

SOME PROPERTIES OF FINITE-DIMENSIONAL SEMISIMPLE HOPF ALGEBRAS

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Kaplansky conjectured that if H is a finite-dimensional semisimple Hopf algebra over an algebraically closed field k of characteristic 0, then H is of Frobenius type (i.e. if V is an irreducible representation of H then $\dim V$ divides $\dim H$) [Ka]. It was proved that the conjecture is true for H of dimension p^n , p prime [MW], and that if H has a 2-dimensional representation then $\dim H$ is even [NR]. It was also proved that the dimension of an irreducible $D(H)$ -submodule of H divides the dimension of H [Z2].

In this paper we first prove in Theorem 1.4 that if V is an irreducible representation of $D(H)$, the Drinfeld double of any finite-dimensional semisimple Hopf algebra H over k , then $\dim V$ divides $\dim H$ (not just $\dim D(H) = (\dim H)^2$). In doing this we use the theory of modular tensor categories (in particular Verlinde formula). We then use Theorem 1.4 to prove in Theorem 1.5 that Kaplansky's conjecture is true for finite-dimensional semisimple quasitriangular Hopf algebras over k . As a result we prove easily in Theorem 1.7 that Kaplansky's conjecture [Ka] on prime dimensional Hopf algebras over k is true by passing to their Drinfeld doubles (compare with [Z1]).

Second, we use a theorem of Deligne [De] to prove in Theorem 2.1 that triangular semisimple Hopf algebras over k are equivalent to group algebras as quasi-Hopf algebras (see formulas (1.11) and (1.12) in [Dr2]).

1. Quasitriangular semisimple Hopf algebras are of Frobenius type

Throughout this paper k will denote an algebraically closed field of characteristic 0.

Let (H, R) be a finite-dimensional quasitriangular Hopf algebra over k , and write $R = \sum_i a_i \otimes b_i$. Let $u = \sum_i S(b_i)a_i$ be the Drinfeld element in H (where S is the antipode of H). Drinfeld showed in [Dr1, Proposition 2.1 1)] that u is invertible and

$$(1) \quad u x u^{-1} = S^2(x)$$

for any $x \in H$. He also showed that

$$(2) \quad \Delta(u) = (R^{21}R)^{-1}(u \otimes u).$$

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Suppose further that H is semisimple or equivalently that H^* is cosemisimple. Then by Theorem 3.3 in [LR2], H^* is semisimple or equivalently H is cosemisimple. Therefore if (H, R) is semisimple then by Theorem 2 in [LR1], $S^2 = 1$, and hence u is a central element in H . Furthermore,

$$(3) \quad u = S(u).$$

Indeed, we have $(S \otimes S)(R) = R$, so $S(u) = \sum S(a_i)S^2(b_i) = \sum a_i S(b_i)$. This shows that $\text{tr}(u) = \text{tr}(S(u))$ in every irreducible representation of H . But u and $S(u)$ are central, so they act as scalars in this representation, which proves (3).

Let H be a finite-dimensional semisimple Hopf algebra over k . Then the Drinfeld double of H , $D(H)$, is semisimple by Proposition 7 in [R1] and the fact that H^* is semisimple too as explained above (this follows from [R1, Theorem 4] and the fundamental [Mo, Theorem 2.2.1]). Furthermore, by Section 6 on page 335 in [Dr1], $D(H)$ is quasitriangular with universal R -matrix $R = \sum_i h_i \otimes h_i^*$, where $\{h_i\}$ and $\{h_i^*\}$ are dual bases of H and H^* respectively. It is moreover a ribbon Hopf algebra (see e.g. Definition XIV.6.1 in [Kas]) with the central Drinfeld element u , defined in (1), as the ribbon element v (note that u is a ribbon element by (2), (3) and the fact that it is central). In particular the special grouplike element defined by $g = uv^{-1}$, equals 1 in this case. Therefore, the category $\text{Rep}(D(H))$ of finite-dimensional representations of $D(H)$ is a semisimple ribbon (i.e. braided, rigid and balanced) category with quantum trace equal to the ordinary trace. Let $\text{Irr}(D(H)) = \{V_i | 0 \leq i \leq m\}$ be the set of all the isomorphism classes of irreducible representations of $D(H)$ with $V_0 = k$, and let $C(D(H)) \subseteq D(H)^*$ be the ring of characters. Clearly, $\{\chi_i = \text{tr}_{|V_i} | 0 \leq i \leq m\}$ forms a linear basis of $C(D(H))$. We also let $\chi_{i^*} = S(\chi_i)$ be the character of the irreducible representation $V_{i^*} = V_i^*$.

Recall that a *modular* category [Ki,T] is a semisimple ribbon category with finitely many (up to isomorphism) irreducible objects $\{V_i | 0 \leq i \leq m\}$ with V_0 as the unit representation, so that the matrix $s = (s_{ij})$, where

$$s_{ij} = (\text{tr}_{|V_i} \otimes \text{tr}_{|V_{j^*}})(R^{21}R)$$

is invertible. Note that s is symmetric and $s_{i0} = s_{0i} = \dim V_i$ for all i .

Lemma 1.1. *Let H be a finite-dimensional semisimple Hopf algebra over k . Then $\text{Rep}(D(H))$ is a modular category.*

Proof. We only have to show that the matrix $s = (s_{ij})$, where $s_{ij} = (\chi_i \otimes \chi_{j^*})(R^{21}R)$ is invertible. Indeed, by the second paragraph on page 226 in [R2], $D(H)$ is factorizable (i.e. the map $F : D(H)^* \rightarrow D(H)$ given by $F(p) = (1 \otimes p)(R^{21}R)$ is an isomorphism of vector spaces), and $F_{|C(D(H))} : C(D(H)) \rightarrow Z(D(H))$ is an isomorphism of algebras, where $Z(D(H))$ is the center of $D(H)$. Let $B = \{\chi_{j^*} | 0 \leq j \leq m\}$ and $C = \{e_j | 0 \leq j \leq m\}$ be bases of $C(D(H))$ and $Z(D(H))$ respectively, where C is the set of central primitive idempotents of $D(H)$. Then $s = DA$ where A is the invertible matrix which represents F with

respect to the bases B and C , and $D = \text{diag}(\dim V_i)$ is the invertible diagonal matrix with entries $\dim V_i$. Indeed, since $F(\chi_j^*) = \sum_i A_{ij} e_i$ and $\chi_i(e_j) = \delta_{ij} \dim V_i$ it follows that

$$\begin{aligned} (DA)_{ij} &= \sum_k D_{ik} A_{kj} = D_{ii} A_{ij} \\ &= (\dim V_i) \left(\frac{1}{\dim V_i} \chi_i(F(\chi_j^*)) \right) = (\chi_i \otimes \chi_j^*)(R^{21} R) = s_{ij}. \end{aligned}$$

Thus, s is invertible. \square

Lemma 1.2. *Let \mathcal{C} be a modular category over k with irreducible objects $\{V_i | 0 \leq i \leq m\}$ with $V_0 = 1$. Set $\mathcal{R} = \bigoplus_{i=0}^m V_i \otimes V_i^*$. Then $\frac{\dim \mathcal{R}}{(\dim V_j)^2}$ is an algebraic integer for all $0 \leq j \leq m$.*

Proof. It is known that $\sum_i s_{ji} s_{ij^*} = \dim \mathcal{R}$ for all j (see e.g. [Ki]), hence $\sum \frac{s_{ji} s_{ij^*}}{s_{j0} s_{0j^*}} = \frac{\dim \mathcal{R}}{(\dim V_j)^2}$. We show that $\frac{s_{ij}}{s_{i0}}$ is an algebraic integer for all $0 \leq i, j \leq m$. Define a map ϕ from $\text{Rep}(\mathcal{C})$ to the algebra of functions $k\{i | 0 \leq i \leq m\}$ by

$$\phi(V_j)(i) = \frac{s_{ij}}{s_{i0}} = \frac{1}{\dim V_i} (\text{tr}_{|V_i} \otimes \text{tr}_{|V_j^*})(R^{21} R).$$

It is straightforward to check that ϕ is an isomorphism of algebras (see e.g. [T] or Remark 1.3 below). Since multiplication by $\phi(V_j)$ has eigenvalues $\{\frac{s_{ij}}{s_{i0}} | 0 \leq i \leq m\}$ it follows that multiplication by V_j in $\text{Rep}(\mathcal{C})$ ($V_i \mapsto V_j \otimes V_i$) has the same eigenvalues (this statement is called ‘‘Verlinde Formula’’ [V]). But, multiplication is represented by an integral matrix (N_{ij}^l) where $V_j \otimes V_i = \sum_l N_{ij}^l V_l$. We thus conclude that $\frac{s_{ji}}{s_{j0}}$ is an algebraic integer. Since s is symmetric [Ki, T] it follows that $\frac{s_{ij^*}}{s_{0j^*}}$ is an algebraic integer too. \square

Remark 1.3. *We demonstrate that the map $\phi : \text{Rep}(D(H)) \rightarrow k\{i | 0 \leq i \leq m\}$ is an algebra isomorphism. Indeed, $\phi(V_j)(i) = \frac{1}{\dim V_i} \chi_i(F(\chi_j^*))$ where F is as in the proof of Lemma 1.1. Since $F|_{\mathcal{C}(D(H))}$ is an isomorphism of algebras onto $Z(D(H))$ we have*

$$\begin{aligned} \phi(V_j \otimes V_i)(i) &= \frac{1}{\dim V_i} \chi_i(F(\chi_j^* \chi_i^*)) = \frac{1}{\dim V_i} \chi_i(F(\chi_j^*) F(\chi_i^*)) \\ &= \frac{1}{(\dim V_i)^2} \chi_i(F(\chi_j^*)) \chi_i(F(\chi_i^*)) = \phi(V_j)(i) \phi(V_i)(i). \end{aligned}$$

So ϕ is an algebra map, and an isomorphism since s is non-degenerate.

Theorem 1.4. *Let H be a finite-dimensional semisimple Hopf algebra over k . If V is an irreducible representation of $D(H)$ then $\dim V$ divides $\dim H$.*

Proof. First note that since $D(H)$ is semisimple, $D(H) = \bigoplus_{i=0}^m V_i \otimes V_i^*$. Now, by Lemmas 1.1 and 1.2, $\frac{(\dim H)^2}{(\dim V)^2}$ is an algebraic integer. This implies that $\frac{\dim H}{\dim V}$ is an algebraic integer too, hence an integer. \square

We are ready now to prove Kaplansky's conjecture for quasitriangular semi-simple Hopf algebras.

Theorem 1.5. *Let (H, R) be a finite-dimensional quasitriangular semisimple Hopf algebra. Then H is of Frobenius type.*

Proof. It is straightforward to check that the map $f : D(H) \rightarrow H$ given by $f(ph) = (p \otimes 1)(R)h$ for all $p \in H^*$ and $h \in H$, is a surjection of Hopf algebras. Therefore, if V is an irreducible representation of H then it is also an irreducible representation of $D(H)$ via pull back along f , and the result follows by Theorem 1.4. \square

Example 1.6. *The group algebra $H = kG$ of a finite group G is quasitriangular with $R = 1 \otimes 1$. In this case Theorem 1.5 is the classical theorem stating that the dimensions of the irreducible representations of a finite group divide its order.*

In the following we show how Kaplansky's conjecture on prime dimensional Hopf algebras follows easily from Theorem 1.4 (compare with [Z1]).

Theorem 1.7. *Let H be a Hopf algebra of prime dimension p . Then $H = k\mathbb{Z}_p$ is the group algebra of the cyclic group of order p .*

Proof. By [NZ], either $|G(H)| = 1$ or $|G(H)| = p$, and the same holds for H^* . Suppose $|G(H)| = |G(H^*)| = 1$. Then it is easy to show that H is semisimple (see e.g. [Z1]). But then by Theorem 1.4, if V is an irreducible representation of $D(H)$ then either $\dim V = 1$ or $\dim V = p$. Since $G(D(H)^*) = 1$ (i.e. $D(H)$ has only one 1-dimensional representation) and $\dim D(H) = p^2$, it follows that $p^2 = 1 + ap^2$ for some positive integer a which is absurd. Therefore, either $|G(H^*)| = p$ or $|G(H)| = p$, and the result follows. \square

2. Semisimple triangular Hopf algebras

By Theorem 1.5, if (H, R) is a finite-dimensional triangular (i.e. $R^{21}R = 1$) semisimple Hopf algebra then it is of Frobenius type. In fact, we can say much more in this case. In the following we show that H is a twisted group algebra in the sense of [Dr2] (see formulas (1.11) and (1.12) there); that is, that H is isomorphic as a Hopf algebra to a group algebra with a modified comultiplication which is obtained by a conjugation of the standard one by an invertible counital cocycle.

Let (H, R) be a finite-dimensional semisimple triangular Hopf algebra. In this case u is a grouplike element by (2), and $u^2 = 1$ by (3). For some purposes it is useful to assume that the Drinfeld element u acts as 1 (see e.g. [CWZ]). Let us demonstrate that it is always possible to replace R with a new R -matrix \tilde{R} so that the new Drinfeld element \tilde{u} equals 1. Indeed, for any irreducible representation V of H define the parity of this representation, $p(V) \in \mathbb{Z}_2$, by $(-1)^{p(V)} = u|_V$. Define $\tilde{R} \in H \otimes H$ by the condition $\tilde{R}|_{V \otimes W} = (-1)^{p(V)p(W)} R|_{V \otimes W}$. It is

straightforward to verify that $\tilde{R} = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)R$ is a new triangular structure on H , with Drinfeld element $\tilde{u} = 1$.

Our main result in this section is:

Theorem 2.1. *Let (H, R) be a finite-dimensional semisimple triangular Hopf algebra over k . Then there exists a finite group G , an invertible element $J \in kG \otimes kG$ which satisfies*

$$(4) \quad (\Delta_0 \otimes 1)(J)J_{12} = (1 \otimes \Delta_0)(J)J_{23}, \quad (\varepsilon_0 \otimes 1)(J) = (1 \otimes \varepsilon_0)(J) = 1,$$

(where Δ_0, ε_0 are the coproduct and the counit of the group algebra), and an algebra isomorphism $\phi : kG \rightarrow H$ such that

$$(5) \quad (\phi^{-1} \otimes \phi^{-1})(\Delta(\phi(a))) = J^{-1}\Delta_0(a)J,$$

and

$$(6) \quad (\phi^{-1} \otimes \phi^{-1})(\tilde{R}) = (J^{21})^{-1}J.$$

That is, (H, \tilde{R}) and $(kG, \Delta, (J^{21})^{-1}J)$ are isomorphic as triangular Hopf algebras, where $\Delta : kG \rightarrow kG \otimes kG$ is determined by $\Delta(g) = J^{-1}(g \otimes g)J$, $g \in G$.

Proof. Let \mathcal{C} be the category of finite-dimensional representations of H . This is a semisimple abelian category over k with finitely many irreducible objects, which has a structure of a rigid symmetric tensor category [DM]. Here the commutativity isomorphism in \mathcal{C} is defined by the operator $\tau_{\tilde{R}} : V \otimes W \rightarrow W \otimes V$, where $\tau : V \otimes W \rightarrow W \otimes V$ is the usual permutation map. Moreover, the categorical dimension [DM] of an object $V \in \mathcal{C}$ is equal to $\text{tr}|_V(\tilde{u})$, so it equals to the ordinary dimension of V as a vector space (since $\tilde{u} = 1$). In particular, all categorical dimensions are non-negative integers.

In this situation we can apply the following deep theorem of Deligne:

Theorem. [De, Theorem 7.1] *Let \mathcal{C} be a semisimple rigid symmetric tensor category over an algebraically closed field k with finitely many irreducible objects, in which categorical dimensions of objects are non-negative integers. Then for a suitable finite group G there exists an equivalence of symmetric rigid tensor categories $F : \mathcal{C} \rightarrow \text{Rep}(G)$ (where $\text{Rep}(G)$ is the category of finite dimensional k -representations of G).*

So let G, F be the group and the functor corresponding to our category \mathcal{C} . Let $K : \text{Rep}(G) \rightarrow \text{Vect}$ be the forgetful functor to the category of vector spaces. Since the functor F preserves dimensions, and the category is semisimple, the functor $K \circ F$ is isomorphic (as an additive functor) to the forgetful functor $L : \mathcal{C} \rightarrow \text{Vect}$. We might as well assume that $K \circ F = L$ as additive functors.

By a standard argument we have $\text{End}(L) = H$. On the other hand, the group G by definition acts on $K \circ F$ as a tensor functor, which defines an algebra homomorphism $\phi : kG \rightarrow H$. It is obvious that ϕ is an algebra isomorphism.

The functor $K \circ F$ has a tensor structure which preserves the commutativity isomorphism. This structure is given by a collection of invertible linear maps

$$\tilde{J}_{VW} : (K \circ F)(V) \otimes (K \circ F)(W) \rightarrow (K \circ F)(V \otimes W)$$

for irreducible V, W , which can be united in an invertible element $\tilde{J} \in \text{End}((K \circ F)^2) = kG \otimes kG$ (since $(K \circ F)(V) \otimes (K \circ F)(W) = L(V) \otimes L(W) = L(V \otimes W) = (K \circ F)(V \otimes W)$). The element $J = \tilde{J}^{-1}$ satisfies (4) and (5) because \tilde{J} is a tensor structure, and satisfies (6) because \tilde{J} preserves the commutativity isomorphism. \square

Remark 2.2. One should distinguish between the categorical dimensions of objects, defined in any rigid braided tensor category, and their quantum dimensions, defined only in a ribbon category. In the diagrammatic language of [Kas, Ki] the quantum dimension corresponds to a loop without self-crossing, and the categorical dimension to a loop with one self-overcrossing. They may be different numbers for a particular irreducible object. For example, in the category of representations of a triangular semisimple Hopf algebra (H, R) , quantum dimensions (for an appropriate ribbon structure) are ordinary dimensions (as in Section 1), while categorical dimensions are $u|_V \dim(V)$, where $u|_V$ is the scalar by which the Drinfeld element u acts on V , i.e. 1 or -1 (as in Section 2).

Remark 2.3. As seen from Remark 2.2, if $u \neq 1$, then the category of representations of (H, R) is equivalent to the category of representations of some group as a rigid tensor category but not as a symmetric category. This was the reason for passing from R to \tilde{R} . It is easy to see that as a symmetric rigid tensor category, the category of representations of (H, R) is equivalent to the category of representations of G on super-vector spaces, such that u acts by 1 on the even part and as -1 on the odd part. For example, if $H = k\mathbb{Z}_2$ with central primitive idempotents a and b , and $R = a \otimes a + b \otimes a + a \otimes b - b \otimes b$, then the category of representations is just the category of super-vector spaces.

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References

- [CWZ] M. Cohen, S. Westreich, and S. Zhu, *Determinants, integrality and Noether's theorem for quantum commutative algebras*, Israel J. Math. **96** (1996).
- [De] P. Deligne, *Catégories tannakiennes*, The Grothendick Festschrift, Vol. II, Prog. Math. **87** (1990), 111–195.
- [DM] P. Deligne and J. Milne, *Tannakian categories*, Lecture Notes in Math., 900, 101–228.
- [Dr1] V. G. Drinfel'd, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990), 321–342.
- [Dr2] ———, *Quasi-Hopf algebras*, Algebra i Analiz **1** (1989), 114–148, English Transl: Leningrad Math. J. **1** (1990), 1419–1457.
- [G] S. Gelaki, *Quantum groups of dimension pq^2* , Israel J. Math. **102** (1997), 227–267.

- [Ka] I. Kaplansky, *Bialgebras*, University of Chicago, 1975.
- [Kas] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, 155, Springer-Verlag, New York, 1995.
- [Ki] A. A. Kirillov Jr., *On an inner product in modular tensor categories*, J. Amer. Math. Soc. **9** (1996), 1135–1169.
- [LR1] R. G. Larson and D. E. Radford, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math. **110** (1988), 187–195.
- [LR2] ———, *Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple*, J. Alg. **117** (1988), 267–289.
- [Mo] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Lecture Notes, 82, Amer. Math. Soc. , 1993.
- [MW] S. Montgomery and S. J. Witherspoon, *Irreducible representations of crossed products*, preprint.
- [NR] W. D. Nichols and B. Richmond, *The Grothendick group of a Hopf algebra*, J. Pure Appl. Alg. **106** (1996), 297–306.
- [NZ] W. D. Nichols and M. B. Zoeller, *A Hopf algebra freeness theorem*, Amer. J. Math. **111** (1989), 381–385.
- [R1] D. E. Radford, *Minimal quasitriangular Hopf algebras*, J. Alg. **157** (1993), 285–315.
- [R2] ———, *On Kauffman's knot invariants arising from finite-dimensional Hopf algebras*, Advances in Hopf algebras (Chicago, IL, 1992), 205–266, Lecture Notes in Pure and Appl. Math., 158, Dekker, New York, 1994.
- [T] V. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, 18, Walter de Gruyter & Co., Berlin, 1994.
- [V] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B **300** (1988), 360–376.
- [Z1] Y. Zhu, *Hopf algebras of prime dimension*, Internat. Math. Res. Notices **1** (1994), 53–59.
- [Z2] ———, *A community pair in Hopf algebras*, Proc. Amer. Math. Soc. **125** (1997), 2847–2851.

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