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ON BLOWUP FORMULAE FOR THE S-DUALITY CONJECTURE OF VAFA AND WITTEN II: THE UNIVERSAL FUNCTIONS

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1. Introduction

This is a continuation of our work [L-Q] on blowup formulae for the S-duality conjecture of Vafa and Witten. In [V-W], Vafa and Witten formulated some mathematical predictions about the Euler characteristics of instanton moduli spaces derived from the S-duality conjecture in physics. From these mathematical predictions, a blowup formula was proposed based upon the work of Yoshioka [Yos]. Roughly speaking, the blowup formula says that there exists a universal relation between the Euler characteristics of instanton moduli spaces for a smooth four manifold and the Euler characteristics of instanton moduli spaces for the blowup of the smooth four manifold. The universal relation is independent of the four manifold and related to some modular forms. In [L-Q], we verified this blowup formula for the gauge group $SU(2)$ and its dual group $SO(3)$ when the underlying four manifold is an algebraic surface. In fact, we proved a stronger blowup formula in [L-Q], i.e. a blowup formula for the virtual Hodge numbers of instanton moduli spaces. However, in [L-Q], we did not find a closed formula for the universal function which appears in this stronger blowup formula. Our goal of the present paper is to determine a closed formula for this universal function.

To state the blowup formulae proved in [L-Q], we recall some standard definitions and notations. Let $\phi : \widetilde{X} \to X$ be the blowing-up of an algebraic surface X at a point $x_0 \in X$, and E be the exceptional divisor. For simplicity, we always assume that X is simply connected. Fix a divisor c_1 on X, $\tilde{c}_1 = \phi^* c_1 - aE$ with $a = 0$ or 1, and an ample divisor H on X with odd $(H \cdot c_1)$. For an integer n, let $\mathfrak{M}_H(c_1, n)$ be the moduli space of Mumford-Takemoto H-stable rank-2 bundles with Chern classes c_1 and n, $\mathfrak{M}_H^G(c_1, n)$ be the moduli space of Gieseker H-semistable rank-2 torsion-free sheaves with Chern classes c_1 and n, and $\mathfrak{M}_H^U(c_1, n)$ be the Uhlenbeck compactification of $\mathfrak{M}_H(c_1, n)$ from gauge theory [Uhl, Don, LiJ]. It is well-known that both the Gieseker moduli spaces and

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the Uhlenbeck compactification spaces are projective. For $r \gg 0$, the divisors $H_r = r\cdot\phi^*H - E$ on $\widetilde{X}$ is ample; moreover, all the moduli spaces $\mathfrak{M}_{H_r}(\tilde{c}_1, n)$ (resp. $\mathfrak{M}^G_{H_r}(\tilde{c}_1, n)$, $\mathfrak{M}^U_{H_r}(\tilde{c}_1, n)$) can be naturally identified. So we shall use $\mathfrak{M}_{H_\infty}(\tilde{c}_1, n)$ (resp. $\mathfrak{M}^G_{H_\infty}(\tilde{c}_1, n)$, $\mathfrak{M}^U_{H_\infty}(\tilde{c}_1, n)$) to denote the moduli space $\mathfrak{M}_{H_r}(\tilde{c}_1, n)$ (resp. $\mathfrak{M}^G_{H_r}(\tilde{c}_1, n)$, $\mathfrak{M}^U_{H_r}(\tilde{c}_1, n)$) with $r \gg 0$.

For a complex algebraic scheme Y (not necessarily smooth, projective, or irreducible), let $e(Y; x, y)$ be the virtual Hodge polynomial of Y. When Y is projective, $e(Y; 1, 1)$ is the topological Euler characteristic of Y_{red}. Our Theorem A in [L-Q] gives the following blowup formula for the Gieseker moduli spaces:

$$(1.1) \quad \sum_n e(\mathfrak{M}^G_{H_\infty}(\tilde{c}_1, n); x, y) q^{n - \frac{\tilde{c}_1^2}{4}} = (q^{\frac{1}{12}} \cdot \widetilde{\widetilde{Z}}_a) \cdot \sum_n e(\mathfrak{M}^G_H(c_1, n); x, y) q^{n - \frac{c_1^2}{4}}$$

where $\widetilde{\widetilde{Z}}_a = \widetilde{\widetilde{Z}}_a(x, y, q)$ is a universal function of x, y, q, a with

$$\widetilde{\widetilde{Z}}_a(1, 1, q) = \frac{\sum_{n \in \mathbb{Z}} q^{(n + \frac{a}{2})^2}}{[q^{\frac{1}{24}} \prod_{n \ge 1}(1 - q^n)]^2}.$$

Assuming that $\mathfrak{M}_H(c_1, n)$ (respectively, $\mathfrak{M}_{H_\infty}(\tilde{c}_1, n)$) is dense in the Gieseker moduli space $\mathfrak{M}^G_H(c_1, n)$ (respectively, $\mathfrak{M}^G_{H_\infty}(\tilde{c}_1, n)$) for every n, we also have a blowup formula for the Uhlenbeck compactification spaces (Theorem B in [L-Q]):

$$\sum_n e(\mathfrak{M}^U_{H_\infty}(\tilde{c}_1, n); x, y) q^{n - \frac{\tilde{c}_1^2}{4}} = (q^{\frac{1}{12}} \cdot \widetilde{Z}_a) \cdot \sum_n e(\mathfrak{M}^U_H(c_1, n); x, y) q^{n - \frac{c_1^2}{4}}$$

where $\widetilde{Z}_a = \widetilde{Z}_a(x, y, q)$ is a universal function of x, y, q, a with

$$\widetilde{Z}_a(1, 1, q) = \frac{\sum_{n \in \mathbb{Z}} q^{(n + \frac{a}{2})^2}}{q^{\frac{1}{12}}(1 - q)}.$$

Our main results are the following closed formulae for $\widetilde{\widetilde{Z}}_a(x, y, q)$ and $\widetilde{Z}_a(x, y, q)$.

Theorem 1.2. *The universal function $\widetilde{\widetilde{Z}}_a(x, y, q)$ is equal to*

$$\frac{\sum_{n \in \mathbb{Z}} (xy)^{\frac{(2n+a)^2 - (2n+a)}{2}} q^{\frac{(2n+a)^2}{4}}}{[q^{\frac{1}{24}} \prod_{n \ge 1}(1 - (xy)^{2n} q^n)]^2}.$$

Theorem 1.3. *The universal function* $\widetilde{Z}_a(x,y,q)$ *is equal to*

$$\frac{1}{q^{\frac{1}{12}}(1-xyq)}\Bigg[\sum_{s\geq 0}(xy)^{\frac{(2s+a)^2+(2s+a)}{2}}q^{\frac{(2s+a)^2}{4}}\prod_{j=1}^{2s+a}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}$$
$$+\sum_{s\geq(1-a)}(xy)^{\frac{(2s+a)^2+(2s+a)-2}{2}}q^{\frac{(2s+a)^2}{4}}\prod_{j=1}^{2s+a-1}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}\Bigg]$$

where we make the convention that $\prod_{j=1}^{0}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}=1$.

The paper is organized as follows. In section two, we verify Theorem 1.2 by taking $X=\mathbb{F}_1$ (the one-point blowup of $\mathbb{P}^2$). In section three, we prove Theorem 1.3 by using a not-closed formula of $\widetilde{Z}_a(x,y,q)$ obtained in [L-Q].

2. The universal function $\widetilde{\widetilde{Z}}_a(x,y,q)$

In this section, we derive a closed formula for the universal function $\widetilde{\widetilde{Z}}_a(x,y,q)$. Our strategy is to compute the virtual Hodge polynomials of the Gieseker moduli spaces of semistable rank-2 sheaves over $\mathbb{F}_1$ and its blownup. These Gieseker moduli spaces are actually smooth and have been studied extensively (see [E-G, F-Q] for example). Adopting a formula of Göttsche [Got], we calculate the (virtual) Hodge polynomials of these Gieseker moduli spaces. Then using the definition of $\widetilde{\widetilde{Z}}_a(x,y,q)$, we can determine a closed formula for $\widetilde{\widetilde{Z}}_a(x,y,q)$.

First of all, we recall virtual Hodge polynomials for complex algebraic schemes (not necessarily smooth, projective, or irreducible). Virtual Hodge polynomials were introduced by Danilov and Khovanskii [D-K]. They can be viewed as a tool for computing the Hodge numbers of smooth projective varieties by reducing to computing those of simpler varieties. For a complex algebraic scheme Y, Deligne [Del] proved that the cohomology $H_c^k(Y,\mathbb{Q})$ with compact support carries a natural mixed Hodge structure which coincides with the classical one if Y is projective and smooth. For each pair of integers (s,t), define the virtual Hodge number

$$e^{s,t}(Y)=\sum_k(-1)^k h^{s,t}(H_c^k(Y,\mathbb{Q})).$$

Then the virtual Hodge polynomials of Y is defined by

$$e(Y;x,y)=\sum_{s,t}e^{s,t}(Y)x^s y^t.$$

Virtual Hodge polynomials satisfy the following properties (see [D-K, Ful, Che]):

(2.1) When Y is projective, $e(Y;1,1)$ is the Euler characteristic $\chi(Y_{\text{red}})$ of Y_{red}. When Y is projective and smooth, $e(Y;x,y)$ is the usual Hodge polynomial.

(2.2) If Z is a Zariski-closed subscheme of Y, then

$$e(Y;x,y) = e(Z;x,y) + e(Y-Z;x,y).$$

So if $Y = \coprod_{i=1}^n Y_i$ is a disjoint union of finitely many locally closed subsets (i.e. each Y_i is the intersection of an open subset and a closed subset), then

$$e(Y;x,y) = \sum_{i=1}^n e(Y_i;x,y).$$

(2.3) If $f: Y \to Z$ is a Zariski-locally trivial bundle with fiber F, then

$$e(Y;x,y) = e(Z;x,y) \cdot e(F;x,y).$$

(2.4) If $f: Y \to Z$ is a bijective morphism, then $e(Y;x,y) = e(Z;x,y)$. In particular, we have $e(Y;x,y) = e(Y_{\text{red}};x,y)$.

Next, we recall a result of Göttsche. Let X be an algebraic surface with effective anti-canonical divisor $-K_X$, and let $q(X)$ be its irregularity. Fix a divisor c_1 on X and an integer n. In [Got], Göttsche studied the change of the virtual Hodge polynomial $e(\mathfrak{M}_H^G(c_1,n);x,y)$ as the ample divisor H crosses walls of type (c_1,n). In addition, a detailed study of the change of the Gieseker moduli space $\mathfrak{M}_H^G(c_1,n)$ as H crosses walls of type (c_1,n) can be found in [E-G, F-Q]. The next lemma follows immediately from the Theorem 3.4 (1) in [Got].

Lemma 2.5. *Assume that X is an algebraic surface with effective $-K_X$. Let H and L be ample divisors not lying on any wall of type (c_1,n). Then*

$$e(\mathfrak{M}_H^G(c_1,n);x,y) = e(\mathfrak{M}_L^G(c_1,n);x,y) + ((1-x)(1-y))^{q(X)} \cdot$$

$$\cdot \sum_\zeta (xy)^{\ell_\zeta - \frac{\zeta^2+\zeta K_X}{2} - \chi(\mathcal{O}_X)} \frac{1-(xy)^{\zeta K_X}}{1-(xy)} \cdot \sum_{s+t=\ell_\zeta} e(\text{Hilb}^s(X);x,y)e(\text{Hilb}^t(X);x,y)$$

where $\ell_\zeta = (4n - c_1^2 + \zeta^2)/4$, and ζ rus over all the classes in $\text{Num}(X)$ *which define walls of type (c_1,n) and satisfy $\zeta H < 0 < \zeta L$.* □

Now let X be a rational ruled surface with effective $-K_X$. Then $q(X) = 0$. Let f be a general fiber of the ruling. Fix a divisor c_1 and an ample divisor H such that both $(f \cdot c_1)$ and $(H \cdot c_1)$ are odd. Fix an integer n. Since $(H \cdot c_1)$ is odd, H does not lie on any wall of type (c_1,n). Since $(f \cdot c_1)$ is odd, it is well-known [H-S, Qi2] that there exists an open chamber $\mathcal{C}_n$ of type (c_1,n) such that $\mathfrak{M}_{L_n}^G(c_1,n) = \emptyset$ for $L_n \in \mathcal{C}_n$ and that the divisor class f is contained in the closure of $\mathcal{C}_n$. Note that since the divisor f is nef and contained in the closure of $\mathcal{C}_n$, the condition $\zeta H < 0 < \zeta L_n$ is equivalent to $\zeta H < 0 < \zeta f$. Let

$$\Lambda_H = \{\zeta \in \text{Pic}(X) | \quad \zeta H < 0 < \zeta f \text{ and } \zeta \equiv c_1 \pmod 2\}. \tag{2.6}$$

Then ζ defines a nonempty wall of type (c_1, n) with $\zeta H < 0 < \zeta L_n$ if and only if $\zeta \in \Lambda_H$ and $\zeta^2 \geq -(4n - c_1^2)$. Applying Lemma 2.5 to H and L_n, we obtain

$$e(\mathfrak{M}_H^G(c_1, n); x, y) = \sum_{\zeta \in \Lambda_H \text{ and } \zeta^2 \geq -(4n-c_1^2)} (xy)^{\ell_\zeta - \frac{\zeta^2 + \zeta K_X}{2} - \chi(\mathcal{O}_X)} \frac{1-(xy)^{\zeta K_X}}{1-(xy)} \cdot$$

$$\cdot \sum_{s+t=\ell_\zeta} e(\mathrm{Hilb}^s(X); x, y) e(\mathrm{Hilb}^t(X); x, y). \tag{2.7}$$

Lemma 2.8. *Let X be a rational ruled surface with effective $-K_X$. Let c_1 be a divisor on X such that both $(f \cdot c_1)$ and $(H \cdot c_1)$ are odd. Then*

$$\sum_n e(\mathfrak{M}_H^G(c_1, n); x, y) q^{n - \frac{c_1^2}{4}} = \frac{[\sum_n e(\mathrm{Hilb}^n(X); x, y)(xyq)^n]^2}{(xy)^{\chi(\mathcal{O}_X)}[1-(xy)]} \cdot$$

$$\cdot \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2 + \zeta K_X}{2}} [1 - (xy)^{\zeta K_X}] q^{-\frac{\zeta^2}{4}}. \tag{2.9}$$

Proof. By definition, $\ell_\zeta = (4n - c_1^2 + \zeta^2)/4 \geq 0$. So $n = \ell_\zeta + (c_1^2 - \zeta^2)/4$. By (2.7),

$$\begin{aligned}
&\sum_n e(\mathfrak{M}_H^G(c_1, n); x, y) q^{n - \frac{c_1^2}{4}} \\
&\quad = \sum_n \sum_{\zeta \in \Lambda_H \text{ and } \zeta^2 \geq -(4n-c_1^2)} (xy)^{\ell_\zeta - \frac{\zeta^2 + \zeta K_X}{2} - \chi(\mathcal{O}_X)} \frac{1-(xy)^{\zeta K_X}}{1-(xy)} \cdot \\
&\qquad \cdot \sum_{s+t=\ell_\zeta} e(\mathrm{Hilb}^s(X); x, y) e(\mathrm{Hilb}^t(X); x, y) q^{n - \frac{c_1^2}{4}} \\
&\quad = \sum_{\zeta \in \Lambda_H} \sum_{\ell \geq 0} (xy)^{\ell - \frac{\zeta^2 + \zeta K_X}{2} - \chi(\mathcal{O}_X)} \frac{1-(xy)^{\zeta K_X}}{1-(xy)} \cdot \\
&\qquad \cdot \sum_{s+t=\ell} e(\mathrm{Hilb}^s(X); x, y) e(\mathrm{Hilb}^t(X); x, y) q^{\ell - \frac{\zeta^2}{4}} \\
&\quad = \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2 + \zeta K_X}{2} - \chi(\mathcal{O}_X)} \frac{1-(xy)^{\zeta K_X}}{1-(xy)} q^{-\frac{\zeta^2}{4}} \cdot \\
&\qquad \cdot \sum_{\ell \geq 0} \sum_{s+t=\ell} e(\mathrm{Hilb}^s(X); x, y) e(\mathrm{Hilb}^t(X); x, y)(xyq)^\ell.
\end{aligned}$$

Here going from the first equality to the second equality, we have changed n to $\ell + (c_1^2 - \zeta^2)/4$ with $\ell \geq 0$. Notice that

$$\sum_{\ell \geq 0} \sum_{s+t=\ell} e(\mathrm{Hilb}^s(X); x, y) e(\mathrm{Hilb}^t(X); x, y)(xyq)^\ell$$

is equal to $[\sum_n e(\mathrm{Hilb}^n(X); x, y)(xyq)^n]^2$. Therefore, we obtain

$$\begin{aligned}
&\sum_n e(\mathfrak{M}_H^G(c_1, n); x, y) q^{n-\frac{c_1^2}{4}} \\
&= \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2+\zeta K_X}{2} - \chi(\mathcal{O}_X)} \frac{1-(xy)^{\zeta K_X}}{1-(xy)} q^{-\frac{\zeta^2}{4}} \cdot \left[\sum_n e(\mathrm{Hilb}^n(X); x, y)(xyq)^n\right]^2 \\
&= \frac{[\sum_n e(\mathrm{Hilb}^n(X); x, y)(xyq)^n]^2}{(xy)^{\chi(\mathcal{O}_X)}[1-(xy)]} \cdot \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2+\zeta K_X}{2}} [1-(xy)^{\zeta K_X}] q^{-\frac{\zeta^2}{4}}.
\end{aligned}$$

□

Next we study the virtual Hodge polynomials of the Gieseker moduli spaces over blownup surfaces. As before, let X be a rational ruled surface with effective $-K_X$. Let f be a general fiber of the ruling. Fix a divisor c_1 and an ample divisor H on X such that both $(f \cdot c_1)$ and $(H \cdot c_1)$ are odd. Let $\phi : \widetilde{X} \to X$ be the blowing-up of X at a point $x_0 \in X$, and E be the exceptional divisor. We assume that $-K_{\widetilde{X}}$ is effective. Let $\tilde{c}_1 = \phi^* c_1 - aE$ with $a = 0$ or 1. It is well-known [F-M, Bru, Qi1] that for $r \gg 0$, all the divisors $H_r = r \cdot \phi^* H - E$ on $\widetilde{X}$ are ample and lie in the same open chamber of type $(\tilde{c}_1, n)$. Thus all the moduli spaces $\mathfrak{M}_{H_r}^G(\tilde{c}_1, n)$ (resp. $\mathfrak{M}_{H_r}(\tilde{c}_1, n)$) with $r \gg 0$ are identical, and shall be denoted by $\mathfrak{M}_{H_\infty}^G(\tilde{c}_1, n)$ (resp. $\mathfrak{M}_{H_\infty}(\tilde{c}_1, n)$). Since $(H_r \cdot \tilde{c}_1) = r(H \cdot c_1) - a$ and $(H \cdot c_1)$ is odd, we can always choose $r \gg 0$ such that $(H_r \cdot \tilde{c}_1)$ is also odd.

Lemma 2.10. *Let $\phi : \widetilde{X} \to X$ be the blowing-up of a rational ruled surface X at one point such that $-K_X$ and $-K_{\widetilde{X}}$ are effective. Let c_1 be a divisor on X such that both $(f \cdot c_1)$ and $(H \cdot c_1)$ are odd, and $\tilde{c}_1 = \phi^* c_1 - aE$ with $a = 0$ or 1. Then*

$$\sum_n e(\mathfrak{M}_{H_\infty}^G(\tilde{c}_1, n); x, y) q^{n-\frac{\tilde{c}_1^2}{4}} = \frac{[\sum_n e(\mathrm{Hilb}^n(\widetilde{X}); x, y)(xyq)^n]^2}{(xy)^{\chi(\mathcal{O}_{\widetilde{X}})}[1-(xy)]} \cdot$$

$$(2.11) \qquad \cdot \sum_{t \in \mathbb{Z}} (xy)^{\frac{(2t+a)^2-(2t+a)}{2}} q^{\frac{(2t+a)^2}{4}} \cdot \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2+\zeta K_X}{2}} [1-(xy)^{\zeta K_X}] q^{-\frac{\zeta^2}{4}}.$$

Proof. Note that the ruling of X induces a ruling of $\widetilde{X}$ and that $\phi^* f$ is the divisor class of a general fiber for the ruling of $\widetilde{X}$. Fix an integer n, and choose $r \gg 0$ such that $(H_r \cdot \tilde{c}_1)$ is odd. Applying (2.7) to $\widetilde{X}$ and H_r, we obtain

$$e(\mathfrak{M}_{H_r}^G(\tilde{c}_1, n); x, y) = \sum_{\tilde{\zeta} \in \Lambda_{H_r} \text{ and } \tilde{\zeta}^2 \geq -(4n-\tilde{c}_1^2)} (xy)^{\ell_{\tilde{\zeta}} - \frac{\tilde{\zeta}^2+\tilde{\zeta} K_{\widetilde{X}}}{2} - \chi(\mathcal{O}_{\widetilde{X}})} \frac{1-(xy)^{\tilde{\zeta} K_{\widetilde{X}}}}{1-(xy)}.$$

$$\cdot \sum_{s+t=\ell_{\tilde{\zeta}}} e(\mathrm{Hilb}^s(\widetilde{X}); x, y) e(\mathrm{Hilb}^t(\widetilde{X}); x, y) \tag{2.12}$$

where by (2.6), $\Lambda_{H_r} = \{\tilde{\zeta} \in \mathrm{Pic}(\widetilde{X}) | \quad \tilde{\zeta} H_r < 0 < \tilde{\zeta} \cdot \phi^* f \text{ and } \tilde{\zeta} \equiv \tilde{c}_1 \pmod 2\}$. Since $(H \cdot c_1)$ and $(H_r \cdot \tilde{c}_1)$ are odd, $\phi^* H$ and H_r are not separated by any wall of type $(\tilde{c}_1, n)$. Thus if $\tilde{\zeta}$ defines a nonempty wall of type $(\tilde{c}_1, n)$, then $\tilde{\zeta} H_r < 0 < \tilde{\zeta} \cdot \phi^* f$ if and only if $\tilde{\zeta} \cdot \phi^* H < 0 < \tilde{\zeta} \cdot \phi^* f$. In view of this observation, we put

$$\Lambda_{H_\infty} = \{\tilde{\zeta} \in \mathrm{Pic}(\widetilde{X}) | \quad \tilde{\zeta} \cdot \phi^* H < 0 < \tilde{\zeta} \cdot \phi^* f \text{ and } \tilde{\zeta} \equiv \tilde{c}_1 \pmod 2\}.$$

Then by (2.12) and the convention for $\mathfrak{M}^G_{H_\infty}(\tilde{c}_1, n)$, we have

$$\begin{aligned} e(\mathfrak{M}^G_{H_\infty}(\tilde{c}_1, n); x, y) &= e(\mathfrak{M}^G_{H_r}(\tilde{c}_1, n); x, y) \\ &= \sum_{\tilde{\zeta} \in \Lambda_{H_\infty} \text{ and } \tilde{\zeta}^2 \ge -(4n - \tilde{c}_1^2)} (xy)^{\ell_{\tilde{\zeta}} - \frac{\tilde{\zeta}^2 + \tilde{\zeta} K_{\widetilde{X}}}{2} - \chi(\mathcal{O}_{\widetilde{X}})} \frac{1 - (xy)^{\tilde{\zeta} K_{\widetilde{X}}}}{1 - (xy)} \\ &\quad \cdot \sum_{s+t=\ell_{\tilde{\zeta}}} e(\mathrm{Hilb}^s(\widetilde{X}); x, y) e(\mathrm{Hilb}^t(\widetilde{X}); x, y). \end{aligned}$$

As in the proof of Lemma 2.8, we conclude that

$$\sum_n e(\mathfrak{M}^G_{H_\infty}(\tilde{c}_1, n); x, y) q^{n - \frac{\tilde{c}_1^2}{4}} = \frac{[\sum_n e(\mathrm{Hilb}^n(\widetilde{X}); x, y)(xyq)^n]^2}{(xy)^{\chi(\mathcal{O}_{\widetilde{X}})}[1 - (xy)]} \cdot$$

$$\cdot \sum_{\tilde{\zeta} \in \Lambda_{H_\infty}} (xy)^{-\frac{\tilde{\zeta}^2 + \tilde{\zeta} K_{\widetilde{X}}}{2}} [1 - (xy)^{\tilde{\zeta} K_{\widetilde{X}}}] q^{-\frac{\tilde{\zeta}^2}{4}}. \tag{2.13}$$

Put $\tilde{\zeta} = \phi^* \zeta + sE$. Then $\tilde{\zeta} \cdot \phi^* H < 0 < \tilde{\zeta} \cdot \phi^* f$ if and only if $\zeta H < 0 < \zeta f$. Moreover, $\tilde{\zeta} \equiv \tilde{c}_1 \pmod 2$ if and only if $\zeta \equiv c_1 \pmod 2$ and $s \equiv a \pmod 2$. So $\tilde{\zeta} = \phi^* \zeta + sE \in \Lambda_{H_\infty}$ if and only if $\zeta \in \Lambda_H$ and $s = (2t - a)$ for some $t \in \mathbb{Z}$. Thus,

$$\begin{aligned} &\sum_{\tilde{\zeta} \in \Lambda_{H_\infty}} (xy)^{-\frac{\tilde{\zeta}^2 + \tilde{\zeta} K_{\widetilde{X}}}{2}} [1 - (xy)^{\tilde{\zeta} K_{\widetilde{X}}}] q^{-\frac{\tilde{\zeta}^2}{4}} \\ &\quad = \sum_{\zeta \in \Lambda_H} \sum_{t \in \mathbb{Z}} (xy)^{-\frac{\zeta^2 - (2t-a)^2 + \zeta K_X - (2t-a)}{2}} [1 - (xy)^{\zeta K_X - (2t-a)}] q^{-\frac{\zeta^2 - (2t-a)^2}{4}} \\ &= \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2 + \zeta K_X}{2}} q^{-\frac{\zeta^2}{4}} \cdot \sum_{t \in \mathbb{Z}} \left[(xy)^{\frac{(2t-a)^2 + (2t-a)}{2}} - (xy)^{\zeta K_X + \frac{(2t-a)^2 - (2t-a)}{2}} \right] q^{\frac{(2t-a)^2}{4}} \\ &= \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2 + \zeta K_X}{2}} q^{-\frac{\zeta^2}{4}} \cdot \sum_{t \in \mathbb{Z}} \left[(xy)^{\frac{(2t+a)^2 - (2t+a)}{2}} - (xy)^{\zeta K_X + \frac{(2t+a)^2 - (2t+a)}{2}} \right] q^{\frac{(2t+a)^2}{4}} \\ &= \sum_{\zeta \in \Lambda_H} (xy)^{-\frac{\zeta^2 + \zeta K_X}{2}} [1 - (xy)^{\zeta K_X}] q^{-\frac{\zeta^2}{4}} \cdot \sum_{t \in \mathbb{Z}} (xy)^{\frac{(2t+a)^2 - (2t+a)}{2}} q^{\frac{(2t+a)^2}{4}}. \end{aligned} \tag{2.14}$$

Here going from the second equality to the third equality, we have changed t to $-t$ in the first term in the brackets and t to $t+a$ in the second term in the brackets. Now the formula (2.11) follows from (2.13) and (2.14). $\square$

Theorem 2.15. *The universal function* $\tilde{\tilde{Z}}_a(x,y,q)$ *is equal to*

$$\frac{\sum_{n\in\mathbb{Z}}(xy)^{\frac{(2n+a)^2-(2n+a)}{2}}q^{\frac{(2n+a)^2}{4}}}{[q^{\frac{1}{24}}\prod_{n\geq 1}(1-(xy)^{2n}q^n)]^2}.$$

Proof. First of all, we notice from [G-S] that for any algebraic surface X,

$$\sum_n e(\text{Hilb}^n(X);x,y)q^n = \prod_{n\geq 1}\prod_{s,t=0}^{2}(1-x^{s+n-1}y^{t+n-1}q^n)^{(-1)^{s+t+1}h^{s,t}(X)} \tag{2.16}$$

where $h^{s,t}(X)$ stands for the Hodge numbers of X. Next, let $X=\mathbb{F}_1$ be the blownup of $\mathbb{P}^2$ at one point, and let σ be the exceptional divisor in X. Then X is a ruled surface with effective $-K_X$. Let f be a fiber of the ruling. Let $\phi:\widetilde{X}\to X$ be the blowing-up of X at one point. Then $-K_{\widetilde{X}}$ is also effective. Let $H=\sigma+2f$ and $c_1=\sigma$. Then $(H\cdot c_1)=1=(f\cdot c_1)$. So $(H\cdot c_1)$ and $(f\cdot c_1)$ are odd. Therefore the conditions in Lemma 2.8 and Lemma 2.10 are satisfied. Note that $\chi(\mathcal{O}_{\widetilde{X}})=\chi(\mathcal{O}_X)$, $h^{s,t}(\widetilde{X})=h^{s,t}(X)$ when $(s,t)\neq(1,1)$, and $h^{1,1}(\widetilde{X})=1+h^{1,1}(X)$. By (2.16),

$$\frac{\sum_n e(\text{Hilb}^n(\widetilde{X});x,y)(xyq)^n}{\sum_n e(\text{Hilb}^n(X);x,y)(xyq)^n} = \frac{1}{\prod_{n\geq 1}(1-(xy)^{2n}q^n)}. \tag{2.17}$$

Combining (2.9), (2.11), (2.17) with (1.1), we see that

$$\begin{aligned}
&\tilde{\tilde{Z}}_a(x,y,q)\\
&=\frac{1}{q^{\frac{1}{12}}}\cdot\frac{\sum_n e(\mathfrak{M}^G_{H_\infty}(\tilde{c}_1,n);x,y)q^{n-\frac{\tilde{c}_1^2}{4}}}{\sum_n e(\mathfrak{M}^G_H(c_1,n);x,y)q^{n-\frac{c_1^2}{4}}}\\
&=\frac{1}{q^{\frac{1}{12}}}\cdot\frac{[\sum_n e(\text{Hilb}^n(\widetilde{X});x,y)(xyq)^n]^2}{[\sum_n e(\text{Hilb}^n(X);x,y)(xyq)^n]^2}\cdot\sum_{t\in\mathbb{Z}}(xy)^{\frac{(2t+a)^2-(2t+a)}{2}}q^{\frac{(2t+a)^2}{4}}\\
&=\frac{\sum_{n\in\mathbb{Z}}(xy)^{\frac{(2n+a)^2-(2n+a)}{2}}q^{\frac{(2n+a)^2}{4}}}{[q^{\frac{1}{24}}\prod_{n\geq 1}(1-(xy)^{2n}q^n)]^2}.
\end{aligned}$$

$\square$

3. The universal function $\widetilde{Z}_a(x, y, q)$

In this section, we prove a closed formula for the universal function $\widetilde{Z}_a(x, y, q)$. Our first goal is to compute the virtual Hodge polynomial of the space $U(m_1, m_2)$ which parameterizes all surjective maps $\mathcal{O}_{\mathbb{P}^1}(-m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_2) \to \mathcal{O}_{\mathbb{P}^1} \to 0$. Then using the results in [L-Q], we obtain a closed formula for $\widetilde{Z}_a(x, y, q)$. We end this section with a remark about this closed formula.

First of all, for two integers $m_1, m_2 \geq 0$, let $U(m_1, m_2)$ be the subset of

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2))) \cong \mathbb{P}^{m_1+m_2+1}$$

parameterizing all pairs (f_1, f_2) of homogeneous polynomials such that $\deg(f_1) = m_1, \deg(f_2) = m_2$, and f_1 and f_2 are coprime. Then $U(m_1, m_2)$ parameterizes all surjective maps $\mathcal{O}_{\mathbb{P}^1}(-m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_2) \to \mathcal{O}_{\mathbb{P}^1} \to 0$. The following result gives the virtual Hodge polynomial of $U(m_1, m_2)$.

Lemma 3.1. *Let m_1 and m_2 be two integers with $0 \leq m_1 \leq m_2$. Then,*
(3.2)

$$e(U(m_1, m_2); x, y) = \begin{cases} (xy) + 1, & \text{if } m_1 = m_2 = 0 \\ (xy)^{m_2+1}, & \text{if } m_1 = 0 \text{ and } m_2 > 0 \\ (xy)^{m_1+m_2-1}[(xy)^2 - 1], & \text{if } m_1 > 0. \end{cases}$$

Proof. We computed $e(U(m_1, m_2); 1, 1)$ in the Lemma 4.13 of [L-Q]. We shall adopt the same approach. First of all, we prove that (3.2) is true for $m_1 = 0$. Indeed, the subset $U(0, 0)$ of $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0))) \cong \mathbb{P}^1$ coincides with $\mathbb{P}^1$. Since $e(\mathbb{P}^d; x, y) = 1 + (xy) + \ldots + (xy)^d$, we have

$$e(U(0, 0); x, y) = e(\mathbb{P}^1; x, y) = (xy) + 1.$$

So (3.2) holds for $m_1 = m_2 = 0$. When $m_2 > 0$, the subset $U(0, m_2)$ of

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2)))$$

is $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2))) - \mathbb{P}(\{0\} \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2))) \cong \mathbb{P}^{m_2+1} - \mathbb{P}^{m_2}$. Thus,

$$e(U(0, m_2); x, y) = e(\mathbb{P}^{m_2+1}; x, y) - e(\mathbb{P}^{m_2}; x, y) = (xy)^{m_2+1}.$$

Hence (3.2) also holds for $m_1 = 0$ and $m_2 > 0$.

Next let $m_1 > 0$. The possible degree of the greatest common divisor of a pair

$$(f_1, f_2) \in \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2))) - \mathbb{P}(\{0\} \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)))$$

can be $0, \ldots, m_1$. For $d = 0, \ldots, m_1$, let Y_d be the subset of

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2))) - \mathbb{P}(\{0\} \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)))$$

parameterizing all pairs (f_1, f_2) such that $\gcd(f_1, f_2)$ has degree d. Then we obtain

$$\begin{aligned}&\mathbb{P}^{m_1+m_2+1} - \mathbb{P}^{m_2}\\ &\qquad \cong \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2))) - \mathbb{P}(\{0\} \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)))\\ &\qquad = \coprod_{d=0,\dots,m_1} Y_d.\end{aligned} \tag{3.3}$$

Let $1 \le d \le m_1$, and $(f_1, f_2) \in Y_d$ with $\gcd(f_1, f_2) = f$. Then we can write $f_1 = fg_1$ and $f_2 = fg_2$ with $f \in \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))) \cong \mathbb{P}^d$ and

$$(g_1, g_2) \in \begin{cases} U(m_1 - d, m_2 - d), & \text{if } 1 \le d < m_1 \\ U(0, m_2 - m_1), & \text{if } d = m_1 < m_2 \\ U(0, 0) - \{\text{a point}\}, & \text{if } d = m_1 = m_2. \end{cases}$$

Thus Y_d is the product of the space $\mathbb{P}^d$ with the space $U(m_1-d, m_2-d)$ when $1 \le d < m_1$ or $d = m_1 < m_2$, or with the space $U(0, 0) - \{\text{a point}\} \cong \mathbb{P}^1 - \{\text{a point}\}$ when $d = m_1 = m_2$. So for $1 \le d \le m_1$, we have

$$e(Y_d; x, y) = e(\mathbb{P}^d; x, y) \cdot \begin{cases} e(U(m_1 - d, m_2 - d); x, y), & \text{if } 1 \le d < m_1 \\ e(U(0, m_2 - m_1); x, y), & \text{if } d = m_1 < m_2 \\ (xy), & \text{if } d = m_1 = m_2. \end{cases}$$

Since $e(U(0, m_2 - m_1); x, y) = (xy)^{m_2-m_1+1}$ when $m_1 < m_2$, we obtain

$$e(Y_d; x, y) = \sum_{i=0}^{d} (xy)^i \cdot \begin{cases} e(U(m_1 - d, m_2 - d); x, y), & \text{if } 1 \le d < m_1 \\ (xy)^{m_2-m_1+1}, & \text{if } d = m_1. \end{cases} \tag{3.4}$$

Note that $Y_0 = U(m_1, m_2)$. From (3.3) and (3.4), we conclude that

$$\begin{aligned}\sum_{i=m_2+1}^{m_1+m_2+1} (xy)^i &= e(U(m_1, m_2); x, y) + \sum_{i=0}^{m_1} (xy)^i \cdot (xy)^{m_2-m_1+1}\\ &\quad + \sum_{1 \le d < m_1} \sum_{i=0}^{d} (xy)^i \cdot e(U(m_1 - d, m_2 - d); x, y).\end{aligned} \tag{3.5}$$

Now we see from (3.5) that $e(U(1, m_2); x, y) = (xy)^{m_2}[(xy)^2 - 1]$. So (3.2) holds for $m_1 = 1$. For $m_1 > 1$, we use (3.5) and induction on m_1:

$$\begin{aligned}e(U(m_1, m_2); x, y) &= \sum_{i=m_2+1}^{m_1+m_2+1} (xy)^i - \sum_{i=0}^{m_1} (xy)^i \cdot (xy)^{m_2-m_1+1}\\ &\quad - \sum_{1 \le d < m_1} \sum_{i=0}^{d} (xy)^i \cdot (xy)^{(m_1-d)+(m_2-d)-1}[(xy)^2 - 1].\\ &= (xy)^{m_1+m_2-1}[(xy)^2 - 1].\end{aligned}$$

□

In section four of [L-Q], we proved the following formula:

$$\widetilde{Z}_a(x,y,q) = \frac{q^{\frac{a}{4}} \cdot \sum_{n\geq 0} B_{a,n}(x,y)q^n}{q^{\frac{1}{12}}(1-xyq)} \tag{3.6}$$

where $B_{0,0}(x,y) = 1$, and $B_{a,n}(x,y)$ with $n \geq (1-a)$ is given by

$$\begin{aligned} B_{0,n}(x,y) = & \sum_{\substack{0\leq d_1, 0\leq d_{2j}\leq d_{2j-1}-1, 0\leq d_{2j+1}\leq d_{2j}(1\leq j\leq s-1), 0\leq d_{2s}\leq d_{2s-1}-1 \\ \sum_{i=1}^{2s} d_i = n}} \\ & \left(\prod_{i=1}^{s-1} e(U(d_{2i-1}-d_{2i}-1, d_{2i-1}+d_{2i}); x,y)\right. \\ & \left. e(U(d_{2i}-d_{2i+1}, d_{2i}+d_{2i+1}); x,y)\right) \\ & e(U(d_{2s-1}-d_{2s}-1, d_{2s-1}+d_{2s}); x,y) e(U(d_{2s}, d_{2s}); x,y) \end{aligned} \tag{3.7}$$

$$\begin{aligned} B_{1,n}(x,y) = & \sum_{\substack{0\leq d_1, 0\leq d_{2i}\leq d_{2i-1}, 0\leq d_{2i+1}\leq d_{2i}-1(1\leq i\leq s) \\ \sum_{i=1}^{2s+1} d_i = n}} \\ & \left(\prod_{i=1}^{s} e(U(d_{2i-1}-d_{2i}, d_{2i-1}+d_{2i}); x,y)\right. \\ & \left. e(U(d_{2i}-d_{2i+1}-1, d_{2i}+d_{2i+1}); x,y)\right) e(U(d_{2s+1}, d_{2s+1}); x,y). \end{aligned} \tag{3.8}$$

Now we can prove a closed formula for $\widetilde{Z}_a(x,y,q)$.

Theorem 3.9. *The universal function $\widetilde{Z}_a(x,y,q)$ is equal to*

$$\begin{aligned} & \frac{1}{q^{\frac{1}{12}}(1-xyq)} \left[\sum_{s\geq 0} (xy)^{\frac{(2s+a)^2+(2s+a)}{2}} q^{\frac{(2s+a)^2}{4}} \prod_{j=1}^{2s+a} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j} \right. \\ & \left. + \sum_{s\geq(1-a)} (xy)^{\frac{(2s+a)^2+(2s+a)-2}{2}} q^{\frac{(2s+a)^2}{4}} \prod_{j=1}^{2s+a-1} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j} \right] \end{aligned} \tag{3.10}$$

where we make the convention that $\prod_{j=1}^{0} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j} = 1$.

Proof. Since the proof for the case $a = 1$ is similar, we shall only prove the case

$a = 0$. By (3.6), it suffices to show that

$$\begin{aligned} \sum_{n\ge 0} B_{0,n}(x,y)q^n = &\sum_{s\ge 0}(xy)^{2s^2+s}q^{s^2}\prod_{j=1}^{2s}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j} \\ &+\sum_{s\ge 1}(xy)^{2s^2+s-1}q^{s^2}\prod_{j=1}^{2s-1}\frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}. \end{aligned} \tag{3.11}$$

First of all, let $\{d_1, d_2, \ldots, d_{2s}\}$ be an indexing sequence in the summation (3.7). So $0 \le d_1, 0 \le d_{2j} \le d_{2j-1} - 1, 0 \le d_{2j+1} \le d_{2j} (1 \le j \le s-1), 0 \le d_{2s} \le d_{2s-1} - 1$, and $\sum_{i=1}^{2s} d_i = n$. We make the following chang of indices:

$$\begin{cases} d'_1 &= d_1 - d_2 - 1, \\ d'_2 &= d_2 - d_3, \\ &\vdots \\ d'_{2s-3} &= d_{2s-3} - d_{2s-2} - 1, \\ d'_{2s-2} &= d_{2s-2} - d_{2s-1}, \\ d'_{2s-1} &= d_{2s-1} - d_{2s} - 1, \\ d'_{2s} &= d_{2s}. \end{cases} \tag{3.12}$$

Thus, $d'_i \ge 0$ for all the i with $1 \le i \le 2s$. Moreover, we have

$$\begin{cases} d_1 &= d'_1 + \ldots + d'_{2s} + s, \\ d_2 &= d'_2 + \ldots + d'_{2s} + (s-1), \\ d_3 &= d'_3 + \ldots + d'_{2s} + (s-1), \\ &\vdots \\ d_{2s-2} &= d'_{2s-2} + d'_{2s-1} + d'_{2s} + 1, \\ d_{2s-1} &= d'_{2s-1} + d'_{2s} + 1, \\ d_{2s} &= d'_{2s}. \end{cases} \tag{3.13}$$

So the condition $\sum_{i=1}^{2s} d_i = n$ becomes $\sum_{i=1}^{2s} i d'_i + s^2 = n$.

Next, let $t = (xy)$, and let $f : \{0, 1, 2, \ldots\} \to \{0, 1\}$ be defined by $f(d') = 0$ if $d' = 0$, and $f(d') = 1$ if $d' > 0$. Then for $0 \le d' \le d''$, (3.2) can be rewritten as:

$$e(U(d', d''); x, y) = \begin{cases} t+1, & \text{if } d'' = 0. \\ t^{d'+d''+1}(1-\frac{1}{t^2})^{f(d')}, & \text{if } d'' > 0. \end{cases} \tag{3.14}$$

Thus by (3.12), (3.13) and (3.14), the typical term in (3.7) is

$$\Big(\prod_{i=1}^{s-1} e(U(d_{2i-1}-d_{2i}-1, d_{2i-1}+d_{2i}); x, y) e(U(d_{2i}-d_{2i+1}, d_{2i}+d_{2i+1}); x, y)\Big)$$
$$e(U(d_{2s-1}-d_{2s}-1, d_{2s-1}+d_{2s}); x, y) e(U(d_{2s}, d_{2s}); x, y)$$
$$= \begin{cases} t^{2d'_1+\ldots+2d'_{2s}+2s}(1-\frac{1}{t^2})^{f(d'_1)}\ldots t^{2d'_{2s}+1}(1-\frac{1}{t^2})^{f(d'_{2s})}, \text{if } d'_{2s}>0, \\ t^{2d'_1+\ldots+2d'_{2s}+2s}(1-\frac{1}{t^2})^{f(d'_1)}\ldots t^{2d'_{2s-1}+2}(1-\frac{1}{t^2})^{f(d'_{2s-1})}(1+t), \text{if } d'_{2s}=0. \end{cases}$$

It follows from (3.7) that $\sum_{n\geq 0} B_{0,n}(x,y)q^n$ is equal to

$$1+\sum_{n\geq 1}\sum_{\substack{s^2+\sum_{i=1}^{2s} id'_i=n,\\ d'_i\geq 0(1\leq i\leq 2s),\\ d'_{2s}\neq 0}} t^{2d'_1+\ldots+2d'_{2s}+2s}(1-\frac{1}{t^2})^{f(d'_1)}\ldots t^{2d'_{2s}+1}(1-\frac{1}{t^2})^{f(d'_{2s})}q^n$$
$$+\sum_{n\geq 1}\sum_{\substack{s^2+\sum_{i=1}^{2s} id'_i=n,\\ d'_i\geq 0(1\leq i\leq 2s),\\ d'_{2s}=0}} t^{2d'_1+\ldots+2d'_{2s}+2s}(1-\frac{1}{t^2})^{f(d'_1)}\ldots t^{2d'_{2s-1}+2}(1-\frac{1}{t^2})^{f(d'_{2s-1})}(1+t)q^n$$
$$=1+\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s), d'_{2s}\neq 0} t^{2\sum_{i=1}^{2s} id'_i+\sum_{i=1}^{2s} i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s} f(d'_i)}q^{\sum_{i=1}^{2s} id'_i+s^2}$$
$$+\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s), d'_{2s}=0} t^{2\sum_{i=1}^{2s} id'_i+\sum_{i=2}^{2s} i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s} f(d'_i)}q^{\sum_{i=1}^{2s} id'_i+s^2}(1+t)$$
$$=1+\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s), d'_{2s}\neq 0} t^{2s^2+s}q^{s^2}(t^2q)^{\sum_{i=1}^{2s} id'_i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s} f(d'_i)}$$
$$+\Big(\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s), d'_{2s}=0} t^{2s^2+s}q^{s^2}(t^2q)^{\sum_{i=1}^{2s} id'_i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s} f(d'_i)}$$
$$+\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s-1)} t^{2s^2+s-1}q^{s^2}(t^2q)^{\sum_{i=1}^{2s-1} id'_i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s-1} f(d'_i)}\Big)$$
$$=1+\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s)} t^{2s^2+s}q^{s^2}(t^2q)^{\sum_{i=1}^{2s} id'_i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s} f(d'_i)}$$
$$+\sum_{s\geq 1}\sum_{d'_i\geq 0(1\leq i\leq 2s-1)} t^{2s^2+s-1}q^{s^2}(t^2q)^{\sum_{i=1}^{2s-1} id'_i}(1-\frac{1}{t^2})^{\sum_{i=1}^{2s-1} f(d'_i)}.$$

Let J be the set consisting of all the j with $d'_j > 0$. Then $\sum_{n\ge 0} B_{0,n}(x,y)q^n$ equals

$$1 + \sum_{s\ge 1} t^{2s^2+s} q^{s^2} \sum_{J\subset\{1,\dots,2s\}} (1-\frac{1}{t^2})^{|J|} \sum_{d'_j>0 (j\in J)} (t^2q)^{\sum_{j\in J} jd'_j}$$

$$+ \sum_{s\ge 1} t^{2s^2+s-1} q^{s^2} \sum_{J\subset\{1,\dots,2s-1\}} (1-\frac{1}{t^2})^{|J|} \sum_{d'_j>0 (j\in J)} (t^2q)^{\sum_{j\in J} jd'_j}$$

$$= 1 + \sum_{s\ge 1} t^{2s^2+s} q^{s^2} \sum_{J\subset\{1,\dots,2s\}} (1-\frac{1}{t^2})^{|J|} \prod_{j\in J} (\frac{1}{1-(t^2q)^j} - 1)$$

$$+ \sum_{s\ge 1} t^{2s^2+s-1} q^{s^2} \sum_{J\subset\{1,\dots,2s-1\}} (1-\frac{1}{t^2})^{|J|} \prod_{j\in J} (\frac{1}{1-(t^2q)^j} - 1)$$

$$= 1 + \sum_{s\ge 1} t^{2s^2+s} q^{s^2} \prod_{j=1}^{2s} (1 + (1-\frac{1}{t^2})\frac{(t^2q)^j}{1-(t^2q)^j})$$

$$+ \sum_{s\ge 1} t^{2s^2+s-1} q^{s^2} \prod_{j=1}^{2s-1} (1 + (1-\frac{1}{t^2})\frac{(t^2q)^j}{1-(t^2q)^j})$$

$$= \sum_{s\ge 0} (xy)^{2s^2+s} q^{s^2} \prod_{j=1}^{2s} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}$$

$$+ \sum_{s\ge 1} (xy)^{2s^2+s-1} q^{s^2} \prod_{j=1}^{2s-1} \frac{1-(xy)^{2j-2}q^j}{1-(xy)^{2j}q^j}.$$

□

Remark 3.15. In view of Yoshioka's results over finite fields (the Remark 4.5 in [Yos]), we think that the following is a better closed formula for $\widetilde{Z}_a(x,y,q)$:

$$\frac{\sum_{n\in\mathbb{Z}} (xy)^{\frac{(2n+a)^2-(2n+a)}{2}} q^{\frac{(2n+a)^2}{4}}}{q^{\frac{1}{12}}(1-xyq)} \cdot \prod_{d\ge 1} \frac{1-(xy)^{2d-1}q^d}{1-(xy)^{2d}q^d}. \tag{3.16}$$

For instance, we can verify that the lower degree terms in (3.10) and (3.16) coincide by using MAPLE. However, we are unable to show that (3.10) and (3.16) are equal.

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