

## RATIONAL HOMOLOGY OF $\text{Aut}(F_n)$

ALLEN HATCHER AND KAREN VOGTMANN

ABSTRACT. We compute the rational homology in dimensions less than seven of the group of automorphisms of a finitely generated free group of arbitrary rank. The only non-zero group in this range is  $H_4(\text{Aut}(F_4); \mathbb{Q})$ , which is one-dimensional.

### 1. Introduction

In [7] a space  $\mathbb{A}_n$  was introduced on which the group  $\text{Aut}(F_n)$  of automorphisms of a free group of rank  $n$  acts with finite stabilizers.  $\mathbb{A}_n$  is a basepointed version of the “Outer space” introduced in [3] to study the group of outer automorphisms  $\text{Out}(F_n)$ ; the definition of Outer space, in turn, was motivated by the definition of the Teichmüller space of a surface, on which of the mapping class group of the surface acts with finite stabilizers.

Like Teichmüller space and Outer space, the space  $\mathbb{A}_n$  is contractible, and it follows from an easy spectral sequence argument that the quotient  $Q_n = \mathbb{A}_n/\text{Aut}(F_n)$  has the same rational homology as  $\text{Aut}(F_n)$ . The space  $Q_n$  is the analog for basepointed graphs of the classical Riemann moduli space, the quotient of Teichmüller space by the mapping class group. There is a natural cell structure on  $Q_n$  in which the vertices are the isomorphism classes of basepointed graphs with fundamental group  $F_n$ , and the higher-dimensional cells have combinatorial descriptions in terms of collapsing forests. Since the description of  $Q_n$  is straightforward, one may attempt to compute the rational homology of  $\text{Aut}(F_n)$  by explicitly computing the cells and boundary maps of  $Q_n$ . However, as  $n$  increases the task of enumerating isomorphism classes of graphs and forests gets out of hand very quickly, and one can do only the very simplest calculations.

In [8] a filtration of  $\mathbb{A}_n$  was introduced by  $\text{Aut}(F_n)$ -invariant subspaces  $\mathbb{A}_{n,k}$  which behave homotopically like the  $k$ -skeleton of  $\mathbb{A}_n$ . In particular, the subspace  $\mathbb{A}_{n,k}$  is  $(k-1)$ -connected, so that for the quotient  $Q_{n,k} = \mathbb{A}_{n,k}/\text{Aut}(F_n)$  we have  $H_i(\text{Aut}(F_n); \mathbb{Q}) \cong H_i(Q_{n,k}; \mathbb{Q})$  if  $i < k$ . These “skeleta”  $Q_{n,k}$  have two distinct computational advantages over  $Q_n$ . First of all, they are much smaller, making many more computations feasible. Secondly the spaces  $Q_{n,k}$  do not continue to grow with  $n$ : there is a natural inclusion  $Q_{n,k} \hookrightarrow Q_{n+1,k}$  which was shown

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in [8] to be a homeomorphism for  $n \geq 2k$  and a homotopy equivalence for  $n \geq 3k/2$ . In particular, this shows that the rational homology of  $\text{Aut}(F_n)$  is *stable*, i.e.  $H_i(\text{Aut}(F_n); \mathbb{Q})$  is independent of  $n$  for  $n \geq 3(i+1)/2$ . We exploit these observations to obtain the main result of this paper, some low-dimensional calculations for  $\text{Aut}(F_n)$ :

**Theorem 1.1.**  $H_4(\text{Aut}(F_4); \mathbb{Q}) \cong \mathbb{Q}$ . For all other  $i$  and  $n$  with  $1 \leq i \leq 6$  and  $n \geq 1$ ,  $H_i(\text{Aut}(F_n); \mathbb{Q}) = 0$ .

It was shown in [7] that the map from  $\text{Aut}(F_n)$  to  $\text{Out}(F_n)$  is an isomorphism on homology in dimensions  $i \ll n$ , so that Theorem 1.1 also shows the stable homology of  $\text{Out}(F_n)$  is zero in dimensions less than seven.

**Question.** *Is the stable rational homology of  $\text{Aut}(F_n)$  and  $\text{Out}(F_n)$  trivial in all dimensions?*

As a byproduct of the proof that  $H_3(\text{Aut}(F_n); \mathbb{Q}) = 0$ , we also improve the rational homology stability range to obtain:

**Proposition 1.2.** *The map  $H_i(\text{Aut}(F_n); \mathbb{Q}) \rightarrow H_i(\text{Aut}(F_{n+1}); \mathbb{Q})$  induced by inclusion is an isomorphism for  $n \geq 5(i+1)/4$ .*

It follows by finiteness of the quotients  $Q_n$  that  $H_i(\text{Aut}(F_n); \mathbb{Q})$  and  $H^i(\text{Aut}(F_n); \mathbb{Q})$  are finite-dimensional vector spaces over  $\mathbb{Q}$ . The stable cohomology with any coefficient field is a Hopf algebra, as we shall show by a standard argument, so Hopf's theorem implies that the stable rational cohomology is a tensor product of a polynomial algebra on even-dimensional generators and an exterior algebra on odd-dimensional generators. In view of our calculations, it would be somewhat surprising if this algebra turned out to be nontrivial. (The stable cohomology with  $\mathbb{Z}_p$  coefficients is known to be nontrivial for all primes  $p$  since it contains the cohomology of the infinite symmetric group  $\Sigma = \bigcup_n \Sigma_n$ , as noted in [7].)

Homology stability and the Hopf algebra structure on stable cohomology are two of a number of formal properties which  $\text{Aut}(F_n)$  shares with both  $SL_n(\mathbb{Z})$  and with mapping class groups. Borel computed the stable rational cohomology of  $SL_n(\mathbb{Z})$  to be an exterior algebra on generators of dimension  $5, 9, 13, \dots$ . The stable rational cohomology of mapping class groups is known to contain a polynomial algebra with one generator in each even dimension ([10],[11]), and to equal this polynomial algebra in dimension less than 6 ([5],[6]). There are natural maps  $\text{Aut}(F_n) \rightarrow GL(n, \mathbb{Z})$  and  $M_{g,1} \rightarrow \text{Aut}(F_{2g})$ , where  $M_{g,1}$  is the mapping class group of a surface of genus  $g$  with one boundary circle. If the stable rational homology of  $\text{Aut}(F_n)$  is non-trivial, we can still ask:

**Question.** *Do the natural maps  $\text{Aut}(F_n) \rightarrow GL(n, \mathbb{Z})$  and  $M_{g,1} \rightarrow \text{Aut}(F_{2g})$  induce trivial maps on stable rational homology?*

This question is also interesting in the unstable range.

There are few previous computations of rational homology for  $\text{Aut}(F_n)$ . It is straightforward to compute that the abelianization of  $\text{Aut}(F_n)$  is finite, so that  $H_1(\text{Aut}(F_n); \mathbb{Q}) = 0$ . Results of Gersten [4] show that  $H_2(\text{Aut}(F_n); \mathbb{Q}) = 0$  for  $n \geq 5$ , and Kiralis [9] showed  $H_2(\text{Aut}(F_4); \mathbb{Q}) = 0$ .

As we shall describe, the calculations for  $H_i(\text{Aut}(F_n); \mathbb{Q})$  were done with computer assistance for  $i = 4, 5, 6$ , and we thank Craig Jensen for checking and extending some of the programs. In particular, the calculation of  $H_6(\text{Aut}(F_n); \mathbb{Q})$  would not have been possible without his help.

## 2. Background

### *Complexes, quotients and degree*

For the convenience of the reader, we briefly recall here the definitions of the various spaces we use. For details and proofs, we refer to [8].

Points of  $\mathbb{A}_n$  are equivalence classes of pairs  $(g, \Gamma)$ , where  $\Gamma$  is a basepointed metric graph whose basepoint  $v_0$  has valence at least 2 and all other vertices have valence at least 3, and  $g: F_n \rightarrow \pi_1(\Gamma, v_0)$  is an isomorphism. Pairs  $(g, \Gamma)$  and  $(g', \Gamma')$  are equivalent if there is a basepoint-preserving homothety  $h: \Gamma \rightarrow \Gamma'$  with  $h_* \circ g = g'$ , where  $h_*$  is the induced map on  $\pi_1$ . An automorphism  $\alpha$  of  $F_n$  acts on  $\mathbb{A}_n$  by changing the markings:  $(g, \Gamma)\alpha = (g\alpha, \Gamma)$ .  $\mathbb{A}_n$  decomposes into a disjoint union of open simplices, where the simplex containing  $(g, \Gamma)$  is obtained by varying the (nonzero) lengths of the edges of  $\Gamma$ . An open simplex  $(g, \Gamma)$  is in the closure of  $(g', \Gamma')$  if it may be obtained by collapsing some edges of  $\Gamma'$  to zero. This face relation gives the open simplices in  $\mathbb{A}_n$  the structure of a partially ordered set, whose geometric realization  $S\mathbb{A}_n$  is called the *spine* of  $\mathbb{A}_n$ .  $Q_n$  is defined to be the quotient of  $S\mathbb{A}_n$  under the action of  $\text{Aut}(F_n)$ .

The *degree* of a basepointed graph  $\Gamma$  with basepoint  $v_0$  is defined by

$$\text{degree}(\Gamma) = \sum_{v \neq v_0} |v| - 2,$$

where the sum is taken over all non-basepoint vertices of  $\Gamma$ , and  $|v|$  denotes the valence of  $v$ . An easy Euler characteristic argument shows that the degree may also be computed by the formula  $\text{degree}(\Gamma) = 2n - |v_0|$ . The complex  $S\mathbb{A}_{n,k}$  is the subcomplex of  $S\mathbb{A}_n$  spanned by graphs of degree at most  $k$ .  $S\mathbb{A}_{n,k}$  is invariant under  $\text{Aut}(F_n)$  with quotient  $Q_{n,k}$ . The main theorem of [8] implies that  $S\mathbb{A}_{n,k}$  is  $(k - 1)$ -connected, giving

**Proposition 2.1.**  $H_i(Q_{n,k}; \mathbb{Q}) \cong H_i(\text{Aut}(F_n); \mathbb{Q})$  for  $k > i$ , and  $H_k(Q_{n,k}; \mathbb{Q})$  maps onto  $H_k(\text{Aut}(F_n); \mathbb{Q})$ . □

Particularly, to compute  $H_k(\text{Aut}(F_n); \mathbb{Q})$  we need compute  $H_k(Q_{n,k+1}; \mathbb{Q})$  only. If we are very lucky, and  $H_k(Q_{n,k}; \mathbb{Q}) = 0$  we can stop there (so far, this has only happened for  $k \leq 3$ ).

### Hopf algebra structure

Let  $\text{Aut} = \bigcup_n \text{Aut}(F_n)$ . Then  $H_i(\text{Aut}(F_n); G) = H_i(\text{Aut}; G)$  for  $n \gg i$  and any coefficient group  $G$  by [7], hence also  $H^i(\text{Aut}(F_n); R) = H^i(\text{Aut}; R)$  for  $n \gg i$  and any commutative coefficient ring  $R$  with identity.

**Proposition 2.2.**  *$H^*(\text{Aut}; R)$  is a commutative associative Hopf algebra.*

**Proof.** The free group  $F_\infty = \bigcup_n F_n$  is the free product of its subgroups  $F_{\text{odd}}$  and  $F_{\text{even}}$  generated by the odd- and even-numbered basis elements of  $F_\infty$ . Using this free product decomposition, define a homomorphism  $\mu : \text{Aut} \times \text{Aut} \rightarrow \text{Aut}$  by  $\mu(\phi, \psi) = \phi * \psi$ . The homomorphism  $\mu$  induces a map of classifying spaces  $B\mu : B\text{Aut} \times B\text{Aut} \rightarrow B\text{Aut}$ . The claim is that the coproduct  $B\mu^* : H^*(\text{Aut}; R) \rightarrow H^*(\text{Aut}; R) \otimes H^*(\text{Aut}; R)$  makes  $H^*(\text{Aut}; R)$  into a Hopf algebra. Note that since  $B\mu^*$  is induced by a map of spaces, it is an algebra homomorphism.

The map  $\phi \mapsto \mu(\phi, 1)$  induces the identity on homology since every homology class in  $\text{Aut}$  is realized in  $\text{Aut}(F_n)$  for some finite  $n$ , and the restriction of  $\phi \mapsto \mu(\phi, 1)$  to  $\text{Aut}(F_n)$  is realizable by conjugation in  $\text{Aut}$ , inducing the identity on homology. Similarly, the map  $\phi \mapsto \mu(1, \phi)$  induces the identity on homology. Both maps then induce the identity on cohomology as well, so the standard proof that the cohomology of an H-space is a Hopf algebra applies to show that  $H^*(\text{Aut}; R)$  is a Hopf algebra (see e.g. [12], p. 267).  $\square$

### 3. The cubical chain complex

The simplices of  $S\mathbb{A}_{n,k}$  naturally group themselves into cubes to give  $S\mathbb{A}_{n,k}$  the structure of a cubical complex, with one cube, denoted  $(g, \Gamma, \Phi)$ , for each pointed marked graph  $(g, \Gamma)$  and forest  $\Phi$  in  $\Gamma$ . Specifically, the cube  $(g, \Gamma, \Phi)$  is the full subcomplex of  $S\mathbb{A}_{n,k}$  spanned by all basepointed marked graphs which can be obtained from  $(g, \Gamma)$  by collapsing a subforest of  $\Phi$ . The dimension of  $(g, \Gamma, \Phi)$  is the number of edges in  $\Phi$ , so that the maximal cubes in  $S\mathbb{A}_{n,k}$  are  $k$ -dimensional and correspond to maximal trees in graphs of degree  $k$  all of whose non-basepoint vertices are trivalent.

The stabilizer of the cube  $(g, \Gamma, \Phi)$  under the action of  $\text{Aut}(F_n)$  is naturally isomorphic to the group  $\text{Aut}(\Gamma, \Phi)$  of automorphisms of  $\Gamma$  which fix the basepoint and send  $\Phi$  to itself. We think of the unmarked cube  $(\Gamma, \Phi)$  as embedded in  $\mathbb{R}^i$  where  $i$  is the number of edges of the forest  $\Phi$ , each unit coordinate vector is an edge of the cube, the graph  $\Gamma_\Phi$  obtained by collapsing each edge of  $\Phi$  is at the origin, and the graph  $\Gamma$  is diagonally opposite. The group  $\text{Aut}(\Gamma, \Phi)$  acts linearly on the cube by permuting the coordinates of  $\mathbb{R}^i$ , fixing the diagonal from  $\Gamma_\Phi$

to  $\Gamma$ . The quotient of the cube  $(\Gamma, \Phi)$  under this action is thus a cone  $C(\Gamma, \Phi)$  with base  $B(\Gamma, \Phi)$ , the quotient of the boundary of the cube by the stabilizer of the cube. The possibilities for the homotopy type of  $B(\Gamma, \Phi)$  are limited by the following proposition.

**Proposition 3.1.** *The quotient of an  $i$ -sphere by a finite linear group has the rational homology of an  $i$ -sphere or ball. The latter possibility holds if and only if the action includes orientation-reversing homeomorphisms.*

**Proof.** Consider the equivariant homology spectral sequence for the action of the group  $G$  on a space  $X$  (cf. [2], p. 173-175): Vertical and horizontal filtrations of the double complex  $C_*(X) \otimes_G C_*(EG)$  give two spectral sequences. The first has  $E_{p,q}^2 = H_p(G; H_q(X, M))$ . The second has  $E_{p,q}^2 = H_p(X/G; \mathcal{H}_q)$ ; here  $\mathcal{H}_q$  is the system of local coefficients  $\sigma \mapsto H_q(G_\sigma; M)$ , where  $G_\sigma$  is the stabilizer of  $\sigma$ . For  $M = \mathbb{Q}$  and  $G$  finite, the second spectral sequence degenerates to  $H_p(X/G; \mathbb{Q})$ . For  $X = S^i$ , the  $E_{p,q}^2$ -term in the first spectral sequence is zero unless  $(p, q) = (0, 0)$  or  $(0, i)$ . If the action of  $G$  on  $S$  preserves orientation,  $E_{0,i}^2 = \mathbb{Q}$ ; otherwise,  $E_{0,i}^2 = 0$ . Since both spectral sequences converge to the same thing,  $S/G$  must have the rational homology of a sphere or of a disk, as was to be shown.  $\square$

**Corollary 3.2.**  *$C(\Gamma, \Phi)$  is a cone on a rational homology ball if and only if there is an automorphism of  $\Gamma$  fixing the basepoint which induces an odd permutation of the (unoriented) edges of  $\Phi$ . If there is no such automorphism, then  $C(\Gamma, \Phi)$  is a cone on a rational homology sphere.*

**Proof.** This is immediate from the description of  $C(\Gamma, \Phi)$ , since an odd permutation of the edges induces an orientation-reversing homeomorphism of the boundary of the cube  $(\Gamma, \Phi)$ .  $\square$

An element of  $\text{Aut}(\Gamma, \Phi)$  which induces an odd permutation of the edges of  $\Phi$  will be called an *odd symmetry* of  $(\Gamma, \Phi)$ .

Two cubes  $(\Gamma, \Phi)$  and  $(\Gamma', \Phi')$  are *isomorphic* if there is a basepoint-preserving isomorphism  $h: \Gamma \rightarrow \Gamma'$  sending  $\Phi$  to  $\Phi'$ . The quotients of the  $i$ -dimensional cubical skeleta of  $S\mathbb{A}_n$  by the action of  $\text{Aut}(F_n)$  give a filtration  $Q^0 \subset Q^1 \subset \dots \subset Q^k$  of  $Q_{n,k}$ , with  $Q^i$  obtained from  $Q^{i-1}$  by attaching one cone  $C(\Gamma, \Phi)$  along its “boundary”  $B(\Gamma, \Phi)$  for each isomorphism type of  $i$ -dimensional cubes  $(\Gamma, \Phi)$ . We define a *cubical chain complex*  $C_*$  with terms

$$C_i = H_i(Q^i, Q^{i-1}) \cong \bigoplus_{(\Gamma, \Phi)} H_i(C(\Gamma, \Phi), B(\Gamma, \Phi)),$$

where the sum is over all isomorphism types of  $i$ -dimensional cubes, and boundary maps the boundary maps of the triple  $(Q^{i+1}, Q^i, Q^{i-1})$ .

**Proposition 3.3.**  $H_i(C_*) = H_i(Q_{n,k})$ .

**Proof.** In light of Proposition 3.1, the proof is identical to the standard proof that one can compute homology using the complex of cellular chains (see e.g. [1], p. 202).

By Corollary 3.2, cubes  $(\Gamma, \Phi)$  with odd symmetry contribute nothing to the cubical chain complex, while cubes with no odd symmetries each contribute one copy of  $\mathbb{Q}$ . In order to do explicit calculations in this chain complex we need to choose basis elements for the various  $\mathbb{Q}$  summands, which we do as follows. For each cube  $(\Gamma, \Phi)$  with no odd symmetries, the projection map from the boundary of this cube to its quotient  $B(\Gamma, \Phi)$  induces an isomorphism on rational homology. Hence, when we cone off, the quotient map induces an isomorphism  $H_*((\Gamma, \Phi), \partial(\Gamma, \Phi); \mathbb{Q}) \approx H_*(C(\Gamma, \Phi), B(\Gamma, \Phi); \mathbb{Q})$ . An orientation of the cube  $(\Gamma, \Phi)$  determines a fundamental class for the cube with  $\mathbb{Z}$  coefficients, whose image in  $H_*(C(\Gamma, \Phi), B(\Gamma, \Phi); \mathbb{Q})$  we take to be the basis element for this summand of the chain complex. With this convention for choosing basis elements, computing the boundary map in the chain complex becomes simply a matter of listing the various codimension one faces of  $(\Gamma, \Phi)$ , with appropriate signs reflecting orientations, and omitting faces which happen to have odd symmetries.

#### 4. $k$ -dimensional cycles and collapsing free faces

In this section we show how to simplify the process of finding the top-dimensional cycles in  $Q_{n,k}$ . Since the inclusion  $Q_{n,k} \rightarrow Q_{n,k+1}$  induces a surjection on homology in dimension  $k$  by Proposition 2.1, these  $k$ -dimensional cycles include all potential homology classes in  $H_k(Q_{n,k+1}) \cong H_k(\text{Aut}(F_n))$

The  $k$ -dimensional cubes in  $S\mathbb{A}_{n,k}$  are marked triples  $(g, \Gamma, T)$ , where  $\Gamma$  is a graph of degree  $k$  with  $k + 1$  vertices, and  $T$  is a maximal tree in  $\Gamma$ . We begin our computations by listing all isomorphism classes of cubes  $(\Gamma, T)$  with  $\Gamma$  and  $T$  as above. We then eliminate all cubes  $(\Gamma, T)$  which have an odd symmetry to obtain a list of all of the  $\mathbb{Q}$ -summands in the cubical  $k$ -chains for  $Q_{n,k}$ , by Corollary 3.2.

We now simplify the chain complex by a process which is the algebraic analog of collapsing cells across free faces. For each remaining cube  $C = (\Gamma, T)$ , we look for a  $(k - 1)$ -dimensional face  $D$  of  $C$  with no odd symmetry such that any marked cube having a face of type  $D$  is either equivalent to one of type  $C$  under the action of  $\text{Aut}(F_n)$  or has already been collapsed in a previous step. If we find such a face  $D$  then  $C$  may be collapsed, i.e. the  $\mathbb{Q}$  summands corresponding to  $C$  and  $D$  can be cancelled without changing the homology of the chain complex.

A cube  $(\Gamma, T)$  has two types of codimension-one faces: those of the form  $(\Gamma, T - \{e\})$  obtained by deleting an edge  $e$  of  $T$ , and faces  $(\Gamma_e, T_e)$  obtained by collapsing an edge  $e$  of  $T$ . A face  $(\Gamma, T - \{e\})$  is free if it has no odd symmetry

and if all other cubes of the form  $(\Gamma, T')$  with  $T' \supset T - \{e\}$  have already been collapsed. A face  $(\Gamma_e, T_e)$  is free if it has no odd symmetry and if every other cube  $(\Gamma', T')$  with a face isomorphic to  $(\Gamma_e, T_e)$  has already been collapsed.

The following lemmas apply the above remarks to successively collapse many cubes  $(\Gamma, T)$ , and thus eliminate the corresponding  $\mathbb{Q}$ -summands from the cubical chain complex.

**Lemma 4.1.** *If  $\Gamma$  contains a wedge summand of degree one or two, then  $(\Gamma, T)$  may be collapsed.*

**Proof.** By Corollary 3.2, we may assume  $(\Gamma, T)$  has no odd symmetry. The only possibilities for wedge summands of degree one or two are shown in Figure 1.

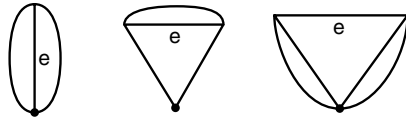


Figure 1

If  $\Gamma$  contains a wedge summand of degree 1, let  $e$  be an edge of this summand. Up to automorphism of  $\Gamma$ ,  $T$  is the only maximal tree which contains  $T - \{e\}$ , so that  $(\Gamma, T - \{e\})$  is a free face of  $(\Gamma, T)$ . If  $\Gamma$  contains a degree 2 wedge summand, let  $e$  be an edge of this summand not adjacent to the basepoint. Then  $T$  must contain  $e$ , since otherwise  $(\Gamma, T)$  has an odd symmetry. The face  $(\Gamma, T - \{e\})$  is a free face, since up to automorphism of  $\Gamma$ , the only other tree that contains  $T - \{e\}$  has an odd symmetry.  $\square$

In particular, since  $H_k(Q_{n,2}; \mathbb{Q})$  maps onto  $H_k(\text{Aut}(F_n); \mathbb{Q})$  for  $k = 1, 2$  by Proposition 2.1, this lemma shows that  $H_1(\text{Aut}(F_n); \mathbb{Q}) = H_2(\text{Aut}(F_n); \mathbb{Q}) = 0$  for all  $n$ .

**Lemma 4.2.** *If  $\Gamma$  contains a double edge joining two non-basepoint vertices, then  $(\Gamma, T)$  may be collapsed.*

**Proof.** We may assume  $(\Gamma, T)$  has no odd symmetry, by Corollary 3.2. Let  $e$  and  $e'$  be distinct edges joining non-basepoint vertices  $v$  and  $w$ . (See Figure 2)

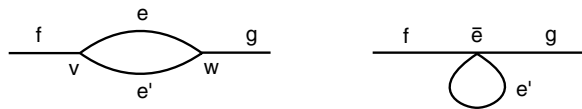


Figure 2

Suppose  $T$  is a tree of  $\Gamma$  which contains one of  $e, e'$ , say  $T$  contains  $e$ . We claim that  $(\Gamma_e, T_e)$  is a free face. Any symmetry of  $\Gamma_e$  must fix the vertex  $\bar{e}$  which is the image of  $e$ , since this is the only non-basepoint valence four vertex. After possibly composing with the automorphism which inverts the image of  $e'$ , such a symmetry lifts to a symmetry of  $\Gamma$  which sends  $e$  to itself. Thus any odd symmetry of  $(\Gamma_e, T_e)$  gives rise to an odd symmetry of  $(\Gamma, T)$ , which cannot

exist by our initial assumption. The only graph which collapses to  $(\Gamma_e, T_e)$  is  $(\Gamma, T)$  itself, since  $S\mathbb{A}_n$  contains no graphs with separating edges, showing that  $(\Gamma_e, T_e)$  is a free face of  $(\Gamma, T)$ .

Now assume that all pairs  $(\Gamma, T)$  with  $T$  containing  $e$  or  $e'$  have been collapsed. If  $T$  does not contain  $e$  or  $e'$ , it must contain edges  $f$  and  $g$  terminating at  $v$  and  $w$  respectively, with  $f, g \neq e, e'$ . Any symmetry of  $(\Gamma, T - \{f\})$  must fix  $v$ , since  $v$  is the only vertex of  $\Gamma$  which is not in  $T - \{f\}$ , and in fact must fix  $f$ . As before, this implies that  $(\Gamma, T - \{f\})$  has no odd symmetries, since any odd symmetry of  $(\Gamma, T - \{f\})$  would induce an odd symmetry of  $(\Gamma, T)$ . The face  $(\Gamma, T - \{f\})$  is contained only in  $(\Gamma, T)$  and in  $(\Gamma, T \cup \{e\} - \{f\})$ , which has been collapsed, i.e.  $(\Gamma, T - \{f\})$  is now a free face.  $\square$

**Lemma 4.3.** *Suppose  $\Gamma$  contains a triangle with no vertices at the basepoint, and that  $T$  contains exactly one edge of this triangle. Then  $(\Gamma, T)$  may be collapsed.*

**Proof.** We may assume that  $(\Gamma, T)$  has no odd symmetries, by Corollary 3.2, and that all pairs  $(\Gamma', T')$  such that  $\Gamma'$  has a double edge away from the basepoint, have been collapsed using Lemma 4.2.

Let  $a$  be the edge of the triangle which is contained in  $T$ , and let  $b$  and  $c$  be the other two edges. (See Figure 3)

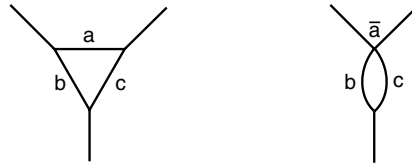


Figure 3

We claim that  $(\Gamma_a, T_a)$  is a free face. Any symmetry of  $(\Gamma_a, T_a)$  must fix the image  $\bar{a}$  of  $a$ , since that is the only non-basepoint vertex of valence 4. After possibly composing with a symmetry exchanging the images of  $b$  and  $c$  in  $\Gamma_a$ , this symmetry can be lifted to a symmetry of  $(\Gamma, T)$  which fixes  $a$ . Since neither  $b$  nor  $c$  are in  $T$ , the symmetry  $(\Gamma_a, T_a)$  is odd if and only if the lifted symmetry is odd. Therefore  $(\Gamma_a, T_a)$  can have no odd symmetries. The only pairs  $(\Gamma, T)$  which may be obtained from  $(\Gamma_a, T_a)$  by blowing up the image of  $a$  are  $(\Gamma, T)$  and a pair  $(\Gamma', T')$  such that  $\Gamma'$  has a double edge. Since this latter pair has already been collapsed,  $(\Gamma_a, T_a)$  is a free face.  $\square$

**Lemma 4.4.** *Suppose  $\Gamma$  contains a triangle with one vertex at the basepoint and one double edge. If  $T$  contains exactly one edge of the triangle, which is either the edge not adjacent to the basepoint or one edge of the double edge, then  $(\Gamma, T)$  may be collapsed.*

**Proof.** If the double edge is not adjacent to the basepoint Lemma 4.2 applies, so



we may assume the double edge has one vertex at the basepoint. If  $T$  contains the edge  $a$  not adjacent to the basepoint (see Figure 4), then any symmetry of  $(\Gamma_a, T_a)$  must fix the image  $\bar{a}$  of  $a$  in  $\Gamma_a$  and, after possibly composing with a permutation of the edges connecting the basepoint to  $\bar{a}$ , can be lifted to a symmetry of  $(\Gamma, T)$  fixing  $a$ .



Figure 4

Thus  $(\Gamma_a, T_a)$  can have no odd symmetries. Furthermore,  $(\Gamma, T)$  is the only cube with a face of type  $(\Gamma_a, T_a)$ , i.e.  $(\Gamma_a, T_a)$  is a free face.

If  $T$  contains one of the double edges  $e$ , then any symmetry of  $(\Gamma, T - \{e\})$  can be lifted to a symmetry of  $(\Gamma, T)$  fixing  $e$ , and the only other maximal tree which contains  $T - \{e\}$  is  $T' = T - \{e\} \cup \{a\}$ . Since  $(\Gamma, T')$  has just been collapsed,  $(\Gamma, T - \{e\})$  is a free face.  $\square$

### 5. Degree 3 computations

**Lemma 5.1.** *If  $\Gamma$  contains a wedge summand of degree 3, then  $(\Gamma, T)$  can be collapsed.*

**Proof.** By Corollary 3.2, we may assume  $(\Gamma, T)$  has no odd symmetry. Write  $\Gamma = \Gamma_0 \vee \Gamma'$ , where  $\Gamma_0$  has degree 3, inducing a decomposition  $T = T_0 \vee T'$ . The possibilities for  $(\Gamma_0, T_0)$  are shown in Figure 5.

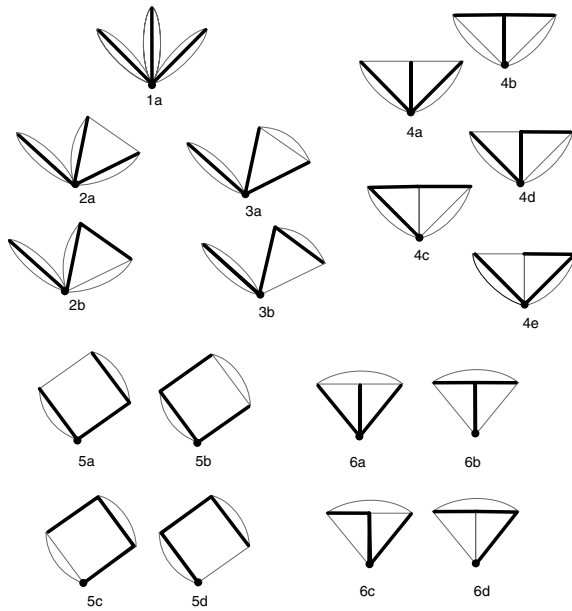


Figure 5

If  $(\Gamma_0, T_0)$  is one of  $4a, 4b, 6a$  or  $6b$  it has an odd symmetry, which extends to an odd symmetry of  $(\Gamma, T)$  contradicting our assumption. If  $(\Gamma_0, T_0)$  is one of  $5a, 5b, 5c$  or  $5d$ , then  $(\Gamma, T)$  may be collapsed by Lemma 4.2, and if  $(\Gamma_0, T_0)$  is one of  $1a, 2a, 2b, 3a$  or  $3b$ , then Lemma 4.1 applies. Lemma 4.3 applies for  $(\Gamma_0, T_0) = 6c$ , and Lemma 4.4 applies for  $(\Gamma_0, T_0) = 4c$  or  $4e$ .

For the remaining possibilities  $(\Gamma_0, T_0) = 4d$  and  $6d$ , every face of the form  $(\Gamma_0, T_0 - \{e\})$  is now free; these correspond to free faces of  $(\Gamma, T)$  which may be used to collapse  $(\Gamma, T)$ .  $\square$

**Corollary 5.2.**  $H_3(\text{Aut}(F_n); \mathbb{Q}) = 0$  for all  $n$ .

**Proof.** Observe that all cubes may be collapsed in order of increasing rank, so that  $H_3(Q_{n,3}; \mathbb{Q}) = 0$  for all  $n$ . The corollary now follows from Proposition 2.1.  $\square$

Lemmas 4.1 and 5.1 say that we need not consider graphs with wedge summands of degree  $\leq 3$  when computing  $k$ -dimensional cycles in  $Q_{n,k}$ . It is also true that we do not need to consider such graphs when determining whether such  $k$ -cycles are boundaries of  $(k+1)$ -chains in  $Q_{n,k+1}$  since applying Lemmas 4.1-4.4 and 5.1 to graphs of degree  $k+1$  involves no graphs of degree  $k$ . In particular, computing  $H_i(\text{Aut}(F_n); \mathbb{Q})$  for  $i \leq 6$  involves only graphs of degree  $\leq 7$ , so we may restrict attention to graphs which are *indecomposable*, i.e. such that the basepoint is not a cut vertex.

Lemma 5.1 can also be used to improve the stability range for rational homology. Recall that the imbedding  $S\mathbb{A}_{n,k} \rightarrow S\mathbb{A}_{n+1,k}$  given by attaching an extra loop at the basepoint of a marked graph and identifying it with the  $(n+1)^{\text{st}}$  generator of  $F_{n+1}$  is natural with respect to the inclusion  $\text{Aut}(F_n) \rightarrow \text{Aut}(F_{n+1})$ . In [8] it is shown that this map is a homotopy equivalence for  $n \geq 3k/2$ , so induces an isomorphism on homology in this range. We improve this to

**Proposition 5.3.** *The stabilization  $H_i(\text{Aut}(F_n); \mathbb{Q}) \rightarrow H_i(\text{Aut}(F_{n+1}); \mathbb{Q})$  is an isomorphism for  $n \geq 5(i+1)/4$ .*

The key point is the following:

**Lemma 5.4.** *If  $n > (m+1)k/m$ , then a graph  $\Gamma$  of degree  $k$  and rank  $n$  must contain a wedge summand of degree  $\leq m-1$ .*

**Proof.** We may assume that  $\Gamma$  has no loops at the basepoint and that all vertices of  $\Gamma$  other than the basepoint are trivalent. Let  $\Gamma_*$  be the full subgraph of  $\Gamma$  spanned by the non-basepoint vertices, so that components of  $\Gamma_*$  correspond to wedge summands of  $\Gamma$ .

Since the basepoint of  $\Gamma$  has valence  $2n - k$ , we have

$$\begin{aligned} \#(\text{components of } \Gamma_*) &\geq \chi(\Gamma_*) = v(\Gamma_*) - e(\Gamma_*) \\ &= (v(\Gamma) - 1) - (e(\Gamma) - (2n - k)) \\ &= \chi(\Gamma) - 1 + 2n - k \\ &= n - k. \end{aligned}$$

Thus if we assume  $n > (m + 1)k/m$ , or equivalently  $n - k > k/m$ , then  $\Gamma_*$  has more than  $k/m$  components. If each component has at least  $m$  vertices,  $\Gamma_*$  will then have more than  $k$  vertices, a contradiction since the degree of  $\Gamma$  is the number of vertices of  $\Gamma_*$ , all non-basepoint vertices of  $\Gamma$  being trivalent. So some component of  $\Gamma_*$  has fewer than  $m$  vertices, in other words some wedge summand of  $\Gamma$  has degree less than  $m$ .  $\square$

**Proof of Proposition.** Since calculating  $H_i(\text{Aut}(F_n); \mathbb{Q})$  involves only graphs of degree  $\leq i + 1$ , we see that after collapsing graphs with wedge summands of degree  $\leq 3$ , the calculation is independent of  $n$  if  $n \geq 5(i + 1)/4$ .  $\square$

For  $n$  large, Proposition 2.1 and Proposition 5.3 together give

$$\begin{aligned} H_1(\text{Aut}(F_n)) &\cong H_1(Q_{2,2}) & H_4(\text{Aut}(F_n)) &\cong H_4(Q_{7,5}) \\ H_2(\text{Aut}(F_n)) &\cong H_2(Q_{4,3}) & H_5(\text{Aut}(F_n)) &\cong H_5(Q_{8,6}) \\ H_3(\text{Aut}(F_n)) &\cong H_3(Q_{5,4}) & H_6(\text{Aut}(F_n)) &\cong H_6(Q_{9,7}) \end{aligned}$$

where all homology groups are taken with coefficients in  $\mathbb{Q}$ .

### 6. Summary of Degree 4 computations

Hand calculations as above begin to be impractical once we reach degree 4. In section 7 we will describe how the calculations can be done by computer, but let us now just describe the end result of the calculations in degree 4, and in particular describe a non-trivial cycle in  $H_4(\text{Aut}(F_4); \mathbb{Q})$ .

The complexes  $Q_{n,4}$  are rationally acyclic for  $n \leq 3$ , so that  $H_4(\text{Aut}(F_n); \mathbb{Q}) = 0$  for  $n \leq 3$ . However, there are two 4-dimensional cycles  $z_1$  and  $z_2$  in  $Q_{4,4}$ , and one additional one  $z_3$  in  $Q_{5,4}$ . The cycle  $z_1$  is the sum of the three cubes  $(\Gamma, T_i)$  shown in Figure 6; its boundary is computed explicitly in Example 7.5.

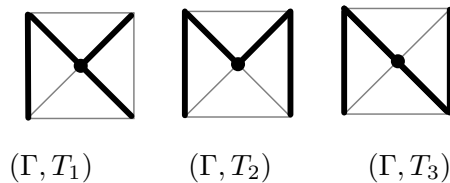


Figure 6

The three trees  $T_i$  are all of the maximal trees contained in the darkened subgraph in Figure 7, so we can indicate the cycle  $z_1$  by simply drawing Figure 7.

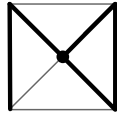


Figure 7

In this shorthand notation, the cycle  $z_2$  is shown in Figure 8; it is the sum of eight different 4-dimensional cubes.

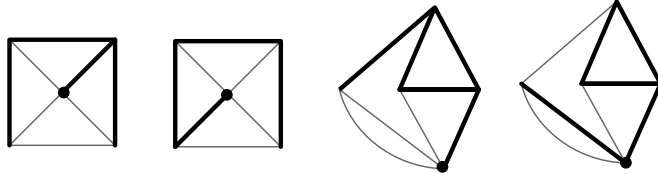


Figure 8

Neither  $z_1$  nor  $z_2$  bounds, but the linear combination  $7z_1 - 3z_2$  is a boundary in  $Q_{4,4}$ , so the rational homology is one-dimensional.

When we increase the rank to 5, the image of  $z_1$  under the imbedding  $Q_{4,4} \rightarrow Q_{5,4}$  now bounds. There is an additional cycle  $z_3$ , shown in Figure 9; it is the sum of six cubes and is a boundary, so is trivial in homology. Stably we have  $H_4(\text{Aut}(F_n); \mathbb{Q}) = 0$  for  $n \geq 5$ .

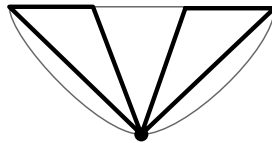


Figure 9

## 7. Computations in degree 4 and higher

There are undoubtedly many different ways to implement on a computer the program described in this paper for computing the homology of  $\text{Aut}(F_n)$ . In this section, we give brief descriptions of the algorithms we used. In many cases, the form of the algorithm was strongly influenced by the ready-made functions available in the version of Mathematica we had at the time.

### *Representing graphs*

We represent a graph with  $r$  vertices as a list of pairs  $\{i, j\}$ , for  $1 \leq i, j \leq r$ , where  $\{i, j\}$  is in the list if and only if there is an edge joining the  $i$ th vertex to the  $j$ th vertex.

For example, the list

$$\{\{1, 2\}, \{2, 3\}, \{3, 3\}, \{3, 1\}, \{1, 2\}\}$$

corresponds to the graph on three vertices shown in Figure 10:

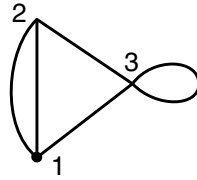


Figure 10

In order to obtain a unique representation of each isomorphism type of base-pointed graph, we always label the basepoint with 1, write the edge  $\{i, j\}$  with  $i \leq j$  and put the list in lexicographical order. This is not yet enough to guarantee uniqueness; for instance the following sorted edge lists both represent graphs isomorphic to the graph shown above:

$$\begin{aligned} & \{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{2, 2\}, \{2, 3\}\} \\ & \{\{1, 2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 3\}\} \end{aligned}$$

To find a unique representative, we start with an arbitrary representative, apply all possible permutations of the labels  $\{2, \dots, r\}$ , sort each resulting edge list, and then sort the list of edge lists lexicographically. The edge list which is first lexicographically is the canonical representative, or *normal form* of the graph. In our example, the normal form is

$$\{\{1, 2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 3\}\}.$$

**Lemma 7.1.** *Two graphs are isomorphic if and only if they have the same normal forms.*

**Proof.** If two graphs have the same normal form, they are clearly isomorphic. Conversely, suppose  $h: \Gamma \rightarrow \Gamma'$  is an isomorphism. Let  $v_1, \dots, v_k$  be the vertices of  $\Gamma$ , and  $v'_1, \dots, v'_k$  be the vertices of  $\Gamma'$ , and define a permutation  $\sigma$  of  $\{1, \dots, k\}$  by sending  $i \rightarrow j$  if  $h(v_i) = v'_j$ . Let  $E$  be the edge list representing  $\Gamma$ , with a copy of  $\{i, j\} \in E$  for each edge connecting  $v_i$  with  $v_j$ ; then  $\sigma(E)$  represents  $\Gamma'$ . We obtain the normal form for  $\Gamma$  (resp.  $\Gamma'$ ) by applying all permutations of  $\{1, \dots, k\}$  to  $E$  (resp.  $\sigma(E)$ ), so that the list of edge lists for  $\Gamma$  and  $\Gamma'$  are the same, resulting in the same normal form.  $\square$

For an example of Mathematica code which computes the normal form of a graph  $G$  given as an edge list, see the Appendix.

Given a graph  $\Gamma$  in normal form, we compute the group of automorphisms of  $\Gamma$  as follows. The subgroup  $\Sigma$  of  $\text{Aut}(\Gamma)$  generated by automorphisms which fix all vertices is normal. It is a direct product of symmetric groups  $\Sigma_s$  for every multiple edge of multiplicity  $s$ , and of  $(\mathbf{Z}/2)^t \times \Sigma_t$  for every bouquet of  $t$  loops at a vertex. Two elements of  $\text{Aut}(\Gamma)$  represent the same element of the quotient  $\text{Aut}_0(\Gamma) = \text{Aut}(\Gamma)/\Sigma$  if and only if they have the same effect on the vertices of  $\Gamma$ . We have

**Lemma 7.2.** *A permutation of the vertices of  $\Gamma$  is induced by an automorphism of  $\Gamma$  if and only if the induced permutation of a sorted edge set  $E$  representing  $\Gamma$ , when sorted, is again equal to  $E$ .*

**Proof.** This follows since a function permuting the vertices of  $\Gamma$ , permuting the edges of  $\Gamma$  and preserving the incidence relations is an automorphism of  $\Gamma$ .  $\square$

To find representatives for all elements of  $\text{Aut}_0(\Gamma)$  then, we start with the normal form representation for  $\Gamma$  and a list of all permutations of  $2, \dots, n$ , apply each permutation to the normal form, sort the resulting list and check whether it is equal to the normal form. If so, we add this permutation to the list of automorphisms of  $\Gamma$  (for Mathematica code, see the Appendix).

We will also need a relative notion of normal form for pairs  $(\Gamma, \Delta)$ , where  $\Delta$  is a subgraph of  $\Gamma$ . Starting with an edge list representing  $\Gamma$  and a sublist representing  $\Delta$ , we apply all permutations of  $2, \dots, r$ , sort the resulting pairs of lists, then take the one which is first lexicographically as the normal form for the pair  $(\Gamma, \Delta)$ . If  $\Gamma$  is already in normal form we need only apply permutations representing elements of  $\text{Aut}_0(\Gamma)$  to  $\Delta$  and sort to obtain the normal form for  $(\Gamma, \Delta)$ . The proof of Lemma 7.1 can be modified slightly to give

**Lemma 7.3.**  *$(\Gamma, \Delta)$  and  $(\Gamma', \Delta')$  have the same normal form if and only if there is an isomorphism  $h: \Gamma \rightarrow \Gamma'$  with  $h(\Delta) = \Delta'$ .*  $\square$

#### Degree $k$ maximal graphs

Recall that a graph represents a maximal vertex of  $SA_{n,k}$  if it has degree  $k$  and exactly  $k+1$  vertices; we call such a graph *maximal* of degree  $k$ . Note that every non-basepoint vertex of a maximal graph is trivalent. In order to generate a list of all degree  $k$  maximal graphs, we must find all indecomposable maximal graphs of degree at most  $k$ . We have:

**Lemma 7.4.** *Every indecomposable maximal graph of degree  $k$  can be obtained from an indecomposable degree  $k-1$  maximal graph  $\Gamma_{k-1}$  by one of the following operations:*

1. *Attach a new edge connecting the basepoint of  $\Gamma_{k-1}$  with a point on the interior of some edge adjacent to the basepoint.*
2. *Partially fold together two edges of  $\Gamma_{k-1}$  which are adjacent to the basepoint, i.e. identify small segments of these edges to form a new edge adjacent to the basepoint.*

**Proof.** Let  $\Gamma_k$  be an indecomposable maximal graph of degree  $k$ . Collapsing any edge adjacent to the basepoint always produces a degree  $k-1$  maximal graph, which may not, however, be indecomposable.

If  $\Gamma_k$  has a double edge at the basepoint, collapse one of the two edges and remove the resulting loop at the basepoint to obtain  $\Gamma_{k-1}$ ; then  $\Gamma_{k-1}$  is indecomposable and  $\Gamma_k$  is obtained from  $\Gamma_{k-1}$  by operation (1).

If  $\Gamma_k$  has no double edges at the basepoint, we claim there is an edge  $e$  adjacent to the basepoint so that collapsing  $e$  produces an indecomposable graph  $\Gamma_{k-1}$ ; then  $\Gamma_k$  can be recovered from  $\Gamma_{k-1}$  by operation (2). To find  $e$ , let  $\Gamma_*$  be the subgraph of  $\Gamma_k$  spanned by all non-basepoint vertices.  $\Gamma_*$  is trivalent except at vertices  $v_1 \dots, v_r$ , which are bivalent and correspond to edges  $e_1, \dots, e_r$  in  $\Gamma_k$  adjacent to the basepoint. Since  $\Gamma_k$  is indecomposable,  $\Gamma_*$  is connected. If any vertex  $v_i$  is not a cut vertex of  $\Gamma_*$ , then the graph obtained from  $\Gamma_k$  by collapsing  $e = e_i$  is indecomposable. So assume that all  $v_i$  are cut vertices, and choose  $i$  such that  $\Gamma_* - v_i$  has a component  $\Gamma_*^+$  which contains no other  $v_j$ . Then the edge of  $\Gamma_*$  adjacent to  $v_i$  and contained in  $\Gamma_*^+$  is a separating edge of  $\Gamma_k$ . This is a contradiction, since graphs in  $S\mathbb{A}_n$  have no separating edges.  $\square$

Given a list of indecomposable degree  $k - 1$  maximal graphs, we can now generate a list of all indecomposable degree  $k$  maximal graphs. To implement this on the computer, we assume our degree  $k - 1$  maximal graphs are given by a list of normal forms. For each  $\Gamma = \{\{1, 2\}, \dots, \{i, j\}, \dots\}$  in this list, we produce a new list of normal forms by:

- for each edge  $\{1, i\}$  of  $\Gamma$ , replace  $\{1, i\}$  by the three edges  $\{1, k\}$ ,  $\{1, k\}$  and  $\{i, k\}$ , then normalize;
- for each pair of edges  $\{1, i\}$ ,  $\{1, j\}$  of  $\Gamma$  with  $i \neq j$ , replace the pair by the three edges  $\{1, k\}$ ,  $\{i, k\}$  and  $\{j, k\}$ , then normalize.

Mathematica code for this is given in the Appendix.

We now remove duplicates from our new list to obtain a list of all indecomposable maximal degree  $k$  graphs. Using the lists for degrees  $\leq k$ , we construct a list of all maximal degree  $k$  graphs. By Lemmas 4.1 and 5.1 we do not need to include graphs with wedge summands of degree 1, 2 or 3. Furthermore, by Lemma 4.2 we may eliminate all graphs with a double edge away from the basepoint;

- If  $\{i, j\}$  occurs twice in a normalized graph, with  $i > 1$ , eliminate this graph.

Mathematica code for this is given in the Appendix.

### Maximal trees in $\Gamma$

A cube  $(g, \Gamma, T)$  in  $S\mathbb{A}_{n,k}$  is maximal-dimensional if  $\Gamma$  is a maximal degree  $k$  graph and  $T$  is a maximal tree in  $\Gamma$ . Thus we now need to find all maximal trees  $T$  in each  $\Gamma$  in our list. There are several standard ways of accomplishing this. We used the fact that a maximal tree has  $k$  edges and has the property that if you prune its free edges, you will eventually prune away the entire tree:

- Consider each subset  $T$  of  $k$  edges of  $\Gamma$ . Eliminate edges  $\{i, j\}$  where  $i$  or  $j$  occurs exactly once (these are free edges) and continue until there are no more free edges. If the resulting list is empty,  $T$  was a maximal tree.

Since we want a unique representative for each cube  $(\Gamma, T)$ , we now normalize each pair in our list and remove duplicates.

By Lemma 3.1, we do not need cubes  $(\Gamma, T)$  with odd symmetry. To identify these, we need to decide whether there is an automorphism of  $\Gamma$  which preserve the basepoint and sends  $T$  to itself, interchanging an odd number of pairs of edges of  $T$ .

- Apply each permutation representing an automorphism of  $\Gamma$  to  $T$  and sort. If the result is  $T$ , the automorphism preserves  $T$ . The automorphism corresponds to an odd symmetry if and only if the sign of the permutation of edges required to sort  $T$  lexicographically is odd.

We now eliminate cubes  $(\Gamma, T)$  from our list in which  $T$  satisfies the conditions of Lemma 4.3 or Lemma 4.4.

- Check whether  $\Gamma$  contains a subset of the form  $\{i, j\}, \{i, k\}, \{j, k\}$  ( $1 < i < j < k$ ). If so, check whether  $T$  contains exactly one of  $\{i, j\}, \{j, k\}$  or  $\{i, k\}$ .
- Check whether  $\Gamma$  contains a subset of the form  $\{1, i\}, \{1, j\}, \{i, j\}$ , and if so whether  $T$  contains exactly one of  $\{1, i\}$  or  $\{i, j\}$ .

We now have a list of all  $k$ -dimensional cubes  $(\Gamma, T)$  which may contribute to cycles in  $Q_{n,k}$ . For  $k = 3$ , this list contains only two cubes, which were denoted  $4d$  and  $6d$  in Figure 5. For  $k = 4$  there are 22 cubes, involving 5 isomorphism types of graphs  $\Gamma$ . For  $k = 5$  there are 144 cubes, using 9 isomorphism types of graphs, for  $k = 6$  there are 864 cubes using 26 isomorphism types of graphs, and for  $k = 7$  there are 5861 cubes using 59 isomorphism types of graphs.

### *The boundary map*

The next task is to compute the boundary map  $\partial_k : C_k(Q_{n,k}) \rightarrow C_{k-1}(Q_{n,k})$ . The boundary of a cube  $(\Gamma, T)$  has two types of faces, those of the form  $(\Gamma, T - \{e\})$  and those of the form  $(\Gamma_e, T_e)$ , with the face  $(\Gamma, T - \{e\})$  opposite to the face  $(\Gamma_e, T_e)$ .

Using the choice of basis for the chain complex given at the end of section 3, the boundary map is given by

$$\partial_k(\Gamma, T) = \sum_{i=1}^k (-1)^i ((\Gamma, T - \{e_i\}) - (\Gamma_{e_i}, T_{e_i}))$$

where  $e_i$  is the  $i$ th edge of  $T$ . It may happen that  $(\Gamma, T - \{e_i\})$  or  $(\Gamma_{e_i}, T_{e_i})$  has odd symmetry even though  $(\Gamma, T)$  does not; in that case, there is no contribution to  $\partial_k(\Gamma, T)$ .

In order to compute the boundary map, we need to put each term on the right-hand side in normal form. To compute  $(\Gamma, T - \{e\})$ , simply remove  $e$  from  $T$ , then normalize. To compute  $(\Gamma_e, T_e)$  for  $e = \{i, j\}$ , we first replace all occurrences of  $j$  by  $i$ , then replace all  $k$  with  $k > j$  by  $k - 1$ , and finally put the result into normal form.



**Example 7.5.** We compute the boundary of the cycle shown in Figure 6-7 of section 6.

The graph  $\Gamma$  has normal form

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}\},$$

and the three trees are

$$T_1 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 5\}\},$$

$$T_2 = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 5\}\},$$

$$T_3 = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}.$$

We have

$$\begin{aligned} \partial(\Gamma, T_1) = & -(\Gamma, \{\{1, 3\}, \{1, 4\}, \{3, 5\}\}) + (\Gamma_{\{1,2\}}, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}) \\ & + (\Gamma, \{\{1, 2\}, \{1, 4\}, \{3, 5\}\}) - (\Gamma_{\{1,3\}}, \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}) \\ & - (\Gamma, \{\{1, 2\}, \{1, 3\}, \{3, 5\}\}) + (\Gamma_{\{1,4\}}, \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}) \\ & + (\Gamma, \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}) - (\Gamma_{\{3,5\}}, \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}). \end{aligned}$$

Up to isomorphism, there are only two possibilities for the graph  $\Gamma_{\{i,j\}}$  obtained by collapsing the edge  $\{i, j\}$ , namely

$$\Gamma_1 = \{\{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\},$$

$$\Gamma_2 = \{\{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

We put all the terms in  $\partial(\Gamma, T_1)$  in normal form (note that switching the order of two edges of the tree changes the sign of the cube) to obtain

$$\begin{aligned} \partial(\Gamma, T_1) = & -(\Gamma, \{\{1, 2\}, \{1, 5\}, \{2, 3\}\}) + (\Gamma_1, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}) \\ & - (\Gamma, \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}) + (\Gamma_1, \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}) \\ & + (\Gamma, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}) + (\Gamma_1, \{\{1, 2\}, \{1, 4\}, \{3, 4\}\}) \\ & - (\Gamma, \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}) + (\Gamma_2, \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}). \end{aligned}$$

The second and fourth rows of this expression vanish, since each term in each row has odd symmetry, leaving

$$\begin{aligned} \partial(\Gamma, T_1) = & -(\Gamma, \{\{1, 2\}, \{1, 5\}, \{2, 3\}\}) + (\Gamma_1, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}) \\ & + (\Gamma, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}) + (\Gamma_1, \{\{1, 2\}, \{1, 4\}, \{3, 4\}\}). \end{aligned}$$

We further compute

$$\begin{aligned} \partial(\Gamma, T_2) = & 2(\Gamma, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}) + 2(\Gamma_2, \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}) \\ & + 2(\Gamma, \{\{1, 2\}, \{2, 3\}, \{4, 5\}\}) + 2(\Gamma_1, \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}), \end{aligned}$$

and

$$\begin{aligned} \partial(\Gamma, T_3) = & 2(\Gamma, \{\{1, 2\}, \{1, 5\}, \{2, 3\}\}) + 2(\Gamma_2, \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}) \\ & + 2(\Gamma, \{\{1, 2\}, \{2, 3\}, \{4, 5\}\}) - 2(\Gamma_1, \{\{1, 2\}, \{1, 4\}, \{3, 4\}\}). \end{aligned}$$

It is now easy to check that  $\partial((\Gamma, T_1) - \frac{1}{2}(\Gamma, T_2) + \frac{1}{2}(\Gamma, T_3)) = 0$ . Note that, since we are using coefficients in  $\mathbb{Q}$ , we might have chosen  $\frac{1}{2}(\Gamma, T_2)$  and  $\frac{1}{2}(\Gamma, T_3)$  as basis elements for the summands of the chain complex corresponding to  $(\Gamma, T_2)$  and  $(\Gamma, T_3)$  respectively, to compensate for the fact that each of these pairs has an even symmetry of order 2.

*Reducing the size of the boundary calculation*

At this point, we could put everything into a giant matrix: List all cubes  $(\Gamma, T)$ , list all cubes  $(\Gamma, T - \{e\})$  and  $(\Gamma_e, T_e)$  obtained by removing and collapsing edges in the pairs  $(\Gamma, T)$  in our list, and compute  $\partial_k$  for each  $(\Gamma, T)$ . The kernel of  $\partial_k$  is the cycles  $Z_k$ . If we find cycles, we must decide whether they are boundaries, so we compute the boundary map  $\partial_{k+1} : C_{k+1}(Q_{n,k+1}) \rightarrow C_k(Q_{n,k})$  in exactly the same way, and see whether our cycles are in the image. This becomes unwieldy very quickly on the computer, so we now make some observations which allow us to break up the computation into smaller pieces.

We divide the orbits of  $(k - 1)$ -dimensional cubes in  $S\mathbb{A}_{n,k}$  into three different types, according to the degree  $d$  and the number of vertices  $v$  of the corresponding graph. Type A has  $d = v = k$ , i.e. type A consists of pairs  $(\Gamma, \Phi)$ , where  $\Gamma$  is maximal of degree  $k$  and  $\Phi$  is a forest with  $k - 2$  edges; type B has  $d = k$  and  $v = k - 1$ , i.e. consists of pairs  $(\Gamma, T)$  where  $\Gamma$  has a basepoint,  $k - 2$  trivalent vertices and one vertex of valence 4, and  $T$  is a maximal tree; and type C has  $d = v = k - 1$ , i.e. consists of pairs  $(\Gamma, T)$  where  $\Gamma$  is maximal of degree  $k - 1$  and  $T$  is a maximal tree. The chains  $C_{k-1} = C_{k-1}(Q_{n,k})$  decompose into a direct sum

$$C_{k-1} = C_{k-1}^A \oplus C_{k-1}^B \oplus C_{k-1}^C,$$

and the boundary map  $\partial_k = (\partial_k^A, \partial_k^B, \partial_k^C)$ . A  $k$ -chain  $z$  is a  $k$ -cycle if and only if  $\partial_k^A(z) = \partial_k^B(z) = \partial_k^C(z) = 0$ . We compute the kernel  $Z_k^A$  of  $\partial_k^A$  first, then compute the kernel  $Z_k^{AB}$  of  $\partial_k^B$  restricted to  $Z_k^A$ , and finally the kernel  $Z_k^{ABC} = Z_k$  of  $\partial_k^C$  restricted to  $Z_k^{AB}$ .

The computation of  $\partial_k^A$  is straightforward; for each cube  $(\Gamma, T)$ , we have

$$\partial_k^A(\Gamma, T) = \sum_i (-1)^i (\Gamma, T - \{e_i\}),$$

where  $e_i$  is the  $i$ th edge of  $T$ . Furthermore, we can do the computation separately for each  $\Gamma$ , since  $C_k$  breaks up into a direct sum  $C_k = \bigoplus_{\Gamma} C_{\Gamma}$  where  $C_{\Gamma}$  has one

$\mathbb{Q}$  summand for each maximal tree  $T$  in  $\Gamma$ , and  $\partial_k^A = \bigoplus_{\Gamma} \partial_{\Gamma}$ , where  $\partial_{\Gamma}$  is  $\partial_k^A$  restricted to  $C_{\Gamma}$ .

The map  $\partial_k^B$  corresponds geometrically to collapsing edges not adjacent to the basepoint, i.e.

$$\partial_k^B(\Gamma, T) = \sum_i (-1)^{i+1} (\Gamma_{e_i}, T_{e_i}),$$

where  $e_i$  is the  $i$ th edge in  $T$  which is of the form  $\{i, j\}$  for  $i > 1$ . Here we cannot do the computation separately for each isomorphism type of graph  $\Gamma$ , since it is possible that  $(\Gamma_e, T_e) = (\Gamma'_e, T'_e)$  even though  $\Gamma$  is not isomorphic to  $\Gamma'$ . It is possible that  $(\Gamma_e, T_e)$  has odd symmetry even though  $(\Gamma, T)$  does not, in which case there is no contribution to  $\partial_k^B$ . Note that all edges of  $T$  may be adjacent to the basepoint, in which case  $\partial_k^B(\Gamma, T) = 0$ .

Similarly, we have

$$\partial_k^C(\Gamma, T) = \sum_i (-1)^{i+1} (\Gamma_{e_i}, T_{e_i}),$$

where  $e_i$  is the  $i$ th edge in  $T$  which is of the form  $\{1, j\}$  (i.e. the  $i$ th edge of  $T$  which is adjacent to the basepoint). Again, it is possible that  $(\Gamma_e, T_e)$  has odd symmetry even though  $(\Gamma, T)$  does not, in which case there is no contribution to the corresponding boundary map.

If we do find a  $k$ -cycle  $z$ , then in order to find out whether it is a boundary we must compute the boundary map  $\partial_{k+1}$ . This task can be simplified by using the decomposition of the boundary map as  $\partial_{k+1} = (\partial_{k+1}^A, \partial_{k+1}^B, \partial_{k+1}^C)$ . If  $z = \partial_{k+1}(w)$  then, since  $z$  only involves maximal graphs of degree  $k$  and maximal trees, we must have  $z = \partial_{k+1}^C(w)$  and  $\partial_{k+1}^A(w) = \partial_{k+1}^B(w) = 0$ . Therefore we compute  $Z_{k+1}^{AB} = \ker(\partial_{k+1}^A) \cap \ker(\partial_{k+1}^B)$  and look for  $w$  in  $Z_{k+1}^{AB}$ , i.e. we decide whether  $z$  is in the span of  $\partial(Z_{k+1}^{AB})$ .

### 8. Results of calculations

The following table summarizes the results of the computer calculations described in the previous section. Let  $Z_{n,k}$  denote the cycles in  $Q_{n,k}$  and  $B_{n,k}$  the subspace of  $Z_{n,k}$  consisting of cycles which are boundaries in  $Q_{n,k+1}$ .

$k \leq 3$	$Z_{n,k} = 0$ for all $n$	
$k = 4$	$Z_{n,k} = 0$	for $n \leq 3$
	$\dim(Z_{n,k}) = 2, \dim(B_{n,k}) = 1$	for $n = 4$
	$\dim(Z_{n,k}) = 3 = \dim(B_{n,k})$	for $n \geq 5$
$k = 5$	$Z_{n,k} = 0$	for $n \leq 4$
	$\dim(Z_{n,k}) = 1 = \dim(B_{n,k})$	for $n = 5$
	$\dim(Z_{n,k}) = 2 = \dim(B_{n,k})$	for $n \geq 6$

$$\begin{array}{lll}
k = 6 & Z_{n,k} = 0 & \text{for } n \leq 4 \\
& \dim(Z_{n,k}) = 2 = \dim(B_{n,k}) & \text{for } n = 5 \\
& \dim(Z_{n,k}) = 7 = \dim(B_{n,k}) & \text{for } n \geq 6
\end{array}$$

Theorem 1.1, stated in the introduction, which gives the homology of  $\text{Aut}(F_n)$  in dimensions  $\leq 6$ , follows since  $H_k(\text{Aut}(F_n); \mathbb{Q}) = Z_{n,k}/B_{n,k}$ .

### Appendix: Examples of Mathematica code

#### Normal form

The following Mathematica code defines a function  $\text{Nf}[G]$ , which finds the normal form of a graph  $G$  given as a list of edges  $\{i, j\}$ . This code requires applying  $(v[G] - 1)$  factorial permutations to  $G$ , where  $v[G]$  is the number of vertices of  $G$ . The number of permutations can be reduced somewhat at the cost of complicating the code, by using the fact that any graph automorphism must preserve distance to the basepoint.

```

v[G_] := Length[Union[Flatten[G]]]
supsort[G_] := Sort[Sort/@G];
one[L_] := Prepend[L, 1];
Sigma[n_] := one/@Permutations[Range[2, n]];
(*List of permutations of {1, ..., n} fixing 1*)
SetAttributes[a, Listable];
Nf[G_] := Block[{i, Z=supsort[G], P=Sigma[v[G]], W},
  For[i=1, i<=Length[P], i++,
    a[k_] := P[[i, k]]; W=supsort[a[G]];
    If[OrderedQ[{W, Z}], Z=W, Null]];
  Return[Z]]

```

#### Automorphisms of a graph

The following code computes the permutations of  $\{1, \dots, v[G]\}$  which give automorphisms of  $G$  fixing all vertices (see Lemma 7.2).  $\text{Sigma}[v[G]]$  is a list of all permutations of  $\{1, \dots, v[G]\}$  which fix 1, and  $\text{supsort}$  is the function which sorts a list of edges lexicographically.

```

SetAttributes[aL, Listable];
aL[k_] := L[[k]];
cd[X_, G_] := Block[{L=X}, aL[G]]
Aut[G_] := Block[{P=Sigma[v[G]]},
  Select[P, supsort[cd[#, G]]==supsort[G]&]]

```

#### Indecomposable maximal degree $k$ graphs

Starting with a list  $Mk$  of all indecomposable maximal degree  $k - 1$  graphs, this produces a list  $Gk$  of all indecomposable maximal degree  $k$  graphs.

```

Gk={ } (* Gk will hold the degree k graphs *)

```

```

For[n=1,n<=Length[Mk], n++, (* Mk is the list of degree k-1
graphs *)
  G= Mk[[n]];
  v[G_]:=Length[Union[Flatten[G]]];k=v[G]+1;
  For[i=1,G[[i,1]]==1, i++,
    NewG=Nf[Join[Drop[G,{i,i}],{,{1,k},{G[[i,1]],k},
      {G[[i,2]],k}}]];
    (*Nf is the normal form function *)
    AppendTo[Gk,NewG]];
  For[j=2, G[[j,1]]==1, j++,
    For[i=1,i<j,i++,
      NewG=Nf[Join[{{1,k},{G[[i,2]],k},{G[[j,2]],k}},
        Drop[Drop[G,{j,j}],{i,i}]]]];
      AppendTo[Gk,NewG]]];
  Gk=Sort[Union[Gk]];
  Gk=Select[Gk,Count[#, {1,3}]>0 &];
  (* This removes any graphs with univalent basepoint *)

```

### Removing double edges

This code removes from the list  $Gk$  any graphs with double edges away from the basepoint

```

For[j=3, j<=v[Gk[[1]]], j++,
  For[i=2, i<j, i++,
    Gk = Select[Gk,Count[#, {i,j}]< 2 &]]]

```

### Forests

This code decides whether a subset of edges of a graph is a forest.

```

notfreeQ[L_,i_,j_]:=
  Count[Flatten[L],i]>1&&Count[Flatten[L],j]>1;
trim[L_]:=Select[L,notfreeQ[L,#1]&];
core[L_]:=FixedPoint[trim,L];
forestQ[L_]:=core[L]=={}

```

### Boundary maps

The function “bdryplus” in the following code computes the  $i$ th term in  $\partial_k^A(G, T)$ . PNormalform puts the pair  $(G, T)$  in normal form, keeping track of the sign.

```

bdryplus[{G_, T_}, i_] :=
  PNormalform[{G, Drop[T, {i}], (-1)^i}];

```

The function “byminus” below computes the  $i$ th term of  $\partial_k^-(G, T)$ , which computes both  $\partial_k^B$  and  $\partial_k^C$ . The list “edges[G]” is a list of the edges of  $G$ , with duplicates removed.

```

c[ij_] [k_] :=
  If[k < ij[[2]], k,
    If[k == ij[[2]], ij[[1]],
      If[k > ij[[2]], k - 1]];
collapse[L_, ij_] := DeleteCases[insert[Map[c[ij], L, {2}], {n_, n_}];
For[i = 1, i <= e[G], i++,
  Q[i] = collapse[G, edges[G][[i]]];
  R[i] = Nperm[Q[i]]
  (* Q[i] is the unnormalized graph obtained by collapsing
     ith edge of G*)
  (* R[i] is the permutations which put Q[i] in normal form *)

byminus[{G_, T_}, i_] :=
  Block[{p = Position[edges[G], T[[i]][[1, 1]]], P = R[p];
    Return[PNormalform[{Q[p], collapse[T, T[[i]]], (-1)^i}]]] }

```

Here PNormalform uses only the permutations in  $R[p]$  to find the normal form; these are all that are needed if  $(G, T)$  was given originally in normal form.

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DEPARTMENT OF MATHEMATICS, WHITE HALL, CORNELL UNIVERSITY, ITHACA, NY 14853  
*E-mail address:* hatcher@math.cornell.edu

DEPARTMENT OF MATHEMATICS, WHITE HALL, CORNELL UNIVERSITY, ITHACA, NY 14853  
*E-mail address:* vogtmann@math.cornell.edu