Mathematical Research Letters **5**, 227–234 (1998)

SIGNATURES, MONOPOLES AND MAPPING CLASS GROUPS

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Abstract. Using the Seiberg–Witten monopole invariants of 4–manifolds, we prove bounds on the signatures of surface bundles over surfaces. From these bounds we derive consequences concerning characteristic classes of flat bundles and the minimal genus of surfaces representing 2–homology classes of mapping class groups.

1. Introduction

It is a classical result of Chern, Hirzebruch and Serre that the signature is multiplicative in fibre bundles in which the fundamental group of the base acts trivially on the cohomology of the fibre. Kodaira and, independently, Atiyah gave examples of surface bundles over surfaces with non–zero signature, showing that some assumption on the monodromy is necessary for multiplicativity of the signature. Given the examples of Atiyah and Kodaira, it is clear that there are surface bundles over surfaces with arbitrarily large signatures, because one can always pull back a given bundle to a finite unramified covering of the base, thereby multiplying the value of the signature by the covering degree. It is then interesting to prove upper bounds on the absolute values of the signatures of surface bundles over surfaces in terms of other invariants, particularly the Euler characteristic, which is what we do in this paper.

The bounds we prove have two interpretations in terms of surface topology. Firstly, they imply new bounds on the characteristic numbers of flat bundles over surfaces whose monodromy representations factor through the mapping class group. Comparing these bounds with the Milnor inequality, we obtain a new obstruction to lifting symplectic representations to the mapping class group. Secondly, and most interestingly, we obtain lower bounds on the Gromov–Thurston norm of 2–dimensional homology classes of mapping class groups. No non–trivial lower bounds were known previously.

Our arguments are based on the observation that, using the Thurston construction, surface bundles over surfaces can be given symplectic structures compatible with both choices of orientation. Results about the Seiberg–Witten invariants of symplectic 4–manifolds then imply bounds on the signature.

Received October 13, 1997.

The author is grateful to the Max–Planck–Institut für Mathematik in Bonn for hospitality and support. He would also like to thank N. A'Campo, A. Adem, F. Hirzebruch and M. Hoster for useful conversations.

In [9] we showed that if a 4–manifold carries complex structures compatible with both choices of orientation, then its signature must vanish. This is no longer true for symplectic manifolds, as shown by surface bundles over surfaces. However, if a closed 4–manifold X is symplectic for both choices of orientation, then the symplectic structures must both be minimal, as X cannot contain embedded spheres of non–zero selfintersection, cf. [9]. Taubes [17, 18] proved the existence of symplectically embedded surfaces representing the canonical class, which implies

$$3\big|\sigma(X)\big| \leq 2\chi(X), \tag{1}$$

unless X is ruled. This shows in particular that X satisfies the $\frac{11}{8}$–conjecture.

For a general 4–manifold which is symplectic for both choices of orientation it is difficult to say more, though we conjecture that such a manifold must be aspherical if it is not ruled. The only examples with non–zero signature that we are aware of are the surface bundles over surfaces studied in this paper. For these we will improve (1) in Theorem 2 below.

2. Signatures of surface bundles

Let X be a smooth closed orientable 4–manifold which is a smooth fibre bundle with base B a surface of genus $g \geq 2$ and fibre F a surface of genus $h \geq 2$. We shall always assume that all the manifolds involved are oriented coherently, and X will denote the total space with its orientation. Note that X is aspherical.

For every such surface bundle X we have:

$$\begin{aligned}\big|\sigma(X)\big| &\leq b_2(X) = \chi(X) - 2 + 2b_1(X) \\ &\leq \chi(X) - 2 + 2b_1(B) + 2b_1(F) = 4gh + 2,\end{aligned} \tag{2}$$

where $\sigma(X)$ denotes the signature of X, and $\chi(X)$ its Euler characteristic.

As $g > 0$, the base B has unramified coverings of arbitrarily high degree d. Pulling back the fibration to such coverings, we obtain surface bundles Y whose signatures are d–fold multiples of the signature of X. Letting d go to infinity, the inequality (2) applied to Y instead of X gives

$$\big|\sigma(X)\big| \leq 4(g-1)h. \tag{3}$$

The first improvement on this trivial inequality is:

Proposition 1. *Let X be an aspherical surface bundle. Then*

$$\big|\sigma(X)\big| \leq \chi(X). \tag{4}$$

Proof. Fix a finite unramified covering $F' \to F$, corresponding to a normal subgroup $\pi_1(F') \subset \pi_1(F)$. There is a finite index subgroup of the mapping class group of F leaving this subgroup invariant, cf. Lemma 4.1 in [14]. This shows that after replacing the fibration $X \to B$ by its pull–back to a finite unramified covering of B, we may assume that X has a finite fibrewise unramified covering of degree > 1. Passing to this covering and iterating the construction, we obtain a sequence of coverings which are surface bundles with growing h. As the fibre genus goes to infinity the inequality (3) applied to these coverings implies $|\sigma(X)| \leq 4(g-1)(h-1)$ as claimed. □

Here is our main result about the signatures of surface bundles.

Theorem 2. *Let X be an aspherical surface bundle over a surface. Then*

$$2\big|\sigma(X)\big| \leq \chi(X). \tag{5}$$

Proof. As $h \neq 1$, the fundamental class of the fibre cannot be zero in $H_2(X, \mathbb{R})$, and so there exists a closed 2–form ϵ on X whose integral over each fibre is positive. It is easy to arrange the restriction of ϵ to be non–degenerate on each fibre, cf. [19]. Let v be a volume form on B, and $\omega_t = \epsilon + t\pi^* v$. Then ω_t is closed for every $t \in \mathbb{R}$, and

$$\omega_t \wedge \omega_t = \epsilon \wedge \epsilon + 2t\epsilon \wedge \pi^* v.$$

Thus, for $t >> 0$ and $t << 0$ respectively, ω_t is a symplectic form on X compatible, respectively not compatible, with the orientation of X.

From Proposition 1 we have $|\sigma(X)| \leq \chi(X)$. Thus

$$b_2(X) - \big|\sigma(X)\big| = \chi(X) - 2 + 2b_1(X) - \big|\sigma(X)\big| \geq 2b_1(X) - 2 \geq 4g - 2 \geq 6,$$

which means that with either choice of orientation X has $b_2^+ \geq 3$.

As X is aspherical, any symplectic structure on it is minimal. Further, for the structure constructed above the canonical class K cannot be zero as $K \cdot F = 2h - 2 \neq 0$ by the adjunction formula applied to the symplectic submanifold $F \subset X$.

It follows from the work of Taubes on the Seiberg–Witten and Gromov invariants [17, 18, 8] that there is a smooth symplectically embedded surface $\Sigma \subset X$ representing K. This surface may not be connected, but in any case it has no spherical component. In the argument below we shall tacitly assume that Σ is connected. In the disconnected case the same argument works, by summing over the components.

The bundle projection $X \to B$ induces a smooth map $\Sigma \to B$ whose degree d is the algebraic intersection number $\Sigma \cdot F = K \cdot F = 2h - 2$. By Kneser's theorem[1], we must have $g(\Sigma) - 1 \geq d(g-1)$. The adjunction formula for Σ gives

$$g(\Sigma) - 1 = \frac{1}{2}(\Sigma^2 + \Sigma \cdot K) = K^2 = 2\chi(X) + 3\sigma(X).$$

Substituting into Kneser's inequality we find $\chi(X) \geq -2\sigma(X)$.

The same argument applied to the manifold X endowed with the opposite orientation gives $\chi(X) \geq 2\sigma(X)$. □

[1]This follows from the results of Milnor [13], see 4.1 below.

The inequality (5) is unlikely to be sharp. Here is an improvement under additional geometric hypotheses:

Theorem 3. *Let X be an aspherical surface bundle over a surface. If X admits a complex structure* (*not necessarily compatible with the orientation*)*, or an Einstein metric, then*

$$3\big|\sigma(X)\big| \leq \chi(X). \tag{6}$$

Proof. Suppose X admits a complex structure. After possibly reversing the orientation, we may assume that the complex structure is compatible with the orientation. As X is aspherical, it is a minimal surface. Its Chern numbers $c_2(X) = \chi(X) = 4(g-1)(h-1) > 0$ and $c_1^2(X) = 2\chi(X) + 3\sigma(X) \geq 2\chi(X) - 3|\sigma(X)| \geq \frac{1}{2}\chi(X) = 2(g-1)(h-1) > 0$ by Theorem 2. Thus X is of general type by the Kodaira classification of surfaces.

Now X is a surface of general type for which the underlying manifold endowed with the other, non–complex, orientation is symplectic and therefore has non–zero Seiberg–Witten invariants [16, 8]. Thus Theorem 1 of [9] gives

$$\sigma(X) \geq 0.$$

This, together with the Miyaoka–Yau inequality

$$3\sigma(X) \leq \chi(X),$$

implies (6).

Suppose that X admits an Einstein metric. As it is also symplectic, it has non–zero Seiberg–Witten invariants and by the result of [10] satisfies $3\sigma(X) \leq \chi(X)$. The same argument for the manifold with the other orientation gives $-3\sigma(X) \leq \chi(X)$. □

I believe that any surface bundle over a surface with g, $h \geq 2$ satisfies (6) strictly. The known examples [1, 7, 14] would allow for even stronger inequalities.

3. The signature as a 2–dimensional characteristic number

The component of the identity of the diffeomorphism group $Diff(F)$ is contractible, so that a surface bundle X over a surface B is determined by its monodromy representation

$$\rho\colon \pi_1(B) \longrightarrow \Gamma_h,$$

where $\Gamma_h = Diff(F)/Diff_0(F)$ is the mapping class group of the fibre F. Composing ρ with the obvious homomorphisms

$$\Gamma_h \xrightarrow{\phi} Sp(2h, \mathbb{Z}) \xrightarrow{i} Sp(2h, \mathbb{R}),$$

we obtain a flat vector bundle $E \to B$, whose fibres are the first homology groups of the fibres of $X \to B$. Applying the Atiyah–Singer index theorem to the family of fibrewise signature operators shows[2]

$$\left|\sigma(X)\right| = 4\left|\langle c_1(E), [B]\rangle\right|, \tag{7}$$

as in [1, 11, 14]. Meyer [11] showed that this formula holds for local coefficient systems $\pi_1(B) \to Sp(2h, \mathbb{R})$ which do not necessarily factor through Γ_h. Then X may not exist as a manifold, but the first homology of the local coefficient system still has a non–degenerate symmetric bilinear form whose signature plays the role of the signature of X.

The image of the monodromy representation ρ will be called the *monodromy group* of the bundle X. By abuse of language, we sometimes also call the image of $\phi \circ \rho$ the monodromy group.

In the special case of surface bundles over surfaces, the result of Chern, Hirzebruch and Serre [2] is that the signature vanishes if $\phi \circ \rho$ is trivial. If $\phi \circ \rho$ has finite image, the same conclusion holds because one can pull back the bundle to the finite covering of B whose fundamental group is the kernel of $\phi \circ \rho$, and use the multiplicativity of the signature in finite unramified coverings. More generally, Morita [15] has proved that a surface bundle with amenable monodromy group has zero signature. However, the signature is *not* usually determined by the monodromy group. The following result shows that the signature depends on the monodromy *representation*, not just its image, the monodromy *group*.

Proposition 4. *Every finitely generated subgroup of Γ_h (respectively of $Sp(2h, \mathbb{Z})$) is the monodromy group of a surface bundle with zero signature. Moreover, the bundle can be chosen to be flat in the sense that its monodromy representation lifts to $Diff(F)$.*

Proof. Given a generating set with k elements for the monodromy group, we choose preimages $\phi_1, \dots, \phi_k$ of the generators in $Diff(F)$. If the genus g of B is at least k, we can map $\pi_1(B)$ onto the free group on k generators in the obvious way, and then map the generators of this free group to the ϕ_i. This gives a monodromy representation for a flat surface bundle with fibre F. The Chern class of the associated local coefficient system over B vanishes because it is pulled back via the classifying space of a free group, which has trivial second cohomology. □

In view of (7), bounding the signatures of surface bundles X amounts to bounding the first Chern numbers of certain flat $Sp(2h, \mathbb{R})$–bundles over the base B. It is a well–known observation usually attributed to Lusztig that there are only finitely many values for the characteristic numbers of flat bundles because the representation variety is algebraic, and therefore has finitely many components. An explicit bound is called a Milnor inequality, because Milnor [13] proved the first such bound in the case $h = 1$. We shall deduce from Theorem 2

[2] The sign ambiguity in (7) can of course be removed.

that flat vector bundles whose monodromy representations factor through the mapping class group Γ_h satisfy much stronger bounds than arbitrary symplectic flat bundles.

4. Milnor inequalities and bounded cohomology

We consider a closed oriented surface B of genus g and a representation of $\pi_1(B)$ in $Sp(2h, \mathbb{R})$. In order to see how liftings of the representation to the mapping class group Γ_h influence the bound on the first Chern number, we deviate temporarily from our assumption $h \geq 2$ and recall the classical case $h = 1$ considered by Milnor [13].

4.1. The case of flat $Sp(2, \mathbb{R})$–bundles. For $h = 1$ we have $\Gamma_1 = Sp(2, \mathbb{Z}) = Sl(2, \mathbb{Z})$ and $Sp(2, \mathbb{R}) = Sl(2, \mathbb{R})$. The first Chern class for $Sp(2, \mathbb{R})$ is the Euler class for $Sl(2, \mathbb{R})$.

Milnor [13] showed that for every flat $Sl(2, \mathbb{R})$–bundle over a surface B of genus $g > 0$ the absolute value of the Euler number is bounded by $g - 1$, and that this bound is best possible. However, for bundles with monodromy group contained in $Sl(2, \mathbb{Z})$, the Euler number vanishes because a torus bundle has zero signature. Thus the Euler number is an obstruction to lifting representations from $Sl(2, \mathbb{R})$ to $Sl(2, \mathbb{Z})$.

4.2. Higher ranks. Consider now a representation $\rho\colon \pi_1(B) \to Sp(2h, \mathbb{R})$, for $g, h \geq 2$. By exhibiting an explicit cocycle representing the first Chern class, Turaev [20] showed

$$\big|\langle c_1(\rho), [B]\rangle\big| \leq (g-1)h. \tag{8}$$

Taking direct sums of Milnor's flat $Sl(2, \mathbb{R})$–bundles shows that this bound is best possible. In a different form, this was also proved by Domic–Toledo [3].

It is remarkable that using (7), the bound (8) translates precisely into the trivial bound $|\sigma(X)| \leq 4(g-1)h$ for a surface bundle with fibre of genus h over a base of genus g – and nothing better.

Theorem 2 has the following equivalent formulation, which shows that most symplectic representations do not factor through the mapping class group.

Theorem 5. *If a representation $\rho\colon \pi_1(B) \to Sp(2h, \mathbb{R})$ factors through the mapping class group Γ_h, then the first Chern number of the associated flat bundle satisfies*

$$\big|\langle c_1(\rho), [B]\rangle\big| \leq \frac{1}{2}(g-1)(h-1). \tag{9}$$

Another way to formulate this, is to say that although the universal first Chern class for $Sp(2h, \mathbb{R})$ has sup norm $||c_1||_\infty = \frac{1}{4}h$ in the sense of Gromov [4], its pullback to the mapping class group has a much smaller sup norm:

$$\big|\big|\phi^* i^* c_1\big|\big|_\infty \leq \frac{1}{8}(h-1). \tag{10}$$

Remark 1. The inequality (9) does not hold if one assumes only that ρ factors through $Sp(2h,\mathbb{Z})$. In [11], Meyer gave an example of a representation $\rho\colon \pi_1(B) \to Sp(4,\mathbb{Z})$ with $g = g(B) = 4$ and with first Chern number 2.

5. Minimal genus in the homology of mapping class groups

The vanishing of the signatures of torus bundles is equivalent to the vanishing of the first Chern class in $H^2(BSl(2,\mathbb{Z}),\mathbb{Q})$. In fact, this cohomology group is well–known to be trivial itself. Passing to higher genus bundles, $H^2(B\Gamma_2,\mathbb{Q})$ also vanishes by a result of Igusa, but $H^2(B\Gamma_h,\mathbb{Q})$ does not, for all $h \geq 3$. Meyer [12] showed that for $h \geq 3$ there are always surface bundles with fiber of genus h and with non–zero signature, so that the first Chern class must be non–zero in $H^2(B\Gamma_h,\mathbb{Q})$ by (7). More precisely, the first Chern class generates $H^2(B\Gamma_h,\mathbb{Z})$ by a result of Harer, who proved that $H_2(B\Gamma_h,\mathbb{Z}) = \mathbb{Z}$ for $h \geq 5$. For $h = 3$ or 4, the second Betti number of Γ_h is also 1, but it is not known whether $H_2(B\Gamma_h,\mathbb{Z})$ is torsion–free in these cases. See [5] and the subsequent correction.

Let $g_h(n)$ denote the minimal genus of a closed oriented surface B admitting a continuous map into $B\Gamma_h$ whose fundamental class represents n times a generator of $H_2(B\Gamma_h,\mathbb{Z})/torsion$. Theorem 2 is equivalent to the following lower bound for the minimal genus:

Theorem 6. *For $n \neq 0$ and $h \geq 3$, we have $g_h(n) \geq 1 + \frac{2|n|}{h-1}$.*

This gives a partial answer to a question raised by G. Mess in [6], Problem 2.18:

Corollary 7. *$lim_{n\to\infty} \frac{g_h(n)}{n} \geq \frac{2}{h-1}$*

This is the first non–trivial lower bound obtained on this limit, which obviously exists and is finite. The known examples of surface bundles [1, 7] give certain upper bounds for this limit. The smallest upper bound, which one gets for certain values of h, is $\frac{44}{5(h-1)}$. In any case, the fibrewise covering argument in the proof of Proposition 1 shows that

$$\lim_{h\to\infty}(\lim_{n\to\infty}\frac{g_h(n)}{n}) = 0,$$

so that the qualitative behaviour of the bound in Corollary 7 is what one expects.

Corollary 7 is a statement about the Gromov–Thurston norm of the generator x of $H_2(\Gamma_h,\mathbb{Z})/torsion$:

$$||x|| \geq \frac{8}{h-1}.$$

This is dual to the upper bound (10) on the Gromov sup norm of the dual generator $\phi^* i^* c_1$.

If one only considers maps $B \to B\Gamma_h$ which are homotopic to maps which are holomorphic for suitable complex structures on B, the minimal genus is likely to be larger than for arbitrary continuous maps, because most surface bundles do not admit complex structures. For fixed g and h there are infinitely many homotopy types of surface bundles with Euler characteristic $4(g-1)(h-1)$, corresponding to conjugacy classes of representations $\pi_1(B) \to \Gamma_h$. At most finitely many can carry complex structures, because those which do are minimal of general type by the proof of Theorem 3, and therefore fall into a bounded family realising at most finitely many diffeomorphism types.

For holomorphic maps from the base B into the moduli space of complex curves of genus h, Theorem 3 gives $g_h^{hol}(n) \geq 1 + \frac{3|n|}{h-1}$ for the minimal genus, which implies $\lim_{n\to\infty} \frac{g_h^{hol}(n)}{n} \geq \frac{3}{h-1}$. This is a non–trivial result about the enumerative algebraic geometry of the moduli space.

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