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AVERAGES OVER CURVES WITH TORSION

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Abstract. We establish L^p Sobolev mapping properties for averages over certain curves in $\mathbb{R}^3$, which improve upon the estimates obtained by $L^2 - L^\infty$ interpolation.

Let T be the operator given by convolution in $\mathbb{R}^3$ against a smooth cutoff of arclength measure on the helix $\gamma(t) = \big(\cos t, \sin t, t\big)$,

$$Tf(x) = \int f(x_1 - \cos t\,,\, x_2 - \sin t\,,\, x_3 - t)\,\phi(t)\,dt\,.$$

For $1 < p < \infty$, let $H^{s,p}(\mathbb{R}^3)$ denote the nonhomogeneous Sobolev space consisting of functions in $L^p(\mathbb{R}^3)$ whose fractional derivative of order s also lies in $L^p(\mathbb{R}^3)$. We consider the following question:

For which values of s (depending on p) does $T : L^p(\mathbb{R}^3) \to H^{s,p}(\mathbb{R}^3)$?

By duality, it suffices to consider $2 \le p < \infty$. As shown by the first two authors in [OS], a necessary condition is that

$$s \;\le\; \tfrac{1}{6} + \tfrac{1}{3p} \quad \text{if} \quad 2 \le p \le 4\,,$$

$$s \;\le\; \tfrac{1}{p} \quad \text{if} \quad 4 \le p < \infty\,.$$

Simple arguments (see for example the lemma below) show that $T : L^2(\mathbb{R}^3) \to H^{\frac{1}{3},2}(\mathbb{R}^3)$. Interpolation with the trivial $L^\infty(\mathbb{R}^3)$ boundedness of T yields a sufficient condition of $s \le \frac{2}{3p}$. In particular, interpolation yields

$$T : L^4(\mathbb{R}^3) \to H^{\frac{1}{6},4}(\mathbb{R}^3)\,. \tag{1}$$

In this note, we combine the arguments of [OS] with Bourgain's [B] improvement of the conic square function estimate of Mockenhaupt [M] to obtain the following.

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Theorem. *There exists $\sigma > 0$ such that*

$$T : L^4(\mathbb{R}^3) \to H^{\frac{1}{6}+\sigma,4}(\mathbb{R}^3)\,. \tag{2}$$

We should point out that T is a model for curve-averaging operators whose canonical relations have two-sided Whitney folds. In two dimensions these operators are much easier to analyze and optimal results are known. See e.g., [SS] and [SW].

In three dimensions, the translation invariant operators of this type are the averages over curves with non-vanishing torsion (a curve $\gamma(t)$ has non-vanishing torsion if the vectors $\{\gamma'(t), \gamma''(t), \gamma'''(t)\}$ are linearly independent for each t.) The helix and the twisted cubic, $\gamma(t) = (t, t^2, t^3)$, are basic examples. We restrict attention here to the helix since this operator has the light cone in ξ as its folding set. A modification of Bourgain's estimate to conic hypersurfaces with one non-vanishing principle curvature would yield the theorem for general curves with torsion.

The value of σ is related to the exponent τ in equation (132) of [B], which is not explicitly determined. Any $\sigma < \frac{1}{3}\tau$ works. In particular, an optimal value $\tau = \frac{1}{4}$ would yield the nearly optimal condition $\sigma < \frac{1}{12}$. Recently, Tao and Vargas [TV] have modified Bourgain's arguments and obtained a definite value of τ. The authors would like to thank T. Tao for a helpful conversation regarding Bourgain's work.

To begin the proof of (2), let

$$\widehat{T}(\xi) = \int e^{-i\xi_1 \cos t - i\xi_2 \sin t - i\xi_3\, t}\, \phi(t)\, dt \tag{3}$$

denote the Fourier multiplier associated to T.

Let $\xi' = (\xi_1, \xi_2)$. The oscillatory integral (3) has no critical points for $|\xi'| < |\xi_3|$. The following thus holds.

$$\big|\widehat{T}(\xi)\big| = \mathcal{O}(|\xi|^{-N}) \quad \forall N\,, \quad \text{if} \quad |\xi'| \le .99\, |\xi_3|\,.$$

For $|\xi'| > |\xi_3|$ there are two, nondegenerate critical points. The following is thus a consequence of Van der Corput's Lemma,

$$\big|\widehat{T}(\xi)\big| \le C\, |\xi|^{-\frac{1}{2}}\,, \quad \text{if} \quad |\xi'| \ge 1.01\, |\xi_3|\,.$$

A simple interpolation argument implies (2) for the operator obtained by conicly restricting $\widehat{T}(\xi)$ to either of the above regions. Indeed, since these bounds imply that these two localized pieces gain a 1/2-derivative on L^2, the interpolation argument behind (1) yields estimates of the form (2) for each term with the desired $\sigma = 1/12$.

It thus suffices to establish (2) for the operator S obtained by restricting the multiplier $\widehat{T}(\xi)$ to the region A, defined by $.98 \le |\xi'|\,/\,|\xi_3| \le 1.02$, via a smooth conic cutoff. Let S_λ denote the operator obtained by further restricting to the region $\lambda \le |\xi_3| \le 2\lambda$. The theorem is then a result of showing that, for some number $a > 0$, for all $\lambda > 2$,

$$\|S_\lambda\|_{4,4} \le C\,\big(\log\lambda\big)^a\,\lambda^{-\frac{1}{6}-\frac{\tau}{3}}\,. \tag{4}$$

We restrict attention to $\xi_3 > 0$. Following [OS], we make a further decomposition of S_λ by decomposing the conic set A into a union of conic sets A_λ^j as follows:

$$\text{for } j \ge 1, \text{ set } A_\lambda^j = \{1 + 2^{j-1}\,\lambda^{-\frac{2}{3}} \le |\xi'|\,/\,\xi_3 \le 1 + 2^j\,\lambda^{-\frac{2}{3}}\}\,;$$

$$\text{set } A_\lambda^0 = \{1 - \lambda^{-\frac{2}{3}} \le |\xi'|\,/\,\xi_3 \le 1 + \lambda^{-\frac{2}{3}}\}\,;$$

$$\text{for } j \le -1, \text{ set } A_\lambda^j = \{1 - 2^{|j|}\,\lambda^{-\frac{2}{3}} \le |\xi'|\,/\,\xi_3 \le 1 - 2^{|j|-1}\,\lambda^{-\frac{2}{3}}\}\,.$$

Introducing a suitable partition of unity on the Fourier transform side leads to the decomposition

$$S_\lambda = \sum_j S_\lambda^j.$$

Inequality (4) will follow from

$$\|S_\lambda^j\|_{4,4} \le C\,\big(\log\lambda\big)^a\,\lambda^{-\frac{1}{6}-\frac{\tau}{3}}\,2^{\frac{|j|}{2}(\tau-\frac{1}{4})} \tag{5}$$

for all j and λ. At this point we make a further decomposition as in [M] of A_λ^j into sets A_λ^{jm} supported in ξ' sectors of angle $\delta \doteq 2^{|j|/2}\lambda^{-\frac{1}{3}}$. This leads to a decomposition

$$S_\lambda^j = \sum_{m=1}^{\delta^{-1}} S_\lambda^{jm}.$$

In the notation of Theorem 1.0 of [M], we have

$$\widehat{S}_\lambda^{jm}(\xi) = \widehat{\psi}_m\big(\lambda^{-1}\xi', \lambda^{-1}(1+\delta^2)\,\xi_3\big)\,\widehat{T}(\xi)\,.$$

The quantity N of that theorem is related to j and λ by $N = \delta^{-1}$.

Lemma.

$$\|S_\lambda^{jm}\|_{4,4} \le C\,\lambda^{-\frac{1}{4}}\,\delta^{\frac{1}{4}}\,.$$

Proof. The proof is almost identical to that of the Lemma in [OS], and is obtained by interpolating the following estimates

$$\begin{aligned} \|S_\lambda^{jm}\|_{2,2} &\le C\,(\lambda\delta)^{-\frac{1}{2}}\,,\\ \|S_\lambda^{jm}\|_{\infty,\infty} &\le C\,\delta\,. \end{aligned} \tag{6}$$

The first estimate in (6) is the bound $|\widehat{S}^j_\lambda(\xi)| \le C\,(\lambda\,\delta)^{-\frac12}$, which follows from Van der Corput's Lemma as shown in [OS]. For the second estimate, we consider the term m corresponding to the ξ' sector along the negative ξ_2 axis. The convolution kernel of S^{jm}_λ, written in the new coordinates

$$\bigl(y_1,y_2,y_3\bigr) = \bigl(x_1, x_2+\alpha x_3, \alpha x_3 - x_2\bigr)\,, \qquad \alpha = (1+\delta^2)^{-1}\,,$$

takes the form

$$K(y) = \lambda^3\,\delta^3 \int \phi(t)\,\theta\bigl(\lambda\,\delta\,(y_1-\cos t)\,,\,\lambda\,\delta^2\,(y_2-\sin t-\alpha t)\,,\,\lambda\,(y_3+\sin t-\alpha t)\bigr)\,dt\,.$$

Here and below, θ denotes a Schwartz function with seminorms bounded independent of j,m,λ, and with $\widehat\theta(\eta)=0$ for $\eta_3\le 1$. We need to show that $\|K\|_{L^1}\le C\,\delta$, and may thus replace $\phi(t)$ by $\phi_\delta(t)$ which vanishes for $|t|\le 10\,\delta$. We write $\theta=\partial_3\theta$ for some new θ to express $K(y)$ as

$$\lambda^2\delta^3\int\biggl(\frac{\phi_\delta(t)}{\alpha-\cos t}\biggr)'\,\theta(\cdots)\,dt + \lambda^3\delta^4\int\frac{\sin t\,\phi_\delta(t)\,\theta(\cdots)}{\alpha-\cos t}\,dt + \lambda^3\delta^5\int\frac{(\alpha+\cos t)\,\phi_\delta(t)\,\theta(\cdots)}{\alpha-\cos t}\,dt\,.$$

The inequality $\alpha-\cos t\ge t^2/10$ for $|t|\in[10\,\delta,\pi]$, together with $|\phi_\delta'(t)|\le C\,\delta^{-1}\le C\,\lambda^{1/3}$, yields the desired $L^1(dy)$ norm bounds on the first and third terms. The desired bound for the second term follows by a further integration by parts of the same kind. □

To conclude the proof of (5), we apply Bourgain's estimate (132) of [B] to obtain

$$\Bigl\|\sum_m S^{jm}_\lambda f\,\Bigr\|_4 \le C\,\delta^{\tau-\frac14}\,\Bigl\|\,\Bigl(\sum_m |S^{jm}_\lambda f|^2\Bigr)^{\frac12}\Bigr\|_4\,.$$

The number of indices m is $O(\delta^{-1})$, so

$$\sum_m \bigl|S^{jm}_\lambda f(x)\bigr|^2 \le C\,\delta^{-\frac12}\Bigl(\sum_m \bigl|S^{jm}_\lambda f(x)\bigr|^4\Bigr)^{\frac12}.$$

With $\widehat{f}_m$ representing the localisation of $\widehat f$ to an appropriate sector in ξ', we thus have

$$\begin{aligned}\Bigl\|\sum_m S^{jm}_\lambda f\,\Bigr\|_4 &\le C\,\delta^{\tau-\frac12}\,\Bigl\|\Bigl(\sum_m |S^{jm}_\lambda f|^4\Bigr)^{\frac14}\Bigr\|_4\\ &\le C\,\lambda^{-\frac16-\frac{\tau}{3}}\,2^{\frac{|j|}{2}(\tau-\frac14)}\,\Bigl\|\Bigl(\sum_m |f_m|^4\Bigr)^{\frac14}\Bigr\|_4\\ &\le C\,\lambda^{-\frac16-\frac{\tau}{3}}\,2^{\frac{|j|}{2}(\tau-\frac14)}\,\Bigl\|\Bigl(\sum_m |f_m|^2\Bigr)^{\frac12}\Bigr\|_4\,.\end{aligned}$$

A result of Córdoba [C] gives

$$\Bigl\|\Bigl(\sum_m |f_m|^2\Bigr)^{\frac12}\Bigr\|_4 \le C\,|\log\delta\,|^a\,\|f\|_4$$

for some positive a, which completes the proof of (5).

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