)$ of pure cohomological degree i. (Assume X equi-dimensional. The general case where one has direct sums of such symbols can be treated similarly.) One has

$$\begin{aligned}\mathrm{Hom}_{CH\mathcal{M}(k)}\big((X,P,r),(Y,Q,s)\big) &= Q\circ CH^{\dim X+s-r}(X\times Y)\circ P \\ &= ({}^tP\times Q)_* CH^{\dim X+s-r}(X\times Y)\ .\end{aligned}$$

The image of (X,P,r) under the canonical functor is (X,p,r) where $p\in H^{2\dim X}(X\times X)$ is the image of P; similarly the image of (Y,Q,s) is (Y,q,s). As above, one has $\mathrm{Hom}_{\mathcal{M}(k)}\big((X,p,r),(Y,q,s)\big) = ({}^tp\times q)_*A^{\dim X+s-r}(X\times Y)$. ${}^tP\times Q$ has pure cohomology degree $2(\dim X+s-r)$ so by (2.3)

$$({}^tP\times Q)_* CH^{\dim X+s-r}(X\times Y)\to ({}^tp\times q)_*A^{\dim X+s-r}(X\times Y)$$

is an isomorphism. Hence the functor is fully faithful.

For the essential surjectivity take an object $\mathcal{M}(k)$, say (X,p,r) where $p\in A^{\dim X}(X\times X)$ is a projector. By Conjecture (Mu) and Lemma (2.2), p can be lifted to a projector $P\in CH^{\dim X}(X\times X)$, and (X,P,r) in $CH\mathcal{M}(k)$ lifts (X,p,r).

We define the full embedding of categories $in_{\mathcal{M}^i}: \mathcal{M}(k)^{\deg i}\to\mathcal{D}(k)$ so that the diagram

$$\begin{array}{ccc} CH\mathcal{M}(k)^{\deg i} & \xrightarrow{in_{CH\mathcal{M}}(-)[i]} & \mathcal{D}(k) \\ \downarrow & \nearrow in_{\mathcal{M}^i} & \\ \mathcal{M}(k)^{\deg i} & & \end{array}$$

is commutative. Concretely (X,p,r) of $\mathcal{M}(k)^{\deg i}$ is sent to $(X,P,r)[i]$ of $\mathcal{D}(k)$ where P is a lifting of p. If $j\neq i$ and (Y,q,s) is in $\mathcal{M}(k)^{\deg j}$, it is sent to $(Y,Q,s)[j]$ by $in_{\mathcal{M}^j}$ and

$$\mathrm{Hom}_{\mathcal{D}(k)}((X,P,r)[i],(Y,Q,s)[j])=0$$

by (Van) – in its reformulation $n=i-j\neq 0$. So we define then the full embedding

$$in_{\mathcal{M}} := \bigoplus in_{\mathcal{M}^i}: \quad \mathcal{M}(k)=\bigoplus_i \mathcal{M}(k)^{\deg i}\to\mathcal{D}(k)\ .$$

This embedding is so arranged that the an object is sent to an object of pure cohomological degree 0. Note that the digram

$$\begin{array}{ccc} CH\mathcal{M}(k) & \xrightarrow{in_{CH\mathcal{M}}} & \mathcal{D}(k) \\ \downarrow & \nearrow in_{\mathcal{M}} & \\ \mathcal{M}(k) & & \end{array}$$

is *not* commutative even after restricted to parts of pure cohomological degrees. The functor $in_{\mathcal{M}}$ takes the object $\mathbb{Q}(r)$ to $\mathbb{Q}(r)$, and it is compatible with the Tate twists.

Proposition 2.9. (Assume Conjectures (Gro), (Mu) and (Van).) *Let* $\underline{H}^j(X)(r)$ *be an object of* $\mathcal{M}(k) \hookrightarrow \mathcal{D}(k)$. *Then*

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}, \underline{H}^j(X)(r)[i]) = 0$$

if

$$\begin{cases} i < 0, & or \\ i = 0 \quad and \quad j - 2r \neq 0, & or \\ i > 0 \quad and \quad i > -j + 2r \ . \end{cases}$$

Remark. If $w := j - 2r$ (the weight of $\underline{H}^j(X)(r)$), the condition reads: $i < 0$, or $i = 0$ and $w \neq 0$, or $i > 0$ and $i > -w$.

Proof. The object $\underline{H}^j(X)(r) = (X, \pi^j, r)$ in $\mathcal{M}(k)^{j-2r}$ is taken by $in_{\mathcal{M}}$ to $(X, \Pi^j, r)[j - 2r]$, so

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}, \underline{H}^j(X)(r)[i]) = \Pi^j_* CH^r(X, -j + 2r - i) \ .$$

The claim is obtained by (Van).

Remark 2.10. Though we will not need in the sequel, the results in this section hold with $CH\mathcal{M}(k)$ and $\mathcal{M}(k)$ replaced by $CH\mathcal{M}_{inf}(k)$ and $\mathcal{M}_{inf}(k)$, respectively. (For these categories of Grothendieck motives of infinite type see [Part II, §2].)

Conjecture(Gro) implies the category $\mathcal{M}_{inf}(k)$ is semi-simple and abelian: take any object M of $\mathcal{M}_{inf}(k)$ and we have to show that it is a direct sum of simple objects. This is the case if $M = \oplus(X_\alpha, r_\alpha)$ since each (X_α, r_α) is a sum of simple objects by the semi-simplicity of $\mathcal{M}(k)$. Hence the same holds if M is a direct summand of $\oplus(X_\alpha, r_\alpha)$.

(2.5)-(2.8) hold for $CH\mathcal{M}_{inf}(k)$ and $\mathcal{M}_{inf}(k)$; the proofs are the same.

§3. The motivic t-structure.

We refer to [BBD] for details on t-structures.

Definition 3.1. *Let D be a triangulated category. A* t-structure *on D consists of a pair of full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ such that, letting $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$, one has:*

(i) For X in $D^{\leq 0}$ and Y in $D^{>0}$, one has $\mathrm{Hom}_D(X, Y) = 0$.

(ii) One has $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 0} \supset D^{\geq 1}$.

(iii) For an object X in D, there is a distinguished triangle $A \to X \to B \xrightarrow{[1]}$ with $A \in D^{\leq 0}$ and $B \in D^{>0}$.

Definition 3.2. *Let D and D' be triangulated categories with t-structures. A functor $F : D \to D'$ is* t-exact *if it is an exact functor of triangulated categories, and*

$$F(D^{\leq 0}) \subset {D'}^{\leq 0} \quad \textit{and} \quad F(D^{\geq 0}) \subset {D'}^{\geq 0} .$$

We give two examples of triangulated categories with t-structures. Let $\mathcal{A}$ be an abelian category, and $D = D(\mathcal{A})$ be the derived category (of unbounded complexes, for example). Define $D^{\leq 0}$ (resp.$D^{\geq 0}$) to be the full subcategory consisting of complexes K with $H^i(K) = 0$ for $i > 0$ (resp. $H^i(K) = 0$ for $i < 0$). This defines a t-structure on $D(\mathcal{A})$, which one refers to as the natural t-structure. Note $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is the category $\mathcal{A}$ fully embedded in the derived category as complexes concentrated in degree 0.

As a second example, if X is a variety over k (k is algebraically closed or finite), then the category $D^b_c(X, \mathbb{Q}_\ell)$ of "complexes of $\mathbb{Q}_\ell$-sheaves with constructible cohomology" is defined in [BBD]. One has the perverse t-structure on $D^b_c(X, \mathbb{Q}_\ell)$, the heart of which is the category of perverse sheaves on X.

3.3 Properties of a t-structure.

1. The inclusion $D^{\leq n} \subset D$ has a right adjoint functor $\tau_{\leq n}$. There is thus the adjunction morphism $\tau_{\leq n} X \to X$ for X in D. For $a \leq b$, there is a functorial morphism $\tau_{\leq a} X \to \tau_{\leq b} X$ compatible with the adjunction morphisms:

$$\begin{tikzcd} \tau_{\leq a} X \arrow[rr] \arrow[dr] & & \tau_{\leq b} X \arrow[dl] \\ & X & \end{tikzcd}$$

Dually, the inclusion $D^{\geq n} \subset D$ has a right adjoint functor $\tau_{\geq n}$. There is the adjunction morphism $X \to \tau_{\geq n} X$ for X in D. For $a \leq b$, there is a functorial morphism $\tau_{\geq b} X \to \tau_{\geq a} X$ compatible with the adjunction morphisms.

We let $\tau_{[a,b]} X = \tau_{\geq a} \tau_{\leq b} X = \tau_{\leq b} \tau_{\geq a} X$. Also we set $H^0(X) := \tau_{[0,0]} X$, and $H^a(X) = H^0(X[a])$.

2. For any object X in D, there exists a unique morphism $d : \tau_{\geq 1}X \to \tau_{\leq 0}X[1]$ such that the triangle

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X \xrightarrow{d}$$

is distinguished. Up to unique isomorphism of distinguished triangles, this is the unique distinguished triangle $A \to X \to B \xrightarrow{[1]}$ with $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

3. Let $C = D^{\geq 0} \cap D^{\leq 0}$, a full subcategory of D. We call this the *heart* of the t-structure. The heart is an abelian category. A sequence $0 \to X \to Y \to Z \to 0$ in C is exact iff there is a distinguished triangle $X \to Y \to Z \xrightarrow{[1]}$ in D. The functor $H^0 : D \to C$ is a cohomological functor, namely a distinguished triangle induces a long exact sequence.

Denote by H^* one of Betti, etale, or de Rham cohomology. Define full subcategories of $\mathcal{D}(k)$ as follows: $\mathcal{D}(k)^{\leq p}$ (resp.$\mathcal{D}(k)^{\geq p}$) consists of objects K in $\mathcal{D}(k)$ such that $H^j(K) = 0$ for $j > p$ (resp. $j < p$). These are independent of the cohomology theory by comparison isomorphisms. Let

$$M\mathcal{M}(k) := \mathcal{D}(k)^{\leq 0} \cap \mathcal{D}(k)^{\geq 0} \ .$$

This is the candidate of the abelian category of mixed motives.

In this section we prove the following:

Theorem 3.4. (Assume Conjectures (Gro), (Mu) and (Van).) *The pair of subcategories $(\mathcal{D}(k)^{\leq 0}, \mathcal{D}(k)^{\geq 0})$ is a t-structure on $\mathcal{D}(k)$. $M\mathcal{M}(k)$ is an abelian category. Any one of Betti, etale, or de Rham cohomology functor $\mathbb{R}\Gamma : \mathcal{D}(k) \to D^b(\Lambda - Vect)$ is a t-exact functor with respect to the t-structure and the natural t-structure on $D^b(\Lambda - Vect)$.*

Remark. On the other hand, under the presumption of Conjecture (Gro), the existence of the t-structure implies Conjectures (Mu) and (Van). cf. [Ja].

For each $K \in \mathcal{D}_{finite}(k)$, the V-truncation induces a filtration on its cohomology $H^*(K)$. We put

$$W_pH^i(K) = \mathrm{Im}\left(H^i(V_{p-i}K) \to H^i(K)\right),$$

the image of the map induced by the canonical morphism $V_{p-i}K \to K$. Then $W_\bullet$ is an increasing filtration on $H^i(K)$ as a Λ-vector space; this is called the *weight filtration.* There is a convergent spectral sequence induced by the V-truncation

$$(\star) \qquad E_1^{a,b} = H^{a+b}(Gr^V_{-a}K) = H^b(K^a) \Rightarrow E^{a+b} = H^{a+b}(K) \ .$$

This spectral sequence is functorial for formal symbols K. Also, the terms E_r, $r \geq 2$, are functorial in K in $\mathcal{D}_{finite}(k)$.

Proposition 3.5. *The above spectral sequence $E_r^{a,b}$ is functorial in K in $\mathcal{D}_{finite}(k)$ for $r \geq 2$. In particular, the weight filtration on $H^*(K)$ is functorial.*

Proof. Let $(u^{m,n})$ be a representative of the zero morphism $K \to L$. By definitions in [Part II,§4] there exist $U^{m,n} \in \operatorname{Hom}(K,L)^{-(n-m-1)}$ such that

$$u^{m,n} = (-1)^n \partial U^{m,n} + \sum_{m<\ell<n} (-1)^{m+\ell} U^{\ell,n} \circ f^{m,\ell} + \sum_{m<\ell<n} (-1)^{\ell+n} g^{\ell,n} \circ U^{m,\ell} \ ;$$

in particular,

$$u^{m,,m} = (-1)^m \partial U^{m,m} - U^{m+1,m} \circ f^{m,m+1} - g^{m-1,m} \circ U^{m,m-1} \ .$$

$\partial U^{m,n}$ induces zero on cohomology. So the maps they induce between the cohomology groups satisfy

$$u_*^{m,,m} = U_*^{m+1,m} \circ f_*^{m,m+1} - g_*^{m-1,m} \circ U_*^{m,m-1} \ .$$

$$\begin{tikzcd}
& H^i(K^m) \arrow[r, "f_*^{m,m+1}"] \arrow[dl, "U_*^{m,m-1}"'] \arrow[d, "u_*^{m,m}"] & H^i(K^{m+1}) \arrow[dl, "U_*^{m+1,m}"] \\
H^i(L^{m-1}) \arrow[r, "g_*^{m-1,m}"] & H^i(L^m) &
\end{tikzcd}$$

Therefore $u_*^{m,m}$ induces the zero map :$\operatorname{Ker} f_*^{m,m+1} \to H^i(L^m)/\operatorname{Im} g_*^{m-1,m}$, and also between the E_2-terms.

Proposition 3.6. *The spectral sequence degenerates at E_2: $E_2^{a,b} = E_\infty^{a,b}$.*

Proof. Consider first the case where k is a finite field and H^* is $\mathbb{Q}_\ell$-cohomology. The realization functor

$$\mathcal{D}(k) \to D^b(\mathbb{Q}_\ell), \quad K \mapsto \mathbb{R}\Gamma(K \otimes_k \bar{k}),$$

factors through $\mathcal{D}(k) \to D_c^b(\operatorname{Spec} k, \mathbb{Q}_\ell)$. In fact, in [Part II,§5], the definitions may be refined so that the functor $\mathbb{R}\Gamma$ takes values in $D_c^b(\operatorname{Spec} k, \mathbb{Q}_\ell)$:

$$\begin{aligned}
C^{a,b}(X,r) &:= C^{a+2b}(\mathbb{Q}_{\ell,\bar{X}\times\square^{-b}})(r), \\
\Gamma C^{a,b}(X,r) &:= \Gamma(\bar{X}\times\square^{-b}, C^{a+2b}(\mathbb{Q}_{\ell,\bar{X}\times\square^{-b}})(r)\,)^{alt}, \\
\Gamma C(X,r) &:= \text{the associated simple complex}\ .
\end{aligned}$$

Extending by linearity to formal symbols and then to diagrams, one has the functor $K \mapsto \Gamma C(K \otimes_k \bar{k}) \in D_c^b(\operatorname{Spec} k, \mathbb{Q}_\ell)$.

If k is a finite field, by [De 3], $H^i\Gamma C(X,r) = H^{i+2r}(\bar{X},\mathbb{Q}_\ell)(r)$ has pure weight i. Hence

$$H^i(Gr_a^V\Gamma C(K_{\bar{k}})) = H^{i+a}\Gamma C(K_{\bar{k}}^{-a}),$$

has pure weight a. In the spectral sequence, $E_1^{a,b}$ has weight a, so $d_r = 0$ for $r \geq 2$.

If k is an arbitrary field, take a subfield k_0 which is finitely generated over the prime field and over which K is defined. One is reduced to the case over k_0. By a specialization argument one is then reduced to the case over a finite field.

In the case of singular cohomology or de Rham cohomology, the claim holds as well by comparison isomorphisms.

For the rest of this section, we assume Conjectures (Gro), (Mu) and (Van). To show that $(\mathcal{D}(k)^{\leq 0}, \mathcal{D}(k)^{\geq 0})$ is a t-structure, we need to verify the three conditions in (3.1). The condition (ii) is obvious.

Proposition 3.7. *To an object K of* Symb(k)*, functorially associated are objects $K \mapsto \tau_{\leq p}K$ (resp. $K \mapsto \tau_{\geq p}K$). The object $\tau_{\leq p}K$ (resp. $\tau_{\geq p}K$) is in $\mathcal{D}(k)^{\leq p}$ (resp. $\mathcal{D}(k)^{\geq p}$). There is a distinguished triangle*

$$\tau_{\leq p}K \to K \to \tau_{>p}K \xrightarrow{[1]},$$

in the category $\mathcal{D}(k)$, where the morphism $\tau_{>p}K \to \tau_{\leq p}K\ [1]$ is zero.

Proof. See (2.7). For the last claim, note that K is non-canonically isomorphic to the direct sum of $\tau_{\leq p}K$ and $\tau_{>p}K$.

Lemma 3.8. *In the category $\mathcal{D}(k)$ one has*

$$\mathrm{Hom}_{\mathcal{D}(k)}(K,L) \cong \mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}, K^\vee \otimes L)\ .$$

Proof. If K and L are in $\mathcal{D}_{finite}(k)$, this is so by definition. In general, take two objects of the form $[K,p]$, $[L,q]$ where K, L are in $\mathcal{D}_{finite}(k)$ and p, q are projectors. Let $u : K \to L$ and $u' : \mathbb{Q} \to K^\vee \otimes L$ be the corresponding morphism. Then $q \circ u \circ p = (p^\vee \otimes q) \circ u'$ by a formula after (4.5). Hence

$$\begin{aligned} q\circ \mathrm{Hom}(K,L)\circ p &\cong (p^\vee \otimes q)\circ \mathrm{Hom}(\mathbb{Q}, K^\vee \otimes L) \\ &= \mathrm{Hom}(\mathbb{Q}, [K^\vee \otimes L, p^\vee \otimes q]) \\ &= \mathrm{Hom}(\mathbb{Q}, [K,p]^\vee \otimes [L,q])\ . \end{aligned}$$

Lemma 3.9. *For K, L in $\mathcal{D}(k)$, there is an isomorphism (Kunneth formula)*

$$H^n(K \otimes L) = \bigoplus_{a+b=n} H^a(K) \otimes H^b(L) \ .$$

There is a functorial isomorphism

$$H^n(K) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(H^{-n}(K^\vee), \Lambda) \ .$$

Proof. The first follows from $\mathbb{R}\Gamma(K \otimes L) = \mathbb{R}\Gamma(K) \otimes \mathbb{R}\Gamma(L)$.

There is a canonical morphism $K \otimes K^\vee \to \mathbb{Q}$ which corresponds to $id_K : K \to K$ under

$$\mathrm{Hom}_{\mathcal{D}(k)}(K \otimes K^\vee, \mathbb{Q}) \cong \mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}, K^\vee \otimes K) \cong \mathrm{Hom}_{\mathcal{D}(k)}(K, K) \ .$$

Induced is a pairing $H^n(K) \otimes H^{-n}(K^\vee) \to \Lambda$, or a map $H^n(K) \to \mathrm{Hom}_\Lambda(H^{-n}(K^\vee), \Lambda)$. The latter map is functorial in K. It is shown to be an isomorphism by using the spectral sequence $(\star)$ and reducing to the case of a formal symbol.

3.10 Proof of $\mathrm{Hom}_{\mathcal{D}(k)}(K, L) = 0$ for $K \in \mathcal{D}(k)^{\leq 0}$ and $L \in \mathcal{D}(k)^{>0}$.

Since $K^\vee \in \mathcal{D}(k)^{\geq 0}$ and $K^\vee \otimes L \in \mathcal{D}(k)^{>0}$, one may assume $K = \mathbb{Q}$ and $L \in \mathcal{D}(k)^{>0}$. First we assume $L \in \mathcal{D}_{finite}(k)^{>0}$. Let $L = (L^m, f^{m,n})$.

Consider the spectral sequence $E_1^{a,b} = H^{a+b}(Gr^V_{-a} L) = H^b(L^a) \Rightarrow E^{a+b} = H^{a+b}(L)$. By the degeneracy of this at E_2 and the assumption that $H^i(L) = 0$ for $i \leq 0$, one has: the complex of E_1-terms and d_1's

$$\cdots \to E_1^{a-1,b} \xrightarrow{d_1} E_1^{a,b} \xrightarrow{d_1} E_1^{a+1,b} \to \cdots$$

is exact at $E_1^{a,b}$ for $a + b \leq 0$. Note that, since L^a is a formal symbol, $E_1^{a,b} = H^b(L^a)$ comes from a Grothendieck motive, namely the object $\underline{H}^b(L^a)$ of $\mathcal{M}(k)$. The d_1 differentials (which are induced by the morphisms $f^{a,a+1} : L^a \to L^{a+1}$) are morphisms of Grothendieck motives, and

(1) The complex of Grothendieck motives

$$\cdots \to \underline{H}^b(L^{a-1}) \xrightarrow{d_1} \underline{H}^b(L^a) \xrightarrow{d_1} \underline{H}^b(L^{a+1}) \to \cdots ,$$

is exact at $\underline{H}^b(L^a)$ for $a + b \leq 0$.

To calculate $\mathrm{Hom}(\mathbb{Q}, L)$, consider the spectral sequence induced from the filtered complex $\big(\mathcal{Z}^0(L, \cdot), \mathcal{Z}^0(V_{-a}L, \cdot)\big)$

$$E_1^{a,b} = H^{a+b}\mathcal{Z}^0(Gr^V_{-a} L, \bullet) \Rightarrow H^{a+b}\mathcal{Z}^0(L, \bullet),$$

which may be written

$$E_1^{a,b} = \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, L^a[b]) \Rightarrow E^{a+b} = \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, L[a+b]) \ . \tag{2}$$

Recall on $Hom_{\mathcal{D}}(\mathbb{Q}, L^a[b])$ there is the filtration $F^\bullet$ whose graded quotients are

$$Gr_F^{i+b} \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, L^a[b]) = \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, \underline{H}^i(L^a)[-i+b]) \ .$$

The map $d_1^{a,b} = f_*^{a,a+1} : \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, L^a[b]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, L^{a+1}[b])$ respects the filtrations and

$$Gr_F^{i+b} d_1^{a,b} : \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, \underline{H}^i(L^a)[-i+b]) \to \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, \underline{H}^i(L^{a+1})[-i+b]),$$

is induced from the map $\underline{H}^i(f^{a,a+1}) : \underline{H}^i(L^a) \to \underline{H}^i(L^{a+1})$.

That $\mathrm{Hom}(\mathbb{Q}, L) = 0$ follows from (2) and the following claim.

(3) The complex

$$\cdots \xrightarrow{Gr_F d_1^{a-1,b}} \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, \underline{H}^i(L^a)[-i+b]) \xrightarrow{Gr_F d_1^{a,b}} \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, \underline{H}^i(L^{a+1})[-i+b]) \to \cdots ,$$

is exact at $\mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}, \underline{H}^i(L^a)[-i+b])$ for $a+b \le 0$.

In fact, if $a+b \le 0$, either $i+a \le 0$ or $i-b > 0$. If $i+a \le 0$, the exactness follows from the claim (1) and the exactness of the functor $\mathrm{Hom}(\mathbb{Q}, (-)[j])$ on the category of Grothendieck motives, which is semi-simple abelian. If $i-b > 0$, the terms in the complex are zero by the consequence (2.9) of the vanishing conjecture.

Proposition 3.11. *Let K be an object of $\mathcal{D}(k)$. There is an object $\tau_{\ge p}K$ in $\mathcal{D}(k)^{\ge p}$ together with a morphism $K \to \tau_{\ge p}K$ (resp. $\tau_{\le p}K$ in $\mathcal{D}(k)^{\le p}$ together with $\tau_{\le p}K \to K$) satisfying the properties:*

(i) *$H^i(K) \xrightarrow{\sim} H^i(\tau_{\ge p}K)$ for $i \ge p$ (resp. $H^i(\tau_{\le p}K) \xrightarrow{\sim} H^i(K)$ for $i \le p$).*

(ii) *There is a unique distinguished triangle*

$$\tau_{\le p}K \to K \to \tau_{>p}K \xrightarrow{[1]} \ .$$

(iii) *The morphism $K \to \tau_{\ge p}K$ has the universal property: if $u : K \to L$ is a morphism where $L \in \mathcal{D}(k)^{\ge p}$, it factors uniquely through $\tau_{\ge p}K$. The association $K \mapsto \tau_{\ge p}K$ is functorial. Similarly for $\tau_{\le p}K$.*

Proof. Once (i) and (ii) are verified for an object K, (iii) follows using (3.10). If $u : K \to L$, there is a unique morphism $\tau_{\leq p}K \to \tau_{\leq p}L$ making the following diagram commutative.

$$\begin{array}{ccccccc} \tau_{\leq p}K & \longrightarrow & K & \longrightarrow & \tau_{>p}K & \xrightarrow{[1]} & \\ \downarrow & & \downarrow u & & & & \\ \tau_{\leq p}L & \longrightarrow & L & \longrightarrow & \tau_{>p}L & \xrightarrow{[1]} & . \end{array}$$

Then one can also define an object $\underline{H}^p(K)$ in $\mathcal{D}(k)^{\leq 0} \cap \mathcal{D}(k)^{\geq 0}$ with distinguished triangles

$$\tau_{\leq p-1}K \to \tau_{\leq p}K \to \underline{H}^p(K)[-p] \xrightarrow{[1]},$$
$$\underline{H}^p(K)[-p] \to \tau_{\geq p}K \to \tau_{\geq p+1}K \xrightarrow{[1]}$$

(see [BBD, p.30]).

For an arbitrary object K, we will show (i) and (ii). The object is of the form $[K, p]$ with K in $\mathcal{D}_{finite}(k)$ and p a projector. Suppose one has $\tau_{\geq p}K$ and $\tau_{\leq p}K$ satisfying (i), (ii). Then $\tau_{\geq p}p$ is a projector of $\tau_{\geq p}K$ and one defines $\tau_{\geq p}[K, p] = [\tau_{\geq p}K, \tau_{\geq p}p]$; similarly for $\tau_{\leq p}[K, p]$. The required properties are satisfied.

Assume K in $\mathcal{D}_{finite}(k)$, and $m \leq n$ are such that $K^i = 0$ for $i \notin [m, n]$. We construct $\tau_{\geq p}K$ by induction on $n - m$. By shifting we may assume $p = 0$. If $m = n$, K is a formal symbol K^m concentrated in degree m, so let

$$\tau_{\geq 0}K = (\tau_{\geq -m}K^m)[-m],$$

where $\tau_{\geq -m}K^m$ is as in (3.7).

Assume $m < n$ and consider the objects

$$\begin{aligned} K &= [K^m \to \cdots \to K^{n-1} \to K^n], \\ K' &= [K^m \to \cdots \to K^{n-1} \to 0] = V_{[-n+1,-m]}K, \\ K'' &= [0 \to \cdots \quad \to K^n \to 0] = (V_{[-n,-n]}K)[1], \\ & \qquad\qquad (K^n \text{ concentrated in deg} = n-1) \end{aligned}$$

and the distinguished triangle $K \to K' \xrightarrow{\beta} K'' \xrightarrow{[1]}$. By induction hypothesis, we have objects $\tau_{\leq p}K'$ and $\tau_{\leq p}K''$.

The image of the map that β induces on cohomology

$$\mathrm{Im} = \mathrm{Im}[H^{-1}(K') \to H^{-1}(K'')] \subset H^{-1}(K''),$$

comes from a Grothendieck motive. Indeed by the spectral sequence $(\star)$ for $H^*(K')$ and by (3.6) one has

$$\mathrm{Im} = \mathrm{Im}[H^{-1}(K^{n-1}[-n+1]) \to H^{-1}(K'')] \ .$$

Denote by $\underline{\mathrm{Im}}$ the object of $\mathcal{M}(k)$ which gives Im. Consider the composite $\underline{\mathrm{Im}}[1] \to \underline{H}^{-1}(K'')[1] \to \tau_{\geq -1}K''$. The cone of this has cohomology as follows:

$$H^i\,\mathrm{Cone}(\underline{\mathrm{Im}}[1] \to \tau_{\geq -1}K'') = \begin{cases} 0 & i < -1 \\ H^{-1}(K'')/\,\mathrm{Im} & i = -1 \\ H^i(K'') & i > -1. \end{cases}$$

One has a diagram

$$\begin{array}{ccccccc} \underline{H}^{-1}(K')[1] & \to & \tau_{\geq -1}K' & \to & \tau_{\geq 0}K' & \xrightarrow{[1]} \\ \downarrow u & & \downarrow v & & \downarrow w & \\ \underline{I}^{-1}(K'')[1] & \to & \tau_{\geq -1}K'' & \to & \mathrm{Cone}\left(\underline{I}^{-1}(K'')[1] \to \tau_{\geq -1}K''\right) & \xrightarrow{[1]} \end{array}$$

where $v = \tau_{\geq -1}\beta$. The left square being commutative, there is a morphism w which makes (u, v, w) a morphism between distinguished triangles. (Since $\mathrm{Hom}(\tau_{\geq -1}K', \underline{I}^{-1}(K'')[1]) = 0$, such a w is unique.) We take

$$\tau_{\geq 0}K = \mathrm{Cone}\,(w)[-1] \ .$$

There is a unique morphism $K[1] \to \mathrm{Cone}(w)$ making the following a morphism of distinguished triangles.

$$\begin{array}{ccc} K' & \to & \tau_{\geq 0}K' \\ \downarrow \beta & & \downarrow \\ K'' & \to & C \\ \downarrow & & \downarrow \\ K[1] & \to & \mathrm{Cone}(w) \\ \downarrow [1] & & \downarrow [1] \end{array}$$

One verifies $H^i(\tau_{\geq 0}K) = H^i(K)$ for $i \geq 0$ and $= 0$ for $i < 0$.

Define $\tau_{<0}K$ by the distinguished triangle

$$\tau_{<0}K \to K \to \tau_{\geq 0}K \xrightarrow{[1]} \ .$$

The properties (i) and (ii) are satisfied, and this completes the proof of the proposition and of Theorem (3.4).

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