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EQUIVARIANT COHOMOLOGY AND WALL CROSSING FORMULAS IN SEIBERG-WITTEN THEORY

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Abstract. We use localization formulas in the theory of equivariant cohomology to rederive the wall crossing formulas of Li-Liu [10] and Okonek-Teleman [11] for Seiberg-Witten invariants.

One of the difficulties in the study of Donaldson invariants or Seiberg-Witten invariants for closed oriented 4-manifolds with $b_2^+ = 1$ is that one has to deal with reducible solutions. There have been a lot of work in this direction in the Donaldson theory context (see e.g. Göttsche [7] and the references therein). In the Seiberg-Witten theory, the case of $b_1 = 0$ and the moduli space with dimension zero was discussed in Witten [17] and Kronheimer-Mrowka [9]. The general case with even b_1 was solved by Li-Liu [10]. The definition of Seiberg-Witten invariant used by Li and Liu is due to Taubes [14] who defined it as the pairing of a suitable power of the first Chern class of a natural principal S^1-bundle on the moduli space with the fundamental class of the moduli space. The dimension formula for Seiberg-Witten moduli space then forces b_1 to be even to get a nontrivial invariant when $b_2^+ = 1$. Very recently, Okonek-Teleman [11] extended the definition of Seiberg-Witten invariants when $b_1 \neq 0$ and obtained a universal wall crossing formula for the invariants. (See §4 for a review of the invariants defined by Okonek and Teleman.) A common feature in these works is that the equations used to define the invariants depend on some parameters. The parameter spaces are divided into chambers by "walls", where reducible solutions can occur. Within the same chamber, the invariants do not change. When the parameters change smoothly from one chamber to another, the usual approach is to examine what happens when they cross the wall. The result is expressed as a wall crossing formula.

The above complications actually all come from one source: the configuration spaces are singular. This leads one to consider other cohomology theories for the configuration spaces which are more suitable for spaces with singularities. For example, Li-Tian [13] have used intersection homology to study the wall crossing phenomenon addressed in Li-Liu [10]. In this paper, we use equivariant cohomology to take advantage of the special features of the singularities. The configuration spaces are quotients of contractible spaces by gauge groups,

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but the reducible solutions and irreducible solutions have different orbit types. The influential papers by Atiyah-Bott [1] and Witten [16] have generated a lot of works which demonstrate the power of equivariant cohomology in two-dimensional gauge theory. For four-dimensional gauge theory, Atiyah-Jeffrey [3] and Baulieu-Singer [4] used different versions of equivariant cohomology to interpret Witten's Lagrangian [15] for a physical theory of Donaldson theory. There have been many papers in physics literature defining topological quantum field theories by equivariant cohomology related to the action of gauge groups following these approaches. However, in contrast to the fruitful applications in two-dimensional gauge theory, equivariant cohomology has not been applied extensively in four-dimensional gauge theories. This paper provides an example indicating the usefulness of equivariant cohomology in four-dimensional gauge theory.

In gauge theory, one often encounters problems of infinite dimensional in nature. A well-known procedure is to break the action of the gauge group into a free action by an infinite dimensional group, then followed by a finite dimensional compact group. In our case, we first consider the quotient by the action of the based gauge group on the space of solutions to Seiberg-Witten equations, and then use Kalkman's formula on the cobordism for the based moduli spaces to study the wall crossing. In the proof, we borrow some technical results from Li-Liu [10].

The rest of the paper is arranged as follows. §1 reviews the definition of equivariant cohomology. A localization formula due to Kalkman [8] and two special cases are discussed in §2. §3 and §4 describe, respectively, how to use the localization formulas to derive the wall crossing formulas for Seiberg-Witten invariants due to Li-Liu [10] and Okonek-Teleman [11].

1. Equivariant cohomology

For simplicity of the presentation, we review only what we will use later about equivariant cohomology. For the general theory on equivariant cohomology, the reader is referred to [12, 2, 5]. We shall only consider the case of an S^1-action on a compact smooth manifold W, with fixed point set F. We allow W to have boundary, but require that $F \cap \partial W = \emptyset$. The action of S^1 generates a vector field X on W. In fact, for any $x \in W$, if we let $c(t) = \exp(\sqrt{-1}t) \cdot x$ then $X(x)$ is the tangent vector to $c(t)$ at $t = 0$. Denote by $\Omega^*(W)^{S^1}$ the space of differential forms on W fixed under the S^1-action. Let u be an indeterminate of degree 2 and consider the space $\Omega^*(W)^{S^1} \otimes \mathbb{R}[u]$. Define

$$d_{S^1} = d - u \cdot i_X : \Omega^*(W)^{S^1} \otimes \mathbb{R}[u] \to \Omega^*(W)^{S^1} \otimes \mathbb{R}[u],$$

as a derivation, whose action on u is zero and $d_{S^1}\alpha = d\alpha - u i_X \alpha$ for an invariant form $\alpha \in \Omega^*(W)^{S^1}$. Now $d_{S^1}^2 = -u(di_X + i_X d) = -uL_X = 0$ on $\Omega^*(W)^{S^1} \otimes \mathbb{R}[u]$.

The equivariant cohomology of the S^1-space W is defined by

$$H^*_{S^1}(W) = \frac{\mathrm{Ker}\, d_{S^1}}{\mathrm{Im}\ d_{S^1}}.$$

Notice that there is a $\mathbb{R}[u]$-linear operator

$$\int_W : \Omega^*(W)^{S^1} \otimes \mathbb{R}[u] \to \mathbb{R}[u],$$

which is defined by sending differential forms of degree $\dim(W)$ to its integral over W, and all other forms to zero.

Let P be a connected closed oriented manifold, and $\pi : E \to P$ be a smooth complex vector bundle over P. Assume that there is an S^1-action on E by bundle homomorphisms, which covers an S^1-action on P. Following Atiyah-Bott [2], one can define the equivariant Euler class as

$$\epsilon(E) = i^* i_* 1,$$

where $i : P \to E$ is the zero section, i_* and i^* are the push-forward and pullback homomorphisms in equivariant cohomology respectively. It is routine to verify that

$$\epsilon(E_1 \oplus E_2) = \epsilon(E_1)\epsilon(E_2),$$

for two S^1 bundles E_1 and E_2 over P. We will be concerned with the case when the action of S^1 on P is trivial. In this case, by splitting principle [6], we can assume without loss of generality that E has a decomposition as S^1 bundles

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r,$$

where each L_j is a line bundle, such that the action of $\exp(\sqrt{-1}t)$ on L_j is multiplication by $\exp(\sqrt{-1}m_j t)$, for some weight $m_j \in \mathbb{Z}$. By formula (8.8) in Atiyah-Bott [2],

$$\epsilon(L_j) = m_j u + c_1(L_j).$$

Hence we have

$$\epsilon(E) = \prod_{j=1}^{r} (m_j u + c_1(L_j)).$$

2. Localization formulas

For an S^1-space W with fixed point set F, let $\{P_k\}$ be the decomposition of F into connected components. It is well-known that each P_k is a smooth submanifold of W, hence F has only finitely many components. The S^1-action on W induces an action on the normal bundle ν_k of P_k in W. The equivariant Euler class of ν_k can be computed as in §1. Now endow W with an S^1-invariant metric. Define a 1-form θ on $W - F$ in the following way: $\theta(X) = 1$, $\theta|_{X^\perp} = 0$. Here we use $X^\perp$ to denote the orthogonal complement of X in the tangent space. It is easy to see that θ is a connection on the principal bundle $W-F \to (W-F)/S^1$. Following Kalkman [8], we define for any $\alpha = \sum \alpha_j u^j \in \Omega^*(M)^{S^1} \otimes \mathbb{R}[u]$,

$$r(\alpha) = \sum \alpha_j (d\theta)^j - \theta \wedge (i_X \alpha_j)(d\theta)^j.$$

It is easy to see that $r(\alpha)$ is S^1-invariant and $i_X r(\alpha) = 0$. So $r(\alpha)$ is the lifting of a form on $(W-F)/S^1$ via the projection $W-F \to (W-F)/S^1$. Now we can state a theorem due to Kalkman [8] which can be proved by Witten's localization principle:

Theorem 2.1. *Let W be an S^1-manifold with an invariant boundary ∂W, and fixed point set $F = \{P_k\}$, such that $F \cap \partial W = \emptyset$. Let α be an equivariant closed form on W of total degree dim$(W) - 2$. Then*

$$\int_{\partial W/S^1} r(\alpha) = \mathrm{Res}_0 \sum_k \int_{P_k} \frac{\alpha}{\epsilon(\nu_k)},$$

where Res_0 *means taking the residue at $u = 0$.*

We now give a construction of an equivariant closed form on W. Let $f : W \to \mathbb{R}$ be an S^1-invariant smooth function which vanishes near F, and $f \equiv 1$ outside a tubular neighborhood of F. Then $f\theta$ can be extended over F. It is straightforward to see that $d(f\theta) - u(-1 + i_X(f\theta)) = d(f\theta) - u(-1+f)$ is an equivariant closed form. Assume now that $\dim(W) = 2(n+1)$ and that S^1-action on the normal bundles ν_k all have weight 1. Let $\alpha = [d(f\theta) - u(-1+f)]^n$, then near ∂W we have $f \equiv 1$, and so

$$r(\alpha) = r((d\theta)^n) = (d\theta)^n.$$

Denote by c the first Chern class of the principal S^1-bundle $\partial W \to \partial W/S^1$. Then in our normalization of θ, $c = [-d\theta]$. An application of Theorem 2.1 then yields

$$(*) \qquad \int_{\partial W/S^1} c^n = (-1)^n \mathrm{Res}_0 \sum_k \int_{P_k} \frac{u^n}{\sum_{j=1}^{r_k} c_{r_k - j}(\nu_k) u^j},$$

where r_k is the complex rank of ν_k.

There is a slight generalization of the above formula. Let $\dim(W) = d+2$, d is not necessarily even. Let k be a number between 1 and d, which has the same parity as d. Let $\beta_1, \cdots, \beta_k$ be S^1-equivariant closed 1-forms on W, such that for each $j = 1, \cdots, k$, $\beta_j|_{\partial W}$ is the pullback of a 1-form on ∂W, which we also denote by β_j. Now let $l = \frac{1}{2}(d-k)$, and let

$$\alpha = \beta_1 \wedge \cdots \wedge \beta_k \wedge [d(f\theta) - u(-1+f)]^l.$$

Then near ∂W,

$$r(\alpha) = r(\beta_1 \wedge \cdots \wedge \beta_k \wedge (d\theta)^l) = \beta_1 \wedge \cdots \wedge \beta_k \wedge (d\theta)^l.$$

So by Theorem 2.1, we get

$$(**) \qquad \int_{\partial W/S^1} \beta_1 \wedge \cdots \wedge \beta_k \wedge c^l = (-1)^n \sum_k \mathrm{Res}_0 \int_{P_k} \frac{u^l \beta_1 \wedge \cdots \wedge \beta_k}{\sum_{j=1}^{r_k} c_{r_k - j}(\nu_k) u^j}.$$

3. Applications to Seiberg-Witten theory

Let X be a closed oriented 4-manifold. Given a Riemannian metric g and a $Spin_c$ structure $\mathcal{S}$ on X, there are associated Hermitian rank 2 vector bundles V_+ and V_-, and a bundle isomorphism

$$\rho : \Lambda_+ \to su(V_+),$$

where $su(V_+)$ is the bundle of anti-Hermitian traceless linear maps on V_+. The Seiberg-Witten equations are for a pair (A, Φ), where A is a unitary connection on the line bundle $L = det(V_+)$, and Φ a section of V_+. For any fixed $\eta \in \Omega^+(X)$, the perturbed Seiberg-Witten equations are

$$(1) \qquad \left\{ \begin{array}{rcl} D_A \Phi & = & 0, \\ \rho(iF_A^+ + \eta) & = & (\Phi \otimes \Phi^*)_0, \end{array} \right.$$

where D_A denotes the Dirac operator defined by the $Spin_c$ structure and the connection A. These equations have a huge degree of symmetries. Let $\mathcal{A}$ denote the set of all unitary connections on L, $\mathcal{G}$ the group $Aut(L) = Map(X, S^1)$. $\mathcal{G}$ is called the gauge group. There is an action of $\mathcal{G}$ on $\mathcal{A} \times \Gamma(V_+)$, which preserves the Seiberg-Witten equations in the sense that solutions are transformed to solutions. It is given by

$$g \cdot (A, \Phi) = (A - 2g^{-1}dg, g\Phi).$$

This action is not free and has two orbit types: if $\Phi \neq 0$, the stabilizer of (A, Φ) is trivial; on the other hand, the stabilizer of $(A, 0)$ is S^1. To fix this problem, we choose an arbitrary point $x_0 \in X$, and let $\mathcal{G}_0 = \{g \in \mathcal{G} | g(x_0) = 1\}$. Then $\mathcal{G} = \mathcal{G}_0 \times S^1$. Furthermore, the action of $\mathcal{G}_0$ on $\mathcal{A} \times \Gamma(V_+)$ is free. We then get the residue S^1 action on the smooth infinite-dimensional manifold $(\mathcal{A} \times \Gamma(V_+))/\mathcal{G}_0$.

Denote by $M(\mathcal{S}, g, \eta)$ and $M^0(\mathcal{S}, g, \eta)$ the quotients of the space $\mathcal{M}(\mathcal{S}, g, \eta)$ of solutions to (1) by $\mathcal{G}$ and $\mathcal{G}_0$ respectively. They have the following well-known properties [9]:

- $M(\mathcal{S}, g, \eta)$ and $M^0(\mathcal{S}, g, \eta)$ are compact in natural topologies.
- For a generic choice of (g, η), $M^0(\mathcal{S}, g, \eta)$ is a smooth manifold of dimension $d + 1 = 1 + \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\tau(X))$, which can be oriented in a natural way.
- For a generic choice of (g, η) with $2\pi c_1^+(L) \neq \eta^h$, the harmonic part of η, $M^0(\mathcal{S}, g, \eta)$ does not contain solutions with $\Phi = 0$ (called reducible solutions). The S^1-action on $M^0(\mathcal{S}, g, \eta)$ then gives rise to a principal S^1-bundle $M^0(\mathcal{S}, g, \eta) \to M(\mathcal{S}, g, \eta)$. We call such a choice of (g, η) a good choice. When $b_2^+(X) > 0$, there are good choices.

- For two good choices (g_0, η_0) and (g_1, η_1), there is a path (g_t, η_t) joining them, such that $M^0(\mathcal{S}, g_t, \eta_t)$, $0 \leq t \leq 1$, form an oriented cobordism W between $M^0(\mathcal{S}, g_0, \eta_0)$ and $M^0(\mathcal{S}, g_1, \eta_1)$. When $b_2^+(X) > 1$, it is possible to choose the path such that none of $M^0(\mathcal{S}, g_t, \eta_t)$ admits a reducible solution.

For a good choice (g, η), the Seiberg-Witten invariant is defined as follows: (a) if $d < 0$, $SW(\mathcal{S}, g, \eta) = 0$; (b) if $d = 0$, then $M(\mathcal{S}, g, \eta)$ is a finite union of signed points, and $SW(\mathcal{S}, g, \eta)$ is the sum of the corresponding ± 1's; (c) if $d > 0$, the Seiberg-Witten invariant can be defined as the coupling of the fundamental class of $M(\mathcal{S}, g, \eta)$ with the suitable power of the first Chern class of the principal S^1-bundle $M^0(\mathcal{S}, g, \eta) \to M(\mathcal{S}, g, \eta)$. So when $b_2^+(X) > 1$, $SW(\mathcal{S}, g, \eta)$ does not depends on the good choice (g, η) and is then a diffeomorphism invariant. However, if $b_2^+(X) = 1$, $c_1^+(L) = \eta^h$ defines a hypersurface in the space of (g, η)'s. It is called a "wall", since it divides the space of (g, η)'s into two connected components, called "chambers". For two good choices (g_0, η_0) and (g_1, η_1) in the same chamber, the Seiberg-Witten invariants are the same. However, when they lie in different chambers the invariants may differ. A formula relating the Seiberg-Witten invariants for good choices in different chambers is called a wall crossing formula.

Since the invariants defined above are nontrivial only when d is even, the formula for d above shows that the only interesting case is when $b_2^+(X) = 1$, and $b_1(X)$ is even. The wall crossing formula of Seiberg-Witten invariants in the case $b_2^+ = 1$, $b_1 = 0$ and $d = 0$ was obtained by Witten [17] and Kronheimer-Mrowka [9]. The general wall crossing formula for $d > 0$ case, proved by Li-Liu [10], can be stated as follows

Theorem 3.1. *Let X be a closed oriented 4-manifold with $b_2^+ = 1$ and b_1 even, $\mathcal{S}$ a $Spin_c$ structure with $det(V_+) = L$, such that $c_1(L)^2 - (2\chi(X) + 3\tau(X)) \geq 0$. Then for any two good choices (g_0, η_0) and (g_1, η_1) in two different chambers, the Seiberg-Witten invariants $SW(\mathcal{S}, g_0, \eta_0)$ and $SW(\mathcal{S}, g_1, \eta_1)$ differ by*

$$\pm \int_{T^{b_1}} \frac{\left(\frac{1}{4}\Omega^2 \cdot c_1(L)[X]\right)^{\frac{b_1}{2}}}{(b_1/2)!},$$

where

$$\Omega = c_1(\mathcal{U}) = \sum_i x_i \cdot y_i,$$

and $\mathcal{U}$ is the universal flat line bundle over $T^{b_1} \times M$, $\{y_i\}$ is any basis of $H^1(X; \mathbb{Z})$ modulo torsion, and $\{x_i\}$ is the dual basis in $H^1(T^{b_1}; \mathbb{Z})$.

We will now reprove this theorem by the method described in §2. Take a path (g_t, η_t) connecting (g_0, η_0) and (g_1, η_1) that goes through the wall transversally once. Then the S^1-action on the induced cobordism W has only one component in the fixed point set F, namely the set of reducible solutions, which are parameterized by the torus $T^{b_1} = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$. We shall assume for now that for each reducible solution $(A,0)$, $Coker D_A = 0$. Under this assumption, the normal bundle of F in W is given by the index bundle ind, whose fiber at each $(A,0) \in F$ is given by $\mathrm{Ker}\, D_A$ (cf. Li-Liu [10].) It is clear that the S^1-action on this normal bundle has only weight 1. Now we use formula $(*)$ in §2 to get

$$\begin{aligned}(2) \qquad SW(\mathcal{S}, g_1, \eta_1) - SW(\mathcal{S}, g_0, \eta_0) &= \int_{\partial W/S^1} c^{d/2}, \\ &= \pm \operatorname{Res}_0 \int_{T^{b_1}} \frac{u^{d/2}}{\sum_{j=1}^{r} c_{r-j}(ind) u^j},\end{aligned}$$

where r is the complex rank of the index bundle ind, and $2r + b_1 = d + 2$. In the proof of Lemma 2.5 in [10], Li-Liu derived, by Atiyah-Singer family index theorem, that

$$\begin{aligned} c_1(ind) &= \frac{1}{4}\Omega^2 \cdot c_1(L)[X], \\ c_j(ind) &= \frac{1}{j!} c_1(ind)^j. \end{aligned}$$

Plugging the above equalities into (2), we see that the difference between the two Seiberg-Witten invariants is the residue at 0 of

$$\begin{aligned}(3) \qquad \pm \int_{T^{b_1}} \frac{u^{d/2}}{u^r exp(c_1(ind)/u)} &= \pm \int_{T^{b_1}} u^{-r+d/2} exp(-c_1(ind)/u), \\ &= \pm \int_{T^{b_1}} \frac{u^{-r+d/2-b_1/2} c_1(ind)^{b_1/2}}{(b_1/2)!}, \\ &= \pm \frac{1}{u} \int_{T^{b_1}} \frac{c_1(ind)^{b_1/2}}{(b_1/2)!}.\end{aligned}$$

Following the suggestion of one of the referees, we also include some details on the discussion for the general case when $Coker D_A \neq 0$. We will use the method of Li and Liu [10], p. 208. To make use of Lemma 9.30 in Berline-Getzler-Vergne [5], we make the following digression. Fix a connection A_0 on L, then any other connection on L can be written as $A + a$ for some $a \in i\Omega^1(X)$. By Hodge theory, after a gauge transformation, one can assume that $d^* a = 0$. Then the system of Seiberg-Witten equations is equivalent to the following system of equations:

$$\begin{cases} d^* a = 0, \\ D_{A_0+a}\Phi = 0, \\ \rho(iF^+_{A_0+a} + \eta) = (\Phi \otimes \Phi^*)_0. \end{cases}$$

This system of equations has the S^1 symmetry given by multiplications of unit length complex numbers on Φ. Hodge theory provides the obvious identifications of the solution space $\mathcal{M}^0_s$ with $\mathcal{M}^0$ and $\mathcal{M}^0_s/S^1$ with $\mathcal{M}$ which show that we can use these spaces to define Seiberg-Witten invariants instead. For a fixed Riemannian metric g, denote by F the set of flat connections $A_0 + a$ such that

$d^*a = 0$. Then there is an integer N and a map $\psi : F \times \mathbb{C}^N \to \Gamma(V_-)$ such that for each $A \in F$, $D_A^{\psi_A} = D_A + \psi_A : \Gamma(V_+) \oplus \mathbb{C}^N \to \Gamma(V_-)$ is surjective. According to the analysis of Li and Liu, the cobordism W obtained by the natural compactification of the cobordism of irreducible solutions may not contain the whole set F, and it may intersect F in a complicated fashion. Nevertheless, near such intersections, the cobordism can be described explicitly as follows: $A \in F \mapsto \operatorname{Ker} D_A^{\psi_A}$ defines a vector bundle $\operatorname{Ker} D^\psi$ on F, and there is an S^1-equivariant map μ from this bundle to $\mathbb{C}^N$, such that near $W \cap F$, W is given by $\mu^{-1}(0)$. Denote by $S(\operatorname{Ker} D^\psi)$ and $B(\operatorname{Ker} D^\psi)$ the sphere bundle and the disk bundle of $\operatorname{Ker} D^\psi$. Notice that the Poincaré dual of $(\mu^{-1}(0) \cap S(\operatorname{Ker} D^\psi))/S^1$ in $S(\operatorname{Ker} D^\psi)/S^1$ is just $r(u^N)$. Then we have

$$\begin{aligned} SW(\mathcal{S}, g_1, \eta_1) - SW(\mathcal{S}, g_0, \eta_0) &= \int_{(W \cap S(\operatorname{Ker} D^\psi))/S^1} c^{d/2}, \\ &= \int_{(\mu^{-1}(0) \cap S(\operatorname{Ker} D^\psi))/S^1} c^{d/2}, \\ &= \int_{S(\operatorname{Ker} D^\psi)/S^1} c^{d/2} \cup r(u^N). \end{aligned}$$

Now S^1 acts fiberwise on $B(\operatorname{Ker} D^\psi)$, with fixed point set F. We can then use Kalkman's formula. Notice that the Chern classes of $\operatorname{Ker}(D^\psi)$ can be computed by family index theory as for *ind* above, but its rank r now satisfies $2(r-N)+b_1 = d+2$. Repeating the calculation as in (3) then completes the proof of Theorem 3.1.

4. Applications to generalized Seiberg-Witten invariants

Okonek-Teleman [11] extended the definition of Seiberg-Witten invariants. They also proved a wall crossing formula for such general Seiberg-Witten invariants. In this section, we will give an equivalent definition of the general Seiberg-Witten invariants, which is along the line of the discussions in the preceding sections. We then reprove Okonek-Teleman's formula by the localization formula $(**)$.

We use the notations of §3. Let $L \to X$ be the Hermitian line bundle associated to a fixed $Spin_c$-structure $\mathcal{S}$. For a good choice (g, η), let $\pi_2 : \mathcal{M}(\mathcal{S}, g, \eta) \times X \to X$ be the projection onto the second factor. Consider the pullback line bundle $\pi_2^* L$. The group $\mathcal{G}$ acts freely on $\pi_2^* L$, which covers a free action of $\mathcal{G}$ on $\mathcal{M}(\mathcal{S}, g, \eta) \times X$. There is therefore a quotient line bundle

$$\mathcal{L} \to M(\mathcal{S}, g, \eta) \times X.$$

We now define a group homomorphism $\mu : H_1(X; \mathbb{Z})/Tor \to H^1(M(\mathcal{S}, g, \eta); \mathbb{R})$ by

$$\mu([A]) = \int_A c_1(\mathcal{L}),$$

where A is a loop in X, and $[A]$ its homology class. It is easy to see that this is well-defined.

Let $d = \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\tau(X))$. When $d < 0$, the Seiberg-Witten invariant $SW(\mathcal{S}, g, \eta)$ is defined to be zero. When $d = 0$, it is defined as in §3. When $d > 0$, $SW(\mathcal{S}, g, \eta)$ is defined as a linear map

$$\Lambda^*(H_1(M, \mathbb{Z})/Tor) \to \mathbb{R}.$$

More precisely, for $0 \le k \le \min\{b_1, d\}$ such that k has the same parity as d, the invariant is defined by

$$SW(\mathcal{S}, g, \eta)([A_1], \cdots, [A_k]) = \int_{M(\mathcal{S}, g, \eta)} \mu([A_1]) \wedge \cdots \wedge \mu([A_k]) \wedge c^l,$$

where $l = \frac{1}{2}(d - k)$ and c is as in §3. For all other values of k, the invariant is defined to be zero. We remark that these invariants are actually integer-valued, even though they are defined as integrals of differential forms. An important point of this definition is that b_1 does not have to be even to get a non-trivial invariant.

Okonek-Teleman's wall crossing formula can be stated as the following:

Theorem 4.1. *For a fixed $Spin_c$-structure $\mathcal{S}$ on a connected closed oriented 4-manifold X with $b_2^+ = 1$, $d = \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\tau(X)) \ge 0$, the Seiberg-Witten invariants for two good choices (g_0, η_0) and (g_1, η_1) in two different chambers are related by*

$$\begin{aligned} &SW(\mathcal{S}, g_1, \eta_1)([A_1], \cdots, [A_k]) - SW(\mathcal{S}, g_0, \eta_0)([A_1], \cdots, [A_k]) = \\ &\qquad \pm \int_{T^{b_1}} \frac{\mu([A_1]) \wedge \cdots \wedge \mu([A_k]) \wedge (\frac{1}{4}\Omega^2 \cdot c_1(L)[X])^{(b_1-k)/2}}{((b_1 - k)/2)!}, \end{aligned}$$

where $[A_1], \cdots, [A_k]$ are homology classes in $H_1(X; \mathbb{Z})/Tor$, Ω is as in Theorem 3.1, $0 \le k \le \min\{b_1, d\}$, and k has the same parity as b_1.

The proof is similar to the proof of Theorem 3.1. To start with, we give a parameterized version of the construction for $\mathcal{L}$. Take a path (g_t, η_t) as in §3. Consider the infinite dimensional cobordism

$$\mathcal{W} = \bigcup_t \mathcal{M}(\mathcal{S}, g_t, \eta_t).$$

Consider the pullback line bundle $\pi_2^* L$ on $\mathcal{W} \times X$, where π_2 is again the projection onto the second factor. Modulo the action by $\mathcal{G}_0$, we get a quotient line bundle

$$\mathcal{L}_0 \to W \times X.$$

This is actually an S^1-bundle, since there is the residue action by $S^1 = \mathcal{G}/\mathcal{G}_0$. Since this S^1-action on ∂W is free, it is straightforward to see that when restricted to $\partial W = M_0(\mathcal{S}, g_0, \eta_0) \sqcup M_0(\mathcal{S}, g_1, \eta_1)$, $\mathcal{L}_0$ can be identified with the pullback of $\mathcal{L}$ on $M(\mathcal{S}, g_j, \eta_j)$ via the projection $M_0(\mathcal{S}, g_j, \eta_j) \to M(\mathcal{S}, g_j, \eta_j)$, for $j = 1, 2$. Let $\epsilon(\mathcal{L}_0)$ be the equivariant first Chern class of $\mathcal{L}_0$. Define $\mu_0 : H_1(X; \mathbb{Z})/Tor \to H^1_{S^1}(W; \mathbb{R})$ by

$$\mu_0([A]) = \int_A \epsilon(\mathcal{L}_0).$$

It is easy to see that when restricted to ∂W, $\mu_0([A])$ is the pullback of $\mu([A])$ on $M(\mathcal{S}, g_0, \eta_0)$ and $M(\mathcal{S}, g_1, \eta_1)$ respectively. Now let $\beta_j = \mu_0([A_j])$, for $j = 1, \cdots, k$, a computation similar to the one in §3 by formula $(**)$ then proves Theorem 4.1.

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