Mathematical Research Letters **5**, 723–730 (1998)

SIMPLY CONNECTED 4-MANIFOLDS NEAR THE BOGOMOLOV-MIYAOKA-YAU LINE

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Abstract. In the following paper, we construct simply connected symplectic 4-manifolds with characteristic numbers satisfying $c_1^2 > 8\frac{9}{10}\chi_h$.

1. Introduction

The geography problem of compact complex surfaces, (i.e., the characterization of pairs $(a, b) \in \mathbb{Z}\times\mathbb{Z}$ corresponding to minimal complex surfaces via $\chi_h = a$ and $c_1^2 = b$,) is a well-studied part of algebraic geometry. (See [Ch1, Ch2, P]. Here χ_h stands for the holomorphic Euler characteristic and c_1^2 for the square of the first Chern class of the surface at hand.) In the light of recent advances in symplectic topology and smooth 4-manifold theory, the same question has been raised for minimal symplectic and for irreducible 4-manifolds. The definitions of the invariants χ_h and c_1^2 has been extended by the formulae

$$\chi_h(X) = \frac{1}{4}(\sigma(X) + e(X)) \quad \text{and} \quad c_1^2(X) = 3\sigma(X) + 2e(X),$$

where X is a closed, oriented 4-manifold with odd $b_1(X) - b_2^+(X)$, $\sigma(X)$ is its signature and $e(X)$ is its Euler characteristic. (Note that for a symplectic 4-manifold X the difference $b_1(X) - b_2^+(X)$ is always odd.) Results of Gompf and Mrowka [GM], Szabó [Sz], Fintushel and Stern [FS1, FS2], and many others show that the answer for the different classification questions (for complex surfaces, symplectic 4-manifolds, and for smooth irreducible 4-manifolds,) is qualitatively different, even if we assume that — for sake of simplicity — our manifolds are simply connected. Gompf and Mrowka [GM] showed the first examples of simply connected irreducible (in fact, symplectic) 4-manifolds not carrying complex structures, then Szabó [Sz] found simply connected (irreducible) 4-manifolds with no symplectic structures. These works were followed by constructions of Fintushel and Stern [FS1, FS2] providing hordes of similar examples. (See also [GS, Pa, S1].) All the above examples shared the property that their signature σ was negative — equivalently, $c_1^2 < 8\chi_h$.

Besides $\mathbb{CP}^2$ (with $c_1^2(\mathbb{CP}^2) = 9$, $\chi_h(\mathbb{CP}^2) = 1$ and $\sigma(\mathbb{CP}^2) = 1$,) complex surfaces of positive signature were hard to find. Using various branched cover constructions, such examples have been constructed in [Ch1, Ch2, H, MT, PPX,

Received January 13, 1998. Revised December 1, 1998.
Supported by the Magyary Zoltán Foundation and OTKA.
Keywords: Symplectic 4-manifolds, geography of 4-manifolds, Lefschetz fibrations

So]. It is known that if S is a complex surface, then $c_1^2(S) \leq 9\chi_h(S)$ (the *Bogomolov-Miyaoka-Yau inequality*); moreover for S differing from $\mathbb{CP}^2$, the equality $c_1^2(S) = 9\chi_h(S)$ holds if and only if the unit disk $U = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$ covers S (implying, in particular, that $c_1^2(S) = 9\chi_h(S)$ for a compact complex surface $S \neq \mathbb{CP}^2$ means $|\pi_1(S)| = \infty$). Examples of surfaces with positive signature found by the above authors are either far from the Bogomolov-Miyaoka-Yau (BMY-)line $c_1^2 = 9\chi_h$ or have large fundamental groups. In the following we will construct a family $B(d)$ ($d \geq 4$) of simply connected symplectic 4-manifolds close to the BMY-line.

Theorem 1.1. *For $d \geq 4$, the simply connected, symplectic 4-manifolds $B(d)$ given in Section 3 satisfy $c_1^2\big(B(d)\big) > 8\frac{9}{10}\chi_h\big(B(d)\big)$ with finitely many exceptions.*

Corollary 1.2. *There are infinitely many irreducible 4-manifolds (admitting nontrivial Seiberg-Witten invariants,) satisfying $c_1^2 > 8\frac{9}{10}\chi_h$.*

It is still an open (and intriguing) question whether simply conneceted irreducible (or symplectic) 4-manifolds exist on the BMY-line or behind it, i.e. X with $c_1^2(X) \geq 9\chi_h(X)$. (Again, we are interested in examples different from $\mathbb{CP}^2$.) One has to be careful about asking the violation of the BMY-inequality by an irreducible 4-manifold. By reversing the orientation of, say a minimal elliptic surface we clearly have such an example. Requiring nonvanishing Seiberg-Witten invariants, for example, the question becomes much harder and more interesting.

In Section 2 of the paper we recall the construction of the complex surfaces $H(n)$ lying on the BMY-line; then using $H(n)$ for appropriate n in Section 3 we will construct $B(d)$ and show the properties announced above. We append two standard constructions in Section 4 for sake of completeness.

2. Surfaces on the BMY-line

In constructing $H(n)$ we will follow the description given in [Ch2]. Assume that D is a Riemann surface (complex 1, real 2 dimensional manifold) of genus 2. We take a $\mathbb{Z}_5$-action on D generated by $\gamma\colon D \to D$ which has exactly 3 fixed points Q_1, Q_2 and Q_3. (The existence of such an action is shown in the Appendix.) It is easy to see that the quotient of D by this $\mathbb{Z}_5$-action is $\mathbb{CP}^1$; let us denote the quotient map by $\varphi\colon D \to \mathbb{CP}^1$. The diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is denoted by Δ; take the inverse image of Δ in $D \times D$ via $\varphi \times \varphi\colon D \times D \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and denote it by $F \subset D \times D$. Note that since $[\Delta] = [\mathbb{CP}^1 \times \{pt.\}] + [\{pt.\} \times \mathbb{CP}^1] \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z})$, we have that $[F] = 5\big([D \times \{pt.\}] + [\{pt.\} \times D]\big) \in H_2(D \times D; \mathbb{Z})$.

Lemma 2.1. *F consists of the union of 5 complex curves $(F_1, \ldots, F_5)$ each diffeomorphic to D. Each F_i goes through (Q_1, Q_1), (Q_2, Q_2) and $(Q_3, Q_3) \in D \times D$. Moreover, F_i intersects F_j in (Q_k, Q_k) transversally, otherwise these curves are disjoint and $[F_i]^2 = -2$ for $1 \leq i \leq 5$.*

Proof. The curve F_i is the graph of the map $\gamma^i\colon D \to D$; consequently $[F_i]^2 = [F_j]^2$. Since F_5 is the diagonal of $D \times D$, we have $[F_5]^2 = -2$. By the fact that $\big(\sum [F_i]\big)^2 = \big(5\big([D \times \{pt.\}] + [\{pt.\} \times D]\big)\big)^2 = 50$, the lemma follows. $\square$

If we blow up $D \times D$ in the points (Q_i, Q_i) $(i = 1, 2, 3)$, the proper transform of F consists of 5 disjoint curves $\tilde{F}_1, \ldots, \tilde{F}_5 \subset D \times D \# 3\overline{\mathbb{CP}^2}$. Since in the second homology group of $D \times D \# 3\overline{\mathbb{CP}^2}$ we have $[\tilde{F}_1] + \ldots + [\tilde{F}_5] = 5\big([D \times \{pt.\}] + [\{pt.\} \times D]\big) - 5e_1 - 5e_2 - 5e_3$ (where e_i is the homology class of the exceptional sphere of the $i^{\underline{th}}$ blow-up), we can take the 5-fold cyclic branched cover of $D \times D \# 3\overline{\mathbb{CP}^2}$ along $\tilde{F}_1 \cup \ldots \cup \tilde{F}_5$. The resulting smooth complex surface is denoted by $H(1)$; the characteristic numbers of $H(1)$ can be easily computed:

Lemma 2.2. *The Euler characteristic $e\big(H(1)\big)$ of $H(1)$ is equal to 75, its signature is $\sigma\big(H(1)\big) = 25$, hence $c_1^2\big(H(1)\big) = 225$ and*

$$\chi_h\big(H(1)\big) = \frac{1}{12}\bigg(c_2\big(H(1)\big) + c_1^2\big(H(1)\big)\bigg) = 25.$$

Consequently, $c_1^2\big(H(1)\big) = 9\chi_h\big(H(1)\big)$, so $H(1)$ is on the Bogomolov-Miyaoka-Yau line. $\square$

Note that the composition of the maps $H(1) \to D \times D \# 3\overline{\mathbb{CP}^2} \to D \times D \xrightarrow{pr} D$ gives a fibration of $H(1)$ over D, the regular fiber being a curve which is a 5-fold cover of D branched in 5 points, hence it is a curve of genus 16. Each $\tilde{F}_i$ gives rise to a section of this fibration, the image of this section is a curve of genus 2 with self-intersection -1.

Any n-fold cover of $H(1)$ will be on the BMY-line as well. This can be seen in two ways: The Euler characteristic and the signature are multiplied by n under an n-fold cover, so direct computation shows the statement. Alternatively, since $H(1)$ is on the BMY-line (and $H(1) \neq \mathbb{CP}^2$), the unit disk U is its universal cover, which is the same for any n-fold cover, implying that the latter is also on the BMY-line. We define $H(n)$ as a particular n-fold cover of $H(1)$: Take $\phi_n\colon D_n \to D$ n-fold cover of D and pull $H(1) \to D$ back via ϕ_n. The resulting complex surface $H(n)$ obviously fibers over the Riemann surface D_n of genus $(n+1)$ with fibers of genus 16 and has the following characteristic numbers:

Proposition 2.3. *The Euler characteristic $e\big(H(n)\big)$ of $H(n)$ is equal to $75n$, $\sigma\big(H(n)\big) = 25n$, hence $c_1^2\big(H(n)\big) = 225n$ and $\chi_h\big(H(n)\big) = 25n$. The inverse image of a section of $H(1)$ gives a section of $H(n) \to D_n$; the corresponding submanifold is a Riemann surface of genus $(n+1)$ with self-intersection $-n$. (For more about $H(n)$ see* [Ch2] *or* [GS].) $\square$

3. Construction of the 4-manifolds

Consider a 4-manifold X admitting a Lefschetz fibration $f\colon X \to \mathbb{CP}^1$ such that the genus of the generic fiber is 16, moreover f admits a simply connected fiber and a section with self-intersection -1. An example of such a Lefschetz fibration is given at the end of the Appendix. (For more about Lefschetz fibrations see [GS].) Taking the fiber connected sum of $H(n)$ and X we get a 4-manifold $Y(n)$ still admitting a Lefschetz fibration over D_n. $Y(n)$ is symplectic [G1], and has the following characteristic numbers:

Lemma 3.1. *Assume that* $\sigma(X) = s$ *and* $e(X) = t$. *Then for the 4-manifold* $Y(n)$ *we have* $\sigma(Y(n)) = 25n + s$, $e(Y(n)) = 75n + t + 60$, *consequently,* $c_1^2\big(Y(n)\big) = 225n + 120 + 3s + 2t = 225n + 120 + c_1^2(X)$, *and* $\chi_h\big(Y(n)\big) = 25n + 15 + \frac{1}{4}(s+t) = 25n + 15 + \chi_h(X)$. □

By sewing a section of $H(n)$ to a section of X we get a section of $Y(n) \to D_n$; its image Σ_n turns out to be an embedded surface of genus $(n+1)$ with self-intersection $-(n+1)$. Since for any Lefschetz fibration and preassigned (finite) set of disjoint sections there is a symplectic structure on the 4-manifold making the sections symplectic [G2, GS], we can assume that $\Sigma_n \subset Y(n)$ is a symplectic submanifold. Applying the result of the next theorem it will be easy to get a hold on the fundamental group of the 4-manifold $Y(n)$.

Theorem 3.2. *If the 4-manifold* M^4 *admits a Lefschetz fibration* $f\colon M \to C$ *over the Riemann surface* C *with connected fibers, with at least one simply connected fiber and with a section* Σ, *then the embedding* $\Sigma \hookrightarrow M$ *induces an isomorphism* $\pi_1(\Sigma) \cong \pi_1(M)$.

Proof. For a Lefschetz fibration there is an exact sequence

$$\pi_1(F) \to \pi_1(M) \to \pi_1(C) \to \pi_0(F),$$

where F is the generic fiber. Since the fibers are connected, we have $\pi_0(F) = 0$; by the existence of a simply connected fiber we have that the homomorphism $\pi_1(F) \to \pi_1(M)$ is the zero homomorphism. This implies that the projection $M \to C$ induces an isomorphism $\pi_1(M) \to \pi_1(C)$. Since the composition of the section $\tau\colon C \to M$ with the above projection as

$$C \xrightarrow{\tau} M \xrightarrow{f} C,$$

results id_C, we get that τ induces an isomorphism on the fundamental groups, and this proves the lemma. □

For $d \geq 4$ let us take $n = \frac{1}{2}(d-1)(d-2) - 1 \geq 1$. The smooth holomorphic curve $C_d \subset \mathbb{CP}^2$ representing d-times the generator is a Riemann surface of genus $n+1$; blowing it up $h = d^2 - n - 1 = \frac{1}{2}(d^2 + 3d - 2)$ times we get the curve $\tilde{C}_d \subset \mathbb{CP}^2 \# h\overline{\mathbb{CP}^2}$ of genus $n+1$ with self-intersection $n+1$. Forming the symplectic normal connected sum (cf. [G1]) of $(Y(n), \Sigma_n)$ with $(\mathbb{CP}^2 \# h\overline{\mathbb{CP}^2}, \tilde{C}_d)$

(recall the definition of n and h in terms of d,) we get a symplectic 4-manifold $B(d)$.

Theorem 3.3. *The symplectic 4-manifold $B(d)$ is simply connected, and its characteristic numbers are given as follows: $\sigma\big(B(d)\big) = 12d^2 - 39d + s + 2$, $e\big(B(d)\big) = 40d^2 - 117d + 62 + t$, hence $c_1^2\big(B(d)\big) = 116d^2 - 351d + 130 + c_1^2(X)$ and $\chi_h\big(B(d)\big) = 13d^2 - 39d + 16 + \chi_h(X)$.*

Proof. Since the normal circle of $\tilde{C}_d \subset \mathbb{CP}^2\#h\overline{\mathbb{CP}^2}$ is contractible along any of the exceptional spheres, the complement of $\tilde{C}_d$ in $\mathbb{CP}^2\#h\overline{\mathbb{CP}^2}$ is obviously simply connected. Lemma 3.2 shows that each element of $\pi_1\big(Y(n)\big)$ can be represented by a loop contained by the section Σ_n. This, however, implies that the map $\pi_1\big(\partial(Y(n) - \nu\Sigma_n)\big) \to \pi_1\big(Y(n) - \nu\Sigma_n\big)$ induced by the embedding is a surjection. Now the application of the Seifert-Van Kampen theorem for $B(d) = (Y(n) - \nu\Sigma_n) \cup_{\partial(Y(n)-\nu\Sigma_n)} (\mathbb{CP}^2\#h\overline{\mathbb{CP}^2} - \tilde{C}_d)$ shows that $B(d)$ is simply connected. The signature of $\mathbb{CP}^2\#h\overline{\mathbb{CP}^2}$ obviously equals $1 - h = 2 - d^2 + n = 1 - d^2 + \frac{1}{2}(d-1)(d-2) = -\frac{1}{2}(d^2 + 3d - 4)$, which implies the formula for $\sigma\big(B(d)\big)$. The Euler characteristic of $\mathbb{CP}^2\#h\overline{\mathbb{CP}^2}$ equals $3 + h = 2 + d^2 - n = 3 + d^2 - \frac{1}{2}(d-1)(d-2) = \frac{1}{2}(d^2 + 3d + 4)$, hence $e\big(B(d)\big)$ can be easily computed; the rest obviously follows. □

Proofs of Theorem 1.1 *and Corollary* 1.2. Now it is easy to see, that

$$\lim_{d\to\infty} \frac{c_1^2\big(B(d)\big)}{\chi_h\big(B(d)\big)} = \frac{116}{13} > 8\frac{9}{10}.$$

Consequently $c_1^2\big(B(d)\big) > 8\frac{9}{10}\chi_h\big(B(d)\big)$ holds with finitely many exceptions, proving Theorem 1.1. By blowing down $B(d)$ if possible, we end up with a minimal (simply connected) symplectic, hence irreducible 4-manifold $\tilde{B}(d)$. The symplectic structure ensures the nontriviality of the Seiberg-Witten invariants, and since blowing down does not change χ_h and increases c_1^2, the resulting irreducible 4-manifolds $\tilde{B}(d)$ obviously satisfy $c_1^2\big(\tilde{B}(d)\big) > 8\frac{9}{10}\chi_h\big(\tilde{B}(d)\big)$ (with finitely many exceptions). This last observation proves Corollary 1.2. □

Remarks 3.4.

- *In the notation we did not record the manifold X, although $B(d)$ depends on the choice of X. Using the 4-manifold provided by the Appendix we get that $c_1^2\big(B(d)\big) = 116d^2 - 351d + 70$ and $\chi_h\big(B(d)\big) = 13d^2 - 39d + 17$. Various choices of X give simply connected irreducible 4-manifolds of positive signature with different characteristic numbers. Combining this freedom with the observation described in* [S1], *one we can prove that most lattice points in $\mathbb{Z} \times \mathbb{Z}$ satisfying $0 \le b \le 8\frac{9}{10}a$ correspond to a simply connected minimal symplectic* (*hence irreducible*) *4-manifold.*

- *It can be shown that for $d \geq 4$ the symplectic 4-manifold $B(d)$ is minimal, hence irreducible: Since $Y(n)$ is a relatively minimal Lefschetz fibration over a Riemann surface of positive genus, the result of* [S2] *applies and shows that $Y(n)$ is minimal. Adapting the method of W. Lorek now the minimality of $B(d)$ follows. For sake of brevity we do not give the complete argument here — the* a priori *necessary blow-downs (resulting in $\tilde{B}(d)$,) provide examples proving our main result.*
- *The routine exercise of determining the bound d_0 for which $d \geq d_0$ implies $c_1^2\big(B(d)\big) > 8\frac{9}{10}\chi_h\big(B(d)\big)$ is left to the reader.*

4. Appendix

First we will show a $\mathbb{Z}_5$-action on D required at the beginning of Section 2: Take the (singular) curve $A = \{[x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0^5 - x_1^3 x_2(x_1 + x_2) = 0\}$ in $\mathbb{CP}^2$ and blow up $\mathbb{CP}^2$ in $[0 : 0 : 1]$ (the singular point of the curve A). The proper transform $\tilde{A}$ still has one singular point, but the proper transform D of an additional blow-up will be smooth. Hence we have found a smooth curve D in $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$; restricting the blow-down map $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2} \to \mathbb{CP}^2$ to D and composing it with the projection $\mathbb{CP}^2 - [1 : 0 : 0] \to \{x_0 = 0\} \approx \mathbb{CP}^1$ (mapping $[x_0 : x_1 : x_2]$ to $[0 : x_1 : x_2]$) we get a map $\varphi \colon D \to \mathbb{CP}^1$. This map is simply an explicit description of the 5-fold cyclic branched cover $D \to \mathbb{CP}^1$ branched in three points $Q_1, Q_2, Q_3 \in \mathbb{CP}^1$. Consequently we have a $\mathbb{Z}_5$-action (the generator is denoted by $\gamma \colon D \to D$) on D; the fixed points of γ are the inverse images of Q_i ($i = 1, 2, 3$) (still denoted by Q_i in D). The above $\mathbb{Z}_5$-action can be explicitely seen on A as multiplication of x_0 by a fifth root of unity (providing fixed points $[0 : 0 : 1], [0 : 1 : 0]$ and $[0 : 1 : -1]$). An easy application of the adjunction formula now shows that D has genus 2.

Next, we show that a Lefschetz fibration with the properties listed at the begining of Section 3 exists. (Recall that we have to find $f \colon X \to \mathbb{CP}^1$ such that the generic fiber has genus 16, f admits a simply connected fiber and it also has a section of square -1.) Let us fix $p_1, \ldots, p_{34}$, 34 distinct points in $\mathbb{CP}^1$ and take the (singular) curve $G = \bigcup_{i=1}^{2}(\mathbb{CP}^1 \times \{p_i\}) \cup \bigcup_{j=1}^{34}(\{p_j\} \times \mathbb{CP}^1) \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. The double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along G is a singular complex surface, the desingularized of which being diffeomorphic to $\mathbb{CP}^2 \# 69\overline{\mathbb{CP}^2}$ [GS]. Composing the double branched cover map $\rho \colon \mathbb{CP}^2 \# 69\overline{\mathbb{CP}^2} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ with the projection $pr_2 \colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ to the second factor, we get a Lefschetz fibration $f \colon \mathbb{CP}^2 \# 69\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$. The generic fiber of f is a curve of genus 16 (the double branched cover of $\mathbb{CP}^1$ branched in 34 points). There are two singular fibers (originated from the parts $\mathbb{CP}^1 \times \{p_i\}$ of the branch locus); each singular fiber is a plumbing of 35 spheres along a star-shaped tree, consequently these singular fibers are simply connected. Moreover, the curves $\{p_j\} \times \mathbb{CP}^1 \subset G$ give rise to 34 sections of f, each being a rational curve with self-intersection -1.

Consequently $X = \mathbb{CP}^2 \# 69\overline{\mathbb{CP}^2}$ with the above fibration provides an example of a 4-manifold required in Section 3. Note that for the above X we have $\sigma(X) = -68$ and $e(X) = 72$ (hence $c_1^2(X) = -60$ and $\chi_h(X) = 1$), so the values of s and t and hence the characteristic numbers of the corresponding $B(d)$ are the ones given in Remark 3.4.

Remark 4.1. *An alternative way for constructing X with the above properties can be carried out in the following way. Assume that the manifold N is defined by the Kirby diagram consisting of a 0-framed torus knot $T(2,33)$ linked geometrically once with a (-1)-framed unknot. Take the compactified Milnor fiber $M_c(2,33,65)$ corresponding to the singularity $x^2+y^{33}+z^{65}=0$ and glue it to N along its boundary, so get $X = N \cup_\partial M_c(2,33,65)$. It can be shown that this closed 4-manifold admits a complex structure and a Lefschetz fibration $f\colon X \to \mathbb{CP}^1$ with fibers of genus 16, such that the torus knot $T(2,33)$ gives a (singular) fiber which is obviously simply connected (since it is homeomorphic to S^2), and the (-1)-framed unlink gives rise to a section of f with self-intersection -1. Consequently this latter construction provides an alternative choice for X. For related constructions of Lefschetz fibrations see* [GS].

Acknowledgements

We would like to thank Zoltán Szabó for helpful discussions.

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