

**THE LONGEST INCREASING SUBSEQUENCE  
IN A RANDOM PERMUTATION AND A  
UNITARY RANDOM MATRIX MODEL**

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ABSTRACT. If  $L_N$  is the expected length of the longest increasing subsequence in a random permutation, then  $L_N \sim 2\sqrt{N}$  as  $N \rightarrow \infty$ . We give a new proof of this result using a connection with a certain unitary random matrix model. The asymptotic formula is directly related to a third order phase transition in this model found by Gross and Witten.

**1. Preliminaries and main results**

Let  $S_N$  be the group of all permutations of  $\{1, 2, \dots, N\}$ . We take the uniform distribution on  $S_N$  as our probability distribution. If  $\pi \in S_N$  we say that  $\pi(i_1), \dots, \pi(i_k)$  is an increasing subsequence in  $\pi$  if  $i_1 < i_2 < \dots < i_k$  and  $\pi(i_1) < \pi(i_2) < \dots < \pi(i_k)$ . We are interested in the random variable  $\ell_N(\pi)$  the length of the longest increasing subsequence in  $\pi$ . Let  $F_N(n) = P[\ell_N(\pi) \leq n]$  be its distribution function. For large  $N$  the function  $F_N(n)$  rises sharply from close to 0 to close to 1 when  $n \sim 2\sqrt{N}$ . A consequence of this is that the expectation  $L_N$  of  $\ell_N(\pi)$  is asymptotically  $2\sqrt{N}$ . The problem of the asymptotics of  $L_N$  has a long history starting with Ulam, [U], [BB] and Hammersley, [Ha], and there are now several proofs, see [LS], [VK], [AD], [DZ1] and [Se1]. In the present paper we will show that the sharp transition of  $F_N(n)$  around  $n \sim 2\sqrt{N}$  is associated with a certain third-order phase transition in a unitary random matrix model first studied by Gross and Witten, [GW], in connection with the large- $N$  limit of 2- $d$   $U(N)$  lattice gauge theory. This connection is interesting since it connects the questions about the distribution of  $\ell_N(\pi)$  with the asymptotic properties of large random matrices a field which has recently advanced considerably.

The connection is through the following formula of Odlyzko et al, [OPWW], and Rains, [Ra],

$$(1.1) \quad F_N(n) = \frac{2^{2N} N!}{(2N)!} \mathcal{J}(2N, n),$$

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where

$$(1.2) \quad \begin{aligned} \mathcal{J}(m, n) &= \frac{1}{(2\pi)^n n!} \int_{[-\pi, \pi]^n} \left( \sum_{j=1}^n \cos \theta_j \right)^m \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 d^n \theta \\ &= \int_{U(n)} \left( \text{Tr} \left( \frac{U + U^*}{2} \right) \right)^m dU. \end{aligned}$$

Here  $dU$  denotes normalized Haar measure on the unitary group. The second equality in (1.2) follows from Weyl's integration formula, [We]. The proof in [OPWW] starts from Gessel's generating function for  $F_N(n)$ , [Ge], see below. A nice derivation, using representation theory, has been given by Rains [Ra].

Clearly

$$L_N = \sum_{n=1}^N n(F_N(n) - F_N(n-1)),$$

and a summation by parts gives

$$(1.3) \quad L_N = \sum_{n=0}^{N-1} (1 - F_N(n)).$$

To simplify the asymptotic analysis we will now assume that  $N$  is a Poisson random variable with mean  $\lambda$ , and consider the expectation (Poissonization)

$$(1.4) \quad \phi_n(\lambda) = \sum_{N=0}^{\infty} \frac{e^{-\lambda} \lambda^N}{N!} F_N(n),$$

where  $F_0(n) \equiv 1$ . We will show below that  $F_N(n)$  is decreasing in  $N$  for a fixed  $n$ , and hence from asymptotic information about  $\phi_n(\lambda)$  we can extract asymptotic information about  $F_N(n)$  (de-Poissonization). Since  $\mathcal{J}(m, n) = 0$  if  $m$  is odd, we get

$$(1.5) \quad \begin{aligned} \phi_n(\lambda) &= e^{-\lambda} \frac{1}{(2\pi)^n n!} \int_{[-\pi, \pi]^n} \exp(2\sqrt{\lambda} \sum_{j=1}^n \cos \theta_j) \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 d^n \theta \\ &= e^{-\lambda} E_n(\exp(2\sqrt{\lambda} \sum_{j=1}^n \cos \theta_j)), \end{aligned}$$

by combining (1.1), (1.2) and (1.4). Here  $E_n(\cdot)$  denotes expectation with respect to the probability density

$$\frac{1}{(2\pi)^n n!} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2$$

on  $[-\pi, \pi]^n$ . This is the probability density for the eigenvalues of a unitary random matrix taken randomly with respect to normalized Haar measure on  $U(n)$ , see [Me]. It is the representation (1.5) that allows us to use random matrix theory. The expectation in the last expression in (1.5) can also be given another interpretation. It is equal to the  $n \times n$  Toeplitz determinant,  $D_n(f)$ , with generating function  $f(\theta) = \exp(2\sqrt{\lambda} \cos \theta)$ . Thus

$$(1.6) \quad \phi_n(\lambda) = D_n(e^{2\sqrt{\lambda} \cos \theta}).$$

Actually in going to formula (1.6) we have gone back to Gessel's generating function, see sect. 7 in [Ge] and [GWW]; the elements in the Toeplitz determinant are  $l_{j-k}(2\sqrt{\lambda})$ , where  $l_j$  is the  $j$ th modified Bessel function. This formula is interesting since it opens the possibility of investigating the asymptotic properties of  $F_N(n)$  through asymptotic properties of Toeplitz determinants. It follows from Szegő's strong asymptotic formula for Toeplitz determinants, [Sz2], that  $\phi_n(\lambda) \rightarrow 1$  as  $n \rightarrow \infty$  for  $\lambda$  fixed. In the present case we are interested in the case when  $n$  and  $\sqrt{\lambda}$  are of the same order, so we are considering a different type of asymptotics. From a statistical mechanical point of view we can think of (1.5) as the partition function of a Coulomb gas of unit charges on the unit circle with logarithmic repulsion and an external potential  $2\sqrt{\lambda} \cos \theta$ . Consider the free energy,

$$f(\gamma) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \log E_n(\exp(\gamma n \sum_j \cos \theta_j)),$$

of this Coulomb gas. We will show below, lemma 2.1, that  $f(\gamma) = \gamma^2/4$  if  $0 \leq \gamma \leq 1$  and  $f(\gamma) = \gamma - 3/4 - (\log \gamma)/2$  if  $\gamma > 1$ . Hence  $d^3 f/d\gamma^3$  is discontinuous at  $\gamma = 1$ , and we have a third order phase transition, [GW]. For  $\gamma < 1$  the asymptotic eigenvalue distribution is supported on the whole unit circle, whereas for  $\gamma > 1$  it is supported on a subset of the circle. If we compare with (1.5) we see that the phase transition occurs when  $n \sim 2\sqrt{\lambda}$ , and the analysis below will show that this entails that  $F_N(n)$  rises sharply from 0 to 1 when  $n \sim 2\sqrt{N}$ , and thus, by (1.3), we expect  $L_N/\sqrt{N} \rightarrow 2$  as  $N \rightarrow \infty$ . Of course, to make this rigorous we have to make precise estimates. The main results are

**Theorem 1.1.** *If  $F_N(n)$  is the distribution of the length of the longest increasing subsequence in a random permutation from  $S_N$ , then*

$$(1.7) \quad \lim_{N \rightarrow \infty} F_N(x\sqrt{N}) = \begin{cases} 0 & \text{if } 0 \leq x < 2, \\ 1 & \text{if } x > 2. \end{cases}$$

We will also give a new proof of

**Theorem 1.2.** *The expected length  $L_N$  of the longest increasing subsequence in a random permutation in  $S_N$  satisfies*

$$(1.8) \quad \lim_{N \rightarrow \infty} \frac{L_N}{\sqrt{N}} = 2.$$

As stated above there are many proofs of this result, see [LS], [VK], [AD] and [Se1], and the purpose of the present paper is to show that the result can also be obtained using analytic tools from the theory of random matrices. Since there has recently been a lot of interest in random matrices and many new results, this opens a new perspective on the problem of understanding the distribution of  $\ell_N(\pi)$ . The formulas (1.5) and (1.6) make it possible to use tools for asymptotic analysis that have been used in connection with random matrix problems. For example to get the variance of  $\ell_N$  we need precise information about the behaviour of  $\phi_n(\lambda)$  near the transition point. This is related to the so called double-scaling limit for the unitary random matrix model, see [PS]. A heuristic analysis using the free energy, lemma 2.1, shows that  $\phi_n(\lambda)$  should, as a function of  $n$ , rise from 0 to 1 in a region of size  $\sim N^{1/6}$ , which leads to  $\text{var}(\ell_N) \sim N^{1/3}$ . That  $\text{var}(\ell_N)$  grows like  $N^{1/3}$  was first conjectured by Odlyzko and Rains about 5 years ago on the basis of Monte Carlo simulations, [Od], see also [Ki] for a discussion of this conjecture. These Monte Carlo simulations also give more detailed information about the mean and the variance, see [OR]. Work to prove this rigorously is in progress, [BDJ].

The theorems will be proved in the next section, but we postpone the proofs of several lemmas to sections 3 and 4. The ideas used in the asymptotic analysis are closely related to those in [Jo], but a more refined analysis is needed in the present case.

## 2. Proof of the main theorems

Put

$$G_n(\gamma) = E_n(\exp(\gamma n \sum_j \cos \theta_j)),$$

for  $\gamma \geq 0$ , so that

$$(2.1) \quad \phi_n(\lambda) = e^{-\lambda} G_n\left(\frac{2\sqrt{\lambda}}{n}\right).$$

We have

**Lemma 2.1.** [GW] *If  $f_n(\gamma) = n^{-2} \log G_n(\gamma)$ , then*

$$\lim_{n \rightarrow \infty} f_n(\gamma) = f(\gamma) = \begin{cases} \gamma^2/4 & \text{if } 0 \leq \gamma \leq 1, \\ \gamma - 3/4 - (\log \gamma)/2 & \text{if } \gamma > 1. \end{cases}$$

The proof will be given in section 3. Using lemma 2.1 we can prove

**Lemma 2.2.** *Let  $\epsilon > 0$  be given. There are positive constants  $C, \delta$ , that only depend on  $\epsilon$ , such that if  $n \leq (1 + \epsilon)^{-1} 2\sqrt{\lambda}$ , then*

$$(2.2) \quad \phi_n(\lambda) \leq C e^{-\delta \lambda}.$$

*Proof.* Clearly  $F_{N,n}$  is increasing in  $n$ , so the same is true for  $\phi_n(\lambda)$  by (1.4). Hence, if we write  $n(\lambda) = \lceil (1 + \epsilon)^{-1} 2\sqrt{\lambda} \rceil$ , then

$$(2.3) \quad \phi_n(\lambda) \leq \phi_{n(\lambda)}(\lambda) = e^{-\lambda} E_{n(\lambda)} \left( \exp \left( \frac{2\sqrt{\lambda}}{n(\lambda)} n(\lambda) \sum_j \cos \theta_j \right) \right).$$

Now,

$$\left| \frac{2\sqrt{\lambda}}{n(\lambda)} - (1 + \epsilon) \right| \leq \frac{2}{n(\lambda)},$$

if  $\lambda$  is sufficiently large and  $\epsilon \leq 1$ . Consequently,

$$\left| \frac{2\sqrt{\lambda}}{n(\lambda)} n(\lambda) \sum_j \cos \theta_j - (1 + \epsilon) n(\lambda) \sum_j \cos \theta_j \right| \leq 2n(\lambda) \leq 4\sqrt{\lambda},$$

and the estimate (2.3) yields

$$(2.4) \quad \phi_n(\lambda) \leq \exp(-\lambda + 4\sqrt{\lambda} + n(\lambda)^2 f_{n(\lambda)}(1 + \epsilon)).$$

Since  $n(\lambda)^2/\lambda \rightarrow 4/(1 + \epsilon)^2$  as  $\lambda \rightarrow \infty$ , (2.4) and lemma 2.1 give

$$\limsup_{\lambda \rightarrow \infty} \phi_n(\lambda)^{1/\lambda} \leq \exp(-1 + \frac{4}{(1 + \epsilon)^2} f(1 + \epsilon)).$$

Expanding in a Taylor series we see that

$$-1 + \frac{4}{(1 + \epsilon)^2} f(1 + \epsilon) = -\frac{2}{3}\epsilon^3 + \frac{11}{6}\epsilon^4 - \dots$$

and thus, if  $0 < \epsilon < \epsilon_0$ ,  $\epsilon_0$  sufficiently small, then

$$\limsup_{\lambda \rightarrow \infty} \phi_n(\lambda)^{1/\lambda} \leq \exp(-\epsilon^3/3).$$

It follows that  $\phi_n(\lambda) \leq \exp(-\epsilon^3 \lambda/4)$  for all sufficiently large  $\lambda$ , and the lemma is proved.  $\square$

Lemma 2.1 can be interpreted as a large deviation result for the Poissonized case,

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \phi_{\lceil x\sqrt{\lambda} \rceil}(\lambda) = -1 + 2x - \frac{3}{4}x^2 + \frac{x^2}{2} \log \frac{x}{2},$$

if  $x < 2$ . A large deviation result for  $\ell_N$  has recently been proved in [DZ2], see also [Se2].

The proof of the next lemma is long and occupies a large part of the paper, so we postpone it to the next two sections.

**Lemma 2.3.** *Let  $\epsilon > 0$  be given. There is a constant  $C$ , which only depends on  $\epsilon > 0$ , such that*

$$(2.5) \quad 0 \leq 1 - \phi_n(\lambda) \leq \frac{C}{n},$$

if  $n \geq \frac{2\sqrt{\lambda}}{1-\epsilon}$ .

The upper bound in (2.5) is not optimal. The methods of the present paper can be used to show that  $1 - \phi_n(\lambda) \leq C_d/n^d$  for any fixed  $d$ . A more precise large deviation result has recently been proved in [Se2], see also [DZ2]. Seppäläinen shows that

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\sqrt{\lambda}} (1 - \phi_{[x\sqrt{\lambda}]}(\lambda)) = -2x \cosh^{-1}\left(\frac{x}{2}\right) + 2\sqrt{x^2 - 4},$$

if  $x \geq 2$ .

In order to extract asymptotic information about  $F_N(n)$  from  $\phi_n(\lambda)$  we will use the fact that  $F_N(n)$  is decreasing in  $N$  for  $n$  fixed:

**Lemma 2.4.** *For all  $n, N \geq 1$ ,*

$$F_{N+1}(n) \leq F_N(n).$$

Using this lemma we can show the following “de-Poissonization”-lemma. Both the lemmas will be proved in section 4.

**Lemma 2.5.** *Write  $\mu_N = N + 4\sqrt{N \log N}$  and  $\nu_N = N - 4\sqrt{N \log N}$ . Then there is a constant  $C$  such that*

$$(2.6) \quad \phi_n(\mu_N) - \frac{C}{N^2} \leq F_N(n) \leq \phi_n(\nu_N) + \frac{C}{N^2},$$

for all sufficiently large  $N$ ,  $0 \leq n \leq N$

We can now give the

*Proof of theorem 1.1.* Let  $x < 2$ . From (2.6) we get

$$F_N([x\sqrt{N}]) \leq \phi_{[x\sqrt{N}]}(\nu_N) + \frac{C}{N^2}.$$

Since  $[x\sqrt{N}]/\sqrt{\nu_N} \leq 1/(1+\epsilon)$ , we see from lemma 2.2 that, if we choose  $\epsilon$  sufficiently small and  $N$  sufficiently large,

$$0 \leq F_N([x\sqrt{N}]) \leq Ce^{-\delta\nu_N} + \frac{C}{N^2}.$$

This proves the first half of (1.7). In the case  $x > 2$  the first inequality in (2.6) gives

$$(2.7) \quad \phi_{[x\sqrt{N}]}(\mu_N) - \frac{C}{N^2} \leq F_N([x\sqrt{N}]) \leq 1.$$

We have  $[x\sqrt{N}]/\sqrt{\mu_N} \geq 1/(1-\epsilon)$  if  $\epsilon$  is small and  $N$  large enough. Thus by lemma 2.3 and (2.7),

$$1 - \frac{C}{[x\sqrt{N}]} - \frac{C}{N^2} \leq F_N(n) \leq 1,$$

which establishes the second half of theorem 1.1.  $\square$

Next we prove theorem 1.2 by using the formula (1.3).

*Proof of theorem 1.2.* Fix  $\epsilon > 0$ . Then, by (2.6) and lemma 2.2,  $F_N(n) \leq C/N^2$  if  $n \leq (1+\epsilon)^{-1}2\sqrt{N}$ , and by (2.6) and lemma 2.3,  $1 - F_N(n) \leq C/N^2 + C/n$ , if  $n \geq 2\sqrt{N}/(1-\epsilon)$ ,  $N$  sufficiently large. Using (1.3) we thus have

$$|L_N - 2\sqrt{N}| \leq C(\epsilon\sqrt{N} + \log N + 1),$$

and the theorem is proved since  $\epsilon$  was arbitrary.  $\square$

### 3. Proofs of the asymptotic formulas

Let  $h_n(\theta)$  be a  $2\pi$ -periodic function, which is  $C^1$  and satisfies  $|h'_n(\theta)| \leq C$  for some constant  $C$ ,  $n \geq 1$ . Let  $u_n^{\xi, h_n}(t)$  be the 1-point function of the probability density

$$(3.1) \quad \frac{1}{Z_n} \prod_{j \neq k} |e^{i\theta_j} - e^{i\theta_k}| \exp(\xi n \sum_j \cos \theta_j + \sum_j h_n(\theta_j))$$

on  $[-\pi, \pi]^n$ . Note that

$$(3.2) \quad u_n^{0,0}(t) = \frac{1}{2\pi}.$$

Now,

$$(3.3) \quad \frac{d}{d\xi} \log G_n(\xi) = n^2 \int_{-\pi}^{\pi} \cos t u_n^{\xi,0}(t) dt,$$

so we need to understand the asymptotics of  $u_n^{\xi,0}$ . Define for  $|z| < 1$ ,

$$U_n(z) = \int_{-\pi}^{\pi} \frac{u_n(t)}{1 - ze^{-it}} dt$$

(we suppress the upper indices of  $u_n$ ), and

$$H_n(z) = \int_{-\pi}^{\pi} \frac{-ih_n(t)u_n(t)}{1 - ze^{-it}} dt.$$

Let  $k_n(t, s) = nu_n(t)u_n(s) - (n-1)u_n(t, s)$ , where  $u_n(t, s)$  is the two-point function of (3.1), and

$$(3.4) \quad K_n(z) = n \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \frac{1}{(1 - ze^{-it})(1 - ze^{-is})} - \frac{1}{(1 - ze^{-it})^2} \right) k_n(t, s) dt ds.$$

We will show in section 4 that  $U_n$  satisfies the equation

$$(3.5) \quad nU_n(z)^2 - n\left(\frac{\xi}{2}\left(z - \frac{1}{z}\right) + 1\right)U_n(z) - \frac{1}{2}\xi nc_n - \frac{\xi n}{2z} = \frac{1}{n}K_n(z) - H_n(z),$$

where

$$c_n = \int_{-\pi}^{\pi} e^{it} u_n(t) dt.$$

**Lemma 3.1.** *Let  $\delta > 0$ . If  $|z| \leq 1 - \delta$  and  $n \geq 1$ , then*

$$|K_n(z)| \leq \frac{2n}{\delta^2}.$$

*Proof.* Let  $\{p_{n,k}(e^{i\theta})\}_{k=0}^{\infty}$  be the sequence of orthonormal polynomials on the unit circle with respect to the weight  $w_n(\theta) = \exp(\gamma_n n \cos \theta + h_n(\theta))$ . Then, [Me],

$$k_n(t, s) = \frac{1}{n} \left| \sum_{k=0}^{n-1} p_{n,k}(e^{it}) p_{n,k}(e^{is}) \right|^2 w_n(t) w_n(s),$$

so  $k_n \geq 0$ . Also,  $\int k_n = 1$ , so the result follows immediately by estimating the integral in (3.4).  $\square$

The sequence  $\{U_n\}$  is a normal family in  $|z| < 1$ . If  $U_{n_j}$  is a convergent subsequence, then the limit  $U$  must satisfy the equation

$$(3.6) \quad U(z)^2 - \left(\frac{\xi}{2}\left(z - \frac{1}{z}\right) + 1\right)U(z) - \frac{\xi}{2}\frac{1}{z} - \frac{\xi}{2}c = 0,$$

where

$$(3.7) \quad U(z) = \int_{-\pi}^{\pi} \frac{d\mu(t)}{1 - ze^{-it}},$$

for some probability measure  $d\mu$ , which is the weak- $*$ -limit of  $u_{n_j} dt$ ;

$c = \int_{[-\pi, \pi]} \exp(it) d\mu(t)$ . This follows from (3.5) by dividing with  $n$ , letting  $n_j \rightarrow \infty$ , using  $|H_n(z)| \leq C/\delta$  for  $|z| \leq 1 - \delta$  and lemma 3.1.



**Lemma 3.2.** ([GW]) *Suppose that  $U(z)$  is given by (3.7) and satisfies the equation (3.6) with  $\xi \geq 0$  and  $c$  a constant.*

- (i) *If  $\xi \leq 1$ , then  $U(z) = 1 + \xi z/2$ ,  $c = \xi/2$  and  $d\mu(t) = (2\pi)^{-1}(1 + \xi \cos t)dt$ .*
- (ii) *If  $\xi > 1$ ,  $d\mu(t) = \pi^{-1}\xi \cos(t/2)\sqrt{1/\xi - \sin^2(t/2)}\chi_{[-t_c, t_c]}(t)dt$ , where  $\sin^2(t_c/2) = 1/\xi$ ,  $0 \leq t_c < \pi$  and  $c = 1 - 1/(2\xi)$ .*

The proof will be given in section 4. Since the limit  $U$  is unique it follows that  $U_n \rightarrow U$  as  $n \rightarrow \infty$  and

$$(3.8) \quad \int_{-\pi}^{\pi} \cos t u_n^{\xi,0}(t) dt \rightarrow \psi(t) = \begin{cases} \xi/2 & \text{if } 0 \leq \xi \leq 1, \\ 1 - 1/(2\xi) & \text{if } \xi > 1. \end{cases}$$

(Note that  $u_n^{\xi,0}$  is an even function.)

We are now ready for the

*Proof of lemma 2.1.* It follows from (3.3) that  $f'_n(\xi) = \int_{[-\pi, \pi]} \cos t u_n^{\xi,0}(t) dt$ , and hence  $|f'_n(\xi)| \leq 1$ . Since  $f'_n \rightarrow \psi$  as  $n \rightarrow \infty$  in  $[0, \gamma]$ , the dominated convergence theorem gives

$$f_n(\gamma) = \int_0^{\gamma} f'_n(\xi) d\xi \rightarrow \int_0^{\gamma} \psi(\xi) d\xi = f(\gamma).$$

and we are done. □

Next, we give the

*Proof of lemma 2.3.* Write  $u^\xi(t) = (2\pi)^{-1}(1 + \xi \cos t)$ . Then, by (3.3),

$$(3.9) \quad \frac{d}{d\xi} \log G_n(\xi) = n^2 \int_{-\pi}^{\pi} \cos t (u_n^{\xi,0}(t) - u^\xi(t)) dt + n^2 \frac{\xi}{2}.$$

Put

$$A_n(\xi) = n^3 \left| \int_{-\pi}^{\pi} \cos t (u_n^{\xi,0}(t) - u^\xi(t)) dt \right|.$$

It follows from (3.2) that  $A_n(0) = 0$ . Let  $\epsilon > 0$  be given and pick  $\gamma \in [0, 1 - \epsilon]$ . Integration of (3.9) from 0 to  $\gamma$  gives

$$(3.10) \quad \left| \log G_n(\gamma) - \frac{n^2 \gamma^2}{4} \right| \leq \frac{1}{n} \int_0^{\gamma} A_n(\xi) d\xi.$$

From the definition it is clear that  $A_n(\xi)$  is a continuous function on  $[0, 1 - \epsilon]$  and hence

$$\max_{0 \leq \xi \leq 1 - \epsilon} A_n(\xi) = A_n(\gamma_n)$$

for some  $\gamma_n \in (0, 1 - \epsilon]$ . We can assume that  $\gamma_n > 0$  since  $A_n(0) = 0$  and  $A_n(\xi) \geq 0$ . The inequality (3.10) gives

$$(3.11) \quad \left| \log G_n(\gamma) - \frac{n^2 \gamma^2}{4} \right| \leq \frac{1}{n} A_n(\gamma_n),$$

if  $\gamma \in [0, 1 - \epsilon]$ . Let  $\limsup_{n \rightarrow \infty} A_n(\gamma_n) = \lim_{k \rightarrow \infty} A_{n_k}(\gamma_{n_k})$ , where we can assume that  $\gamma_{n_k} \rightarrow \gamma_0 \in [0, 1 - \epsilon]$ , after, perhaps, picking yet another subsequence. To simplify notation we will write  $\gamma_n$  instead of  $\gamma_{n_k}$ . The key result is the next lemma which we will prove below.

**Lemma 3.3.** *Let  $\epsilon > 0$  be given. If  $0 < \gamma_n \leq 1 - \epsilon$  and  $\gamma_n \rightarrow \gamma_0$  as  $n \rightarrow \infty$ , then there is a constant  $C$ , which only depends on  $\epsilon$ , such that  $A_n(\gamma_n) \leq C$  for all  $n \geq 1$ .*

If we accept lemma 3.3, the inequality (3.11) gives

$$\left| \log G_n(\gamma) - \frac{n^2 \gamma^2}{4} \right| \leq \frac{C}{n}.$$

If  $n \geq 2\sqrt{\lambda}/(1 - \epsilon)$ , we can take  $\gamma = 2\sqrt{\lambda}/n$  and get

$$\left| \log G_n(2\sqrt{\lambda}/n) - \lambda \right| \leq C/n,$$

which, using (2.1), proves lemma 2.3.  $\square$

*Proof of lemma 3.3.* Below  $h_n$  will be either identically zero, or a certain  $2\pi$ -periodic  $C^1$ -function, (4.3), on  $\mathbb{R}$  satisfying

$$(3.12) \quad \|h'_n\|_\infty \leq C, \quad n \geq 1,$$

for some constant  $C$ . Here and in what follows  $C$  denotes a constant which only depends on  $\epsilon$ , but which may vary from place to place. Then

$$(3.13) \quad |H_n(z)| \leq \frac{C}{\delta}, \quad |z| \leq 1 - \delta.$$

Observe that  $U^{\gamma_n}$  satisfies

$$(3.14) \quad U^{\gamma_n}(z)^2 - \left( \frac{\gamma_n}{2} \left( z - \frac{1}{z} \right) + 1 \right) U^{\gamma_n}(z) - \frac{\gamma_n}{2} \frac{1}{z} - \frac{\gamma_n^2}{4} = 0.$$

Put

$$D_n(z) = D_n^{\gamma_n, h_n}(z) = n(U_n^{\gamma_n, h_n}(z) - U^{\gamma_n}(z)).$$

Taking the difference between the two equations (3.5) and (3.14) we see that  $D_n$  satisfies

$$(3.15) \quad \frac{1}{n^2} D_n^2 - \left( \frac{\gamma_n}{2} \left( z + \frac{1}{z} \right) + 1 \right) \frac{D_n}{n} - x_n = \frac{1}{n^2} K_n - \frac{1}{n} H_n,$$

where  $x_n = \gamma_n(c_n - \gamma_n/2)/2$ . Let  $z_n = -1/\gamma_n + \sqrt{1/\gamma_n^2 - 1}$ , which is the root of  $\gamma_n(z + 1/z)/2 + 1 = 0$  that lies inside the unit circle. Since  $0 < \gamma_n \leq 1 - \epsilon$  for all  $n$  we can choose  $\delta$ ,  $0 < \delta < 1/4$ , so that  $0 < |z_n| < 1 - 2\delta$  for all  $n \geq 1$ .

Equation (3.15) can be written

$$(3.16) \quad \frac{1}{n}D_n = \frac{x_n + K_n/n^2 - D_n^2/n^2 - H_n/n}{\gamma_n(z + 1/z)/2 + 1}.$$

Let

$$M_n = \sup_{|z|=1-\delta} \left| \frac{1}{n}D_n(z) \right|.$$

From equation (3.16), lemma 3.1 and inequality (3.13) we find

$$(3.17) \quad M_n \leq L(|x_n| + 1/n + M_n^2),$$

where the constant  $L$  only depends on  $\epsilon$ , and we can assume  $L \geq 1$ . We now claim that there is an  $n_0 \geq 16L^2$  such that if  $n \geq n_0$ , then

- (i)  $|x_n| \leq 1/16L^2$ ,
- (ii)  $|M_n| \leq 1/16L^2$ .

To see this let

$$\eta_n = \sup_{|z| \leq 1-\delta} \left| \frac{1}{n}D_n(z) \right|, \quad n \geq 1,$$

which gives a bounded sequence. Assume that there is a subsequence  $\eta_{n_k}$  such that  $\eta_{n_k} \rightarrow \eta > 0$  as  $n_k \rightarrow \infty$ . Since  $\{U_{n_k}\}$  is a normal family in the open unit disk, we can extract a subsequence  $\{U_{n_{k_j}}\}$  which converges uniformly in  $|z| \leq 1 - \delta$ . It follows from equation (3.5) and lemma 3.2 that the limit must be  $U^{\gamma_0}(z) = 1 + \gamma_0 z/2$ , and since  $\gamma_n \rightarrow \gamma_0$  it follows that  $\eta_{n_{k_j}} \rightarrow 0$  as  $j \rightarrow \infty$ , and we have a contradiction. Hence  $\eta_n \rightarrow 0$  as  $n \rightarrow \infty$  and we can pick  $n_0$  so that  $|D_n^{\gamma_n, h_n}(z)/n| \leq 1/(16L^2)$  if  $|z| \leq 1 - \delta$  and  $n \geq n_0$ ;  $n_0$  can be chosen so that this hold for both choices of  $h_n$ . Now,

$$x_n = \frac{\gamma_n}{4\pi i} \int_{|z|=1-\delta} \frac{D_n(z)/n}{z^2} dz,$$

and thus  $|x_n| \leq 1/16L^2$  if  $n \geq n_0$ . This proves (i) and (ii).

Using (i) it follows from (3.17) that

$$M_n \leq 2L(|x_n| + 1/n), \quad n \geq n_0.$$

Since  $|z_n| < 1 - \delta$  the maximum principle gives  $|D_n(z_n)|/n \leq 2L(|x_n| + 1/n)$ , and hence taking  $z = z_n$  in (3.15) we obtain

$$|x_n| \leq 4L^2(|x_n| + 1/n)^2 + 2/(n\delta^2) + C/n,$$

or

$$|x_n|^2 - (1/4L^2 - 2/n)|x_n| + C/n \geq 0.$$

Together with the estimate (ii) above this gives  $|x_n| \leq C/n$  and thus

$$(3.18) \quad |D_n^{\gamma_n, h_n}(z)| \leq C,$$

if  $|z| \leq 1 - \delta$  and  $n \geq n_0$ .

The inequality (3.18), with  $h_n$  given by (4.3) below, is the basis for the proof of the next lemma. We postpone the proof to section 4.

**Lemma 3.4.** *There is a constant  $C$  such that*

$$(3.19) \quad |K_n^{\gamma_n, 0}(z)| \leq C,$$

for all  $|z| \leq 1 - \delta$  and  $n \geq n_0$ .

Now, take  $h_n \equiv 0$  so that  $H_n \equiv 0$  in (3.15), and put  $z = z_n$ . The estimates (3.18) and (3.19) then give  $|nx_n| \leq C/n$ ,  $n \geq n_0$ . Combining this estimate with equation (3.16), the estimates (3.18), (3.19) and the maximum principle, we get

$$(3.20) \quad |D_n^{\gamma_n, 0}(z)| \leq C/n,$$

for all  $|z| \leq 1 - \delta$  and  $n \geq n_0$ .

To get further we need a better estimate of  $K_n^{\gamma_n, 0}$  than the one in lemma 3.4, and below we will prove

**Lemma 3.5.** *There is a constant  $C$  such that*

$$(3.21) \quad |K_n^{\gamma_n, 0}(z)| \leq C/n,$$

for all  $|z| \leq 1 - 3\delta/2$  and  $n \geq n_0$ .

We can now repeat the same type of argument one more time. Equation (3.15) with  $z = z_n$  together with the estimates (3.20) and (3.21) give  $|n^2x_n| \leq C/n$ . The same estimates and equation (3.15) then yield

$$(3.22) \quad |n^2D_n^{\gamma_n, 0}(z)| \leq C, \quad n \geq n_0,$$

for  $|z| = 1 - 3\delta/2$ .

Since  $u_n^{\gamma_n, 0}(t)$  is even a straightforward computation shows that

$$A_n(\gamma_n) = \frac{1}{2\pi i} \int_{|z|=1-3\delta/2} \frac{n^2D_n(z)}{z^2} dz.$$

Combining this with (3.22), lemma 3.3 follows.  $\square$

#### 4. The variational formulas and proofs of some lemmas

Let  $h$  and  $g$  be two given  $2\pi$ -periodic,  $C^1$ -functions. In the integral

$$Z_n = \int_{[-\pi, \pi]^n} \left( \sum_j g(\theta_j) \right) e^{\sum_{j \neq k} \log |2 \sin \frac{\theta_j - \theta_k}{2}| + \xi n \sum_j \cos \theta_j + \sum_j h_n(\theta_j)} d^n \theta,$$

we make the change of variables  $\theta_j = x_j + \epsilon \psi(x_j)$ , where  $\psi$  is a  $2\pi$ -periodic,  $C^1$ -function with  $\|\epsilon \psi'\|_\infty < 1$ . The obvious equality  $\frac{d}{d\epsilon} \log Z_n|_{\epsilon=0} = 0$  yields

$$(4.1) \quad E_n^{\gamma_n, h_n} \left[ \left( \frac{1}{2} \sum_{j \neq k} \cot \left( \frac{x_j - x_k}{2} \right) (\psi(x_j) - \psi(x_k)) \right. \right. \\ \left. \left. - \xi n \sum_j \psi(x_j) \sin x_j \sum_j h'_n(x_j) \psi(x_j) \right. \right. \\ \left. \left. + \sum_j \psi'(x_j) \right) \left( \sum_\ell g(x_\ell) \right) + \sum_\ell g'(x_\ell) \psi(x_\ell) \right] = 0.$$

In our first application of (4.1) we choose  $g(\theta) = 1/n$ , and obtain the identity

$$(4.2) \quad \frac{1}{2}n(n-1) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cot \frac{t-s}{2} (\psi(t) - \psi(s)) u_n(t, s) dt ds \\ - \xi^2 n^2 \int_{-\pi}^{\pi} \sin t \psi(t) u_n(t) dt + n \int_{-\pi}^{\pi} h'_n(t) \psi(t) u_n(t) dt \\ + n \int_{-\pi}^{\pi} \psi'(t) u_n(t) dt = 0,$$

where  $u_n(t, s)$  is the 2-point function of (3.1). Now, choose  $\psi(t) = (1 - ze^{-it})^{-1}$  in (4.2). This gives equation (3.5) after some algebraic manipulation.

Next, we will give the

*Proof of lemma 3.3.* Let  $\Omega = \{|z| \leq 1 - \delta\} \times \{|w_1| \leq 1\} \times \{|w_2| \leq 1\} \subseteq \mathbb{C}^3$  and

$$g_n(\theta; z, w_1, w_2) = w_1(e^{i\theta} - \int_{-\pi}^{\pi} e^{it} u^{\gamma_n}(t) dt) + w_2 \left( \frac{1}{1 - ze^{-i\theta}} - \int_{-\pi}^{\pi} \frac{u^{\gamma_n}(t)}{1 - ze^{-it}} dt \right).$$

On  $\Omega$  we define

$$F_n(z, w_1, w_2) = E_n^{\gamma_n, 0}(\exp(\sum_j g_n(\theta_j; z, w_1, w_2))),$$

where  $E_n^{\gamma_n, 0}(\cdot)$  denotes expectation with respect to (3.1) with  $h_n = 0$ . Below we omit the superscripts. Then

$$|F_n(z, w_1, w_2)| \leq E_n(\exp(\sum_j \operatorname{Re} g_n(\theta_j; z, w_1, w_2))).$$

The right hand side is a continuous function of  $(z, w_1, w_2)$  in  $\Omega$ , a compact set, so it assumes its maximum at some point  $(z^{(n)}, w_1^{(n)}, w_2^{(n)})$ . Write

$$(4.3) \quad h_n(\theta) = \operatorname{Re} g_n(\theta; z^{(n)}, w_1^{(n)}, w_2^{(n)}),$$

and consider

$$f_n(\theta) = \log E_n(\exp(\xi \sum_j h_n(\theta_j))),$$

for  $0 \leq \xi \leq 1$ . It is easy to see that  $f_n''(\xi) \geq 0$ , so  $f_n'(\xi)$  is increasing and consequently

$$f_n(1) = \int_0^1 f_n'(\xi) d\xi \leq f_n'(1).$$

Now,

$$f_n'(1) = n \int_{-\pi}^{\pi} h_n(t) u_n^{\gamma_n, h_n}(t) dt \\ = \operatorname{Re} \left( w_1^{(n)} \frac{1}{2\pi i} \int_{|\zeta|=1-\delta} \frac{D_n^{\gamma_n, h_n}(\zeta)}{\zeta^2} d\zeta + w_2^{(n)} D_n^{\gamma_n, h_n}(z^{(n)}) \right),$$

so using the inequality (3.18) we obtain

$$(4.4) \quad |F_n(z, w_1, w_2)| \leq C,$$

for all  $(z, w_1, w_2)$  in  $\Omega$  and  $n \geq n_0$ . Put  $G_n(w) = F_n(z, 0, w)$  for a fixed  $z$ . Cauchy's integral formula gives  $|G'_n(0)| \leq 4C$  and  $|G''_n(0)| \leq 8C$ , and since  $K_n^{\gamma_n, 0}(z) = -(G''_n(0) - G'_n(0)^2)$ , we obtain  $|K_n^{\gamma_n, 0}(z)| \leq 24C^2$ , and the lemma is proved.  $\square$

We will now use equation (4.1) with other choices of  $g$  but with  $h_n = 0$ . Write  $r(z) = 1 + \gamma_n(z - 1/z)/2$  and choose  $\psi(t) = (1 - ze^{-it})^{-1}$  in (4.1). This gives

$$(4.5) \quad E_n \left[ \left( - \sum_{j,k} \frac{1}{(1 - ze^{-ix_j})(1 - ze^{-ix_k})} + nr(z) \sum_j \frac{1}{(1 - ze^{-ix_j})} \right. \right. \\ \left. \left. + \frac{\gamma_n}{2} \sum_j e^{ix_j} + \frac{\gamma_n^2}{2z} \right) \left( \sum_j g(x_j) \right) - i \sum_j g'(x_j) \frac{1}{(1 - ze^{-ix_j})} \right] = 0.$$

Write  $F = \sum_j (1 - ze^{-ix_j})^{-1}$  and  $G = \sum_j g(x_j)$ . Define

$$L_n(z; g) = E_n(FG) - E_n(F)E_n(G),$$

$$K_n^{(2)}(z) = E_n(F^2) - E_n(F)^2 \quad \text{and}$$

$$K_n^{(3)}(z, g) = E_n(FG^2) - E_n(F^2)E_n(G) - 2E_n(F)E_n(FG) + 2E_n(F)^2E_n(G).$$

Note that  $K_n^{(2)}(z) = -K_n^{\gamma_n, 0}(z)$ . Furthermore, let

$$\mu_n(g) = \int_{-\pi}^{\pi} g(t) u_n^{\gamma_n, 0}(t) dt,$$

and

$$\nu_n(g) = E_n \left( \left( \sum_j e^{ix_j} \right) G \right).$$

Equation (4.1) can now be written

$$\begin{aligned} & -K_n^{(3)}(z, g) - n^3 \left[ \frac{1}{n^2} K_n^{(2)}(z) + U_n(z)^2 - r(z)U_n(z) \right. \\ & - \frac{\gamma_n}{2} c_n - \frac{\gamma_n}{2z} \left. \right] \mu_n(g) - \frac{\gamma_n n^3}{2} c_n \mu_n(g) - n(2U_n(z) \\ & - r(z)) L_n(z; g) + \frac{\gamma_n n}{2} \nu_n(g) - in \int_{-\pi}^{\pi} \frac{g'(t) u_n(t)}{1 - ze^{-it}} dt = 0. \end{aligned}$$

Using equation (3.5) this can be simplified to

$$(4.6) \quad -K_n^{(3)}(z; g) - n[(2U_n(z) - r(z))L_n(z; g) - \frac{\gamma_n n^3}{2} \mu_n(g) c_n \\ + \frac{\gamma_n n}{2} \nu_n(g) - in \int_{-\pi}^{\pi} \frac{g'(t) u_n(t)}{1 - ze^{-it}} dt] = 0.$$

If we choose  $g(t) = (1 - ze^{-it})^{-1}$ , then  $\mu_n(g) = U_n(z)$ ,  $\nu_n(g) = L_n(z; e^{it}) + n^2 U_n(z) c_n$ , and  $L_n(z; g) = K_n^{(2)}(z)$ . Writing  $K_n^{(3)}(z) = K_n^{(3)}(z; g)$ , the equation (4.6) gives

$$(4.7) \quad -K_n^{(3)}(z) - n(2U_n(z) - r(z))K_n^{(2)}(z) + \frac{\gamma_n n}{2} L_n(z; e^{it}) \\ - nzU_n'(z) - \frac{nz^2}{2} U_n''(z) = 0,$$

after some simplification. Next, we choose  $g(t) = e^{it}$  in (4.6). Then  $\mu_n(g) = c_n$  and  $\nu_n(g)$  is independent of  $z$ . Equation (4.6) becomes

$$(4.8) \quad -K_n^{(3)}(z; e^{it}) - n(2U_n(z) - r(z))L_n(z; e^{it}) + f_n + nzU_n(z) = 0,$$

where  $f_n = -\gamma_n n^3 c_n^2 / 2 + \gamma_n n \nu_n(e^{it}) / 2 + n c_n$ .

Multiply equation (4.7) with  $2U_n(z) - r(z)$  and use equation (4.8) to get

$$(4.9) \quad -(2U_n(z) - r(z))K_n^{(3)}(z) + \frac{\gamma_n}{2} [-K_n^{(3)}(z; e^{it}) + f_n + nU_n(z)] \\ - n(2U_n(z) - r(z))^2 K_n^{(2)}(z) - n(zU_n'(z) + \frac{z^2}{2} U_n''(z))(2U_n(z) - r(z)) = 0.$$

Note that

$$(4.10) \quad \frac{\gamma_n z}{2} U^{\gamma_n}(z) - [z(U^{\gamma_n})'(z) + \frac{z^2}{2} (U^{\gamma_n})''(z)](2U^{\gamma_n}(z) - r(z)) = 0.$$

Multiply equation (4.10) with  $n$  and subtract from equation (4.9). This gives the following equation

$$(4.11) \quad -\frac{2}{n} D_n(z) K_n^{(3)}(z) - s(z) K_n^{(3)}(z) - n(\frac{2}{n} D_n(z) + s(z))^2 K_n^{(2)}(z) \\ + \frac{\gamma_n}{2} K_n^{(3)}(z; e^{it}) + \frac{\gamma_n}{2} (f_n + D_n(z) + zD_n(z)) \\ + (\frac{2}{n} D_n(z) + s(z))(zD_n'(z) + \frac{z^2}{2} D_n''(z)) = 0,$$

where  $s(z) = \gamma_n(z + 1/z)/2 + 1$ . This equation is the basis for the

*Proof of lemma 3.4.* We use the same notation as in the proof of lemma 3.3. Note that

$$K_n^{(3)}(z) = \frac{\partial^3 F_n}{\partial w_2^3} - 3 \frac{\partial^2 F_n}{\partial w_2^2} \frac{\partial F_n}{\partial w_2} + 2 \left( \frac{\partial F_n}{\partial w_2} \right)^3,$$

and

$$K_n^{(3)}(z; e^{it}) = \frac{\partial^3 F_n}{\partial w_2^2 \partial w_1} - \frac{\partial^2 F_n}{\partial w_2^2} \frac{\partial F_n}{\partial w_1} - 2 \frac{\partial^2 F_n}{\partial w_1 \partial w_2} \frac{\partial F_n}{\partial w_2} + 2 \left( \frac{\partial F_n}{\partial w_2} \right)^2 \frac{\partial F_n}{\partial w_1},$$

where all derivatives are evaluated at the point  $(z, 0, 0)$ . Cauchy's integral formula and inequality (4.4) now yield

$$(4.12) \quad |K_n^{(3)}(z)| \leq C \quad \text{and} \quad |K_n^{(3)}(z; e^{it})| \leq C.$$

If we use (3.20), Cauchy's integral formula to estimate  $D'_n$  and  $D''_n$ , the estimates (4.12) and take  $z = z_n$  in (4.11) ( $s(z_n) = 0$ ), we get

$$(4.13) \quad \left| \frac{\gamma_n f_n}{2} \right| \leq C,$$

if  $n \geq n_0$ . Now, use equation (4.11) again with  $|z| = 1 - 3\delta/2$  together with (3.20) and (4.11) - (4.13) to see that  $|nK_n^{(2)}(z)| \leq C$  if  $n \geq n_0$  and  $|z| = 1 - 3\delta/2$ . Since  $K_n^{\gamma_n, 0}(z) = -K_n^{(2)}(z)$  lemma 3.4 follows using the maximum principle.  $\square$

The rest of this section is devoted to the proofs of the lemmas 2.4, 2.5 and 3.2.

*Proof of lemma 2.4.* Denote a permutation in  $S_N$  by  $\pi^{(N)}$  and put  $g_n(\pi^{(N)}) = 1$  if  $\pi^{(N)}$  does not have an increasing subsequence of length  $> n$  and  $g_n(\pi^{(N)}) = 0$  otherwise. Clearly,

$$(4.14) \quad F_N(n) = \frac{1}{N!} \sum_{\pi^{(N)} \in S_N} g_n(\pi^{(N)}).$$

Let  $S_{N+1}(k)$  denote the set of all  $\pi^{(N+1)}$  such that  $\pi^{(N+1)}(1) = k$ ,  $k = 1, \dots, N+1$ . Each  $S_{N+1}(k)$  contains  $N!$  elements and we can define a bijection  $F_k : S_{N+1}(k) \rightarrow S_N$  as follows. Let  $\psi_k(m) = m$  if  $1 \leq m \leq k-1$  and  $\psi_k(m) = m-1$  if  $k+1 \leq m \leq N+1$ , so that  $\psi_k$  is an increasing map from  $\{1, 2, \dots, k-1, k+1, \dots, N+1\}$  to  $\{1, \dots, N\}$ . Put

$$F_k(\pi^{(N+1)})(m) = \psi_k(\pi^{(N+1)}(m+1)), \quad m = 1, \dots, N,$$

for  $\pi^{(N+1)} \in S_{N+1}(k)$ . Clearly,  $F_k(\pi^{(N+1)})$  maps  $\{1, \dots, N\}$  into itself and, since  $\psi_k$  is strictly increasing,  $F_k(\pi^{(N+1)}) \in S_N$ . Also,  $F_k : S_{N+1}(k) \rightarrow S_N$  is a bijection. Note that if  $F_k(\pi^{(N+1)})$  has an increasing subsequence of length



$> n$ , then  $\pi^{(N+1)}$  has an increasing subsequence of length  $> n$ , because adding  $\pi^{(N+1)}(1) = k$  in the beginning can only increase the length of the increasing subsequence; recall that  $\psi_k$  is strictly increasing. Hence

$$(4.15) \quad g_n(F_k(\pi^{(N+1)})) \geq g_n(\pi^{(N+1)}).$$

From (4.15) we get

$$\begin{aligned} \frac{1}{(N+1)!} \sum_{\pi^{(N+1)} \in S_{N+1}} g_n(\pi^{(N+1)}) &= \frac{1}{(N+1)!} \sum_{k=1}^{N+1} \sum_{\pi^{(N+1)} \in S_{N+1}(k)} g_n(\pi^{(N+1)}) \\ &\leq \frac{1}{(N+1)!} \sum_{k=1}^{N+1} \sum_{\pi^{(N+1)} \in S_{N+1}(k)} g_n(F_k(\pi^{(N+1)})) \\ &= \frac{1}{(N+1)!} \sum_{k=1}^{N+1} \sum_{\pi^{(N)} \in S_N} g_n(\pi^{(N)}), \end{aligned}$$

and hence using (4.14) the lemma is proved.  $\square$

*Proof of lemma 2.5.* Write  $w_N(\lambda) = \lambda^N e^{-\lambda}/N!$ . By Stirling's formula we have

$$w_N(\lambda) \sim \exp\left(-\lambda\left(\frac{N}{\lambda} \log \frac{N}{\lambda} + 1 - \frac{N}{\lambda}\right)\right) \frac{1}{\sqrt{2\pi N}}.$$

Put  $f(x) = x \log x + 1 - x$ . Then

$$w_N(\lambda) \leq C \exp(-\lambda f(N/\lambda)).$$

If  $0 \leq x \leq 2$  it is easy to show that  $f(x) \geq (x-1)^2/4$ , and consequently

$$(4.16) \quad w_N(\lambda) \leq C \exp\left(-\frac{\lambda}{4}\left(\frac{N}{\lambda} - 1\right)^2\right),$$

if  $0 \leq N \leq 2\lambda$ . Also,  $f(x) \geq x/10$  if  $x \geq 2$  and hence

$$w_N(\lambda) \leq C e^{-N/10},$$

if  $N \geq 2\lambda$ . The inequality (4.17) gives

$$\sum_{N \geq 2\lambda} w_N(\lambda) \leq C e^{-\lambda/5}.$$

From (4.16) we get

$$\sum_{N \leq \lambda - 3\sqrt{\lambda \log \lambda}} w_N(\lambda) \leq \frac{C}{\lambda^2},$$

and

$$\sum_{\lambda+3\sqrt{\lambda \log \lambda} \leq N \leq 2\lambda} w_N(\lambda) \leq \frac{C}{\lambda^2}.$$

Since  $0 \leq F_N(n) \leq 1$ , we obtain

$$(4.18) \quad \left| \sum_{|N-\lambda| \leq 3\sqrt{\lambda \log \lambda}} w_N(\lambda) F_N(n) - \phi_N(\lambda) \right| \leq \frac{C}{\lambda^2},$$

for  $\lambda$  sufficiently large and  $0 \leq n \leq N$ . Now, since  $F_N(n)$  is decreasing in  $N$ ,

$$(4.19) \quad \sum_{|N-\lambda| \leq 3\sqrt{\lambda \log \lambda}} w_N(\lambda) F_N(n) \leq F_{\nu(\lambda)}(n),$$

where  $\nu(\lambda) = [\lambda - 3\sqrt{\lambda \log \lambda}]$ , and

$$(4.20) \quad \sum_{|N-\lambda| \leq 3\sqrt{\lambda \log \lambda}} w_N(\lambda) F_N(n) \geq F_{\mu(\lambda)}(n) - \frac{C}{\lambda^2},$$

where  $\mu(\lambda) = [\lambda + 3\sqrt{\lambda \log \lambda}]$ . If we choose  $\lambda = \mu_N$ , then  $\nu(\mu_N) \geq N$  if  $N$  is sufficiently large, and hence combining (4.18), (4.19) and lemma 2.4 we get the left inequality in (2.6). If we take  $\lambda = \nu_N$ , then  $\mu(\nu_N) \leq N$ , if  $N$  is sufficiently large. Combining (4.18), (4.20) and lemma 2.4 we get the right inequality in (2.6). The lemma is proved.  $\square$

*Proof of lemma 3.2.* The function  $U(z)$  is analytic in  $\mathbb{C} \setminus \mathbb{T}$ ,  $\mathbb{T}$  the unit circle and solving equation (3.6) we get

$$(4.21) \quad U(z) = \frac{1}{2} \left[ \frac{\xi}{2} \left( z - \frac{1}{z} \right) + 1 + \sqrt{\left( \frac{\xi}{2} \left( z + \frac{1}{z} \right) + 1 \right)^2 + 2\xi c - \xi^2} \right],$$

and

$$U(z) = \int_{-\pi}^{\pi} \frac{u(t)}{1 - ze^{-it}} dt,$$

where  $u(t)$  is a probability density given by

$$u(t) = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \operatorname{Re} (2U(re^{it}) - 1).$$

From (4.21) we get

$$2U(z) - 1 = \frac{\xi}{2} \left( z - \frac{1}{z} \right) + \sqrt{\left( \frac{\xi}{2} \left( z + \frac{1}{z} \right) + 1 \right)^2 - b^2},$$

where we have written  $2\xi c - \xi^2 = -b^2$ . If we define  $w = \xi(z + 1/z)/2 + 1$ , then  $\sqrt{w^2 - b^2}$  must be defined so that  $2U(z) - 1$  is analytic in  $\mathbb{C} \setminus \mathbb{T}$ . The image of  $\mathbb{T}$  under  $z \rightarrow w$ ,  $[1 - \xi, 1 + \xi]$  must cover the cut  $[-b, b]$  in the  $w$ -plane, so  $b$  is real  $\geq 0$  and

$$(4.22) \quad [-b, b] \subseteq [1 - \xi, 1 + \xi],$$

unless  $b = 0$ . If  $\xi \leq 1$ , then  $b = 0$  and  $2U(z) - 1 = 1 + \xi z$ , which gives  $u(t) = (1 + \xi \cos t)/2\pi$  and  $c = \xi/2$ . If  $\xi > 1$  the inclusion (4.22) gives the inequality  $b \leq \xi - 1$ . Assume that  $b < \xi - 1$ . Write  $\sqrt{w^2 - b^2} = \sqrt{w + b}\sqrt{w - b}$ . As  $z \rightarrow 0$  along the real axis,  $w \rightarrow +\infty$  along the real axis, and since  $2U(z) - 1$  is bounded and  $\xi(z - 1/z)/2 \rightarrow -\infty$ , both roots must be real and  $\geq 0$  when  $w > b$ . Thus we can take the arguments of  $w - b$  and  $w + b$  between  $-\pi$  and  $\pi$ . If  $z = re^{i\theta}$  and  $0 \leq \theta \leq \pi$ , then  $w$  approaches  $[1 - \xi, 1 + \xi]$  from the lower half-plane as  $r \rightarrow 1-$ . Since  $b < \xi - 1$  we can choose  $\theta_0$  in  $[0, \pi]$  so that  $1 - \xi < 1 + \xi \cos \theta_0 < -b$ , and we get

$$\sqrt{w^2 - b^2} \rightarrow |(1 + \xi \cos \theta_0)^2 - b^2|(-i)^2,$$

as  $r \rightarrow 1-$ , which gives  $u(\theta_0) < 0$ . This contradiction shows that  $b = \xi - 1$  if  $\xi > 1$ . Thus  $2\xi c - \xi^2 = -(\xi - 1)^2$ , which gives  $c = 1 - 1/2\xi$  and a straightforward computation gives  $u(\theta)$ .  $\square$

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