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REMARKS ON THE PAPER OF V. GUILLEMIN AND K. OKIKIOLU: "SUBPRINCIPAL TERMS IN SZEGÖ ESTIMATES"

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1. Introduction

Let M be a smooth compact manifold without boundary, $\dim M = d$ and let A and B be pseudodifferential operators (PsDO) acting in the space $L^2(M)$ of half-densities on M. We assume that A is a positive elliptic PsDO of order 1 and that B is a PsDO of order 0. Denote by $a(x,\xi)$ and $b(x,\xi)$, $(x,\xi) \in T^*M\backslash 0$, the principal symbols of the operators A and B respectively. The spectrum of A is discrete and therefore its spectral projection P_λ, $\lambda \geqslant 0$, is an operator of a finite rank. Let

$$\mathcal{V}_a(x,\xi) = \sum_{j=1}^{d} \frac{\partial a}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial}{\partial \xi_j}, \qquad (x,\xi) \in T^*M\backslash 0, \tag{1}$$

be the bicharacteristic vector field on $T^*M\backslash 0$ associated with a. A point $(x,\xi) \in T^*M\backslash 0$ is called periodic with a period t if $\exp(t\mathcal{V}_a)(x,\xi) = (x,\xi)$.

Guillemin and Okikiolu [GO] have recently announced the following result:

Theorem 1. *Let $a(x,\xi) = a(x,-\xi)$ and the subprincipal symbol of A is equal to zero . Suppose that for any $t > 0$ the set of t-periodic points is of measure zero with respect to the invariant measure $dx\,d\xi$ on the cotangent bundle $T^*M\backslash 0$. Then*

$$\operatorname{Tr}(P_\lambda B P_\lambda)^k = \operatorname{Tr} P_\lambda B^k P_\lambda - \lambda^{d-1}(2\pi)^{-d}\gamma_k(A,B) + o(\lambda^{d-1}), \qquad k \geqslant 2, \tag{2}$$

where

$$\gamma_k(A,B) = \frac{d}{8\pi}\sum_{m=1}^{k-1}\frac{k}{m(k-m)}\times \int_{a<1}\int_{-\infty}^{\infty}\frac{\big(b_t^m(x,\xi) - b^m(x,\xi)\big)\big(b_t^{k-m}(x,\xi) - b^{k-m}(x,\xi)\big)}{t^2}\,dt\,dx\,d\xi, \tag{3}$$

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and where $b_t(x,\xi) = (\exp(t\mathcal{V}_a))^* b(x,\xi) = b(\exp(t\mathcal{V}_a)(x,\xi))$.

Obviously this result can be reformulated for the trace $\operatorname{Tr} \mathcal{Q}_k(P_\lambda B P_\lambda)$, where $\mathcal{Q}_k$ is an arbitrary polynomial of degree k and such that $\mathcal{Q}_k(0) = 0$. Moreover, under certain conditions on the pseudodifferential operator B the paper [GO] also contains a corresponding asymptotic formula for $\operatorname{Tr} \log(P_\lambda B P_\lambda)$.

The purpose of this paper is to extend Theorem 1 to the case where instead of $\mathcal{Q}_k$ (or log) one deals with an arbitrary function $\psi \in C^2(R^1)$.

2. The main result

Let

$$K := \bigcup_{0\leqslant t\leqslant 1} t\,\sigma(B) \;\subset\; \mathbb{R}^1\,, \tag{4}$$

where $\sigma(B)$ is the spectrum of the operator B. Clearly, K is a closed bounded interval. In order to formulate our main result we introduce the transformation

$$\mathcal{W}\psi(t,s) = \int_s^t \int_s^t \frac{\psi'(u) - \psi'(v)}{u-v} du\, dv, \qquad \psi \in C^2(K)\,, \quad t,s \in K\,. \tag{5}$$

One can easily see that $\mathcal{W}$ is a linear continuous map from $C^2(K)$ into $C^1(K\times K)$ such that $|\mathcal{W}\psi(t,s)| \leqslant \|\psi''\|_{C(K)} |t-s|^2$. The kernel of the map $\mathcal{W}$ consists of the first degree polynomials.

Theorem 2. *Let* $\psi \in C^2(K)$ *and* B *be a selfadjoint PsDO of order 0. Then under the conditions of Theorem* 1

$$\operatorname{Tr} P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda = \operatorname{Tr} P_\lambda \psi(B) P_\lambda - \lambda^{d-1}(2\pi)^{-d}\gamma_\psi(A,B) + o(\lambda^{d-1}), \tag{6}$$

where

$$\gamma_\psi(A,B) = \frac{d}{8\pi} \int_{a<1} \int_{-\infty}^{\infty} \frac{\mathcal{W}\psi\left(b_t(x,\xi), b(x,\xi)\right)}{t^2} dt\, dx d\xi\,.$$

From the properties of the map $\mathcal{W}$ it follows that

$$\gamma_{\psi_0}(A,B) \min_{u\in K} \psi''(u) \leqslant \gamma_\psi(A,B) \leqslant \gamma_{\psi_0}(A,B) \max_{u\in K} \psi''(u)\,,$$

where $\psi_0(u) = u^2/2$ and

$$\gamma_{\psi_0}(A,B) = \frac{d}{8\pi} \int_{a<1} \int_{-\infty}^{\infty} \left(\frac{b_t(x,\xi) - b(x,\xi)}{t} \right)^2 dt\, dx d\xi\,.$$

This implies that $\gamma_\psi(A,B)$ is a linear continuous functional on the space $C^2(K)$.

If $\psi \in C^\infty(K)$, then $\psi(B)$ is a PsDO of order 0. Its principal symbol coincides with $\psi\left(b(x,\xi)\right)$, and subprincipal symbol is given by

$$\operatorname{sub} \psi(B)(x,\xi) = \psi'\left(b(x,\xi)\right) \operatorname{sub} B(x,\xi).$$

By the methods of [DG] one can prove that under the conditions of Theorem 1

$$(7)\quad \operatorname{Tr} P_\lambda \psi(B) P_\lambda = (2\pi)^{-d} \int_{a<1} \Big(\lambda^d \psi(b(x,\xi)) + \lambda^{d-1} \operatorname{sub} \psi(B)(x,\xi) \Big)\, dx\, d\xi + o(\lambda^{d-1})\,.$$

This result can be deduced from (4.2.6) in [SV] in the same way as the two-term asymptotic formula for the counting function $N(\lambda)$. It also follows from Proposition 29.1.2 in [H] (Hörmander's formula contains an extra term which is, as was pointed out by D. Vassiliev, actually equal to zero).

Combining Theorem 2 with (7) we obtain

Corollary 3. *Let $\psi \in C^\infty(\mathbb{R}^1)$. Then under the conditions of Theorem 2*

$$(8)\quad \operatorname{Tr} P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda = (2\pi)^{-d} \Big[\lambda^d \int_{a<1} \psi(b(x,\xi))\, dx d\xi - \lambda^{d-1} \Big(\gamma_\psi(A,B) - \int_{a<1} \operatorname{sub} \psi(B)(x,\xi)\, dx d\xi \Big) \Big] + o(\lambda^{d-1}).$$

3. Auxiliary statements

The proof of Theorem 2 is based on a version of an abstract result obtained in [LS1] (see also [LS2]). Let B be a bounded selfadjoint operator, P be an orthogonal projection in a Hilbert space $\mathcal{H}$, and K be the compact set defined by (4). Denote by $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively the trace class and the Hilbert-Schmidt class of operators in H.

Proposition 4. *Let $PB \in \mathfrak{S}_2$. Then for any function ψ whose second derivative lies in $L^\infty(K)$ we have*

$$P\psi(B)P - P\psi(PBP)P \in \mathfrak{S}_1$$

and

$$(9)\quad \Big| \operatorname{Tr}\Big(P\psi(B)P - P\psi(PBP)P \Big) \Big| \leqslant \frac{1}{2}\, \|\psi''\|_{L^\infty(K)} \|PB(I-P)\|^2_{\mathfrak{S}_2}.$$

The next statement concerns the map $\mathcal{W}$ defined in (5).

Proposition 5. *For an arbitrary polynomial*

$$\mathcal{Q}_k(x) = \sum_{m=0}^{k} a_m x^m,$$

we have

$$(10)\quad \mathcal{W}\mathcal{Q}_k(t,s) = \sum_{m=2}^{k} a_m \sum_{n=1}^{m-1} \frac{m}{n(m-n)} (t^n - s^n)(t^{m-n} - s^{m-n}).$$

Proof. It is sufficient to check (10) for $\mathcal{Q}_k(x) = x^k$, $k \geqslant 2$. In this case

$$\mathcal{W}\mathcal{Q}_k(t,s) = k\int_s^t\int_s^t \frac{u^{k-1}-v^{k-1}}{u-v}\,du\,dv$$
$$= k\int_s^t\int_s^t \Big(\sum_{n=1}^{k-1} u^{n-1}v^{k-1-n}\Big)\,du\,dv = k\sum_{n=1}^{k-1}\frac{1}{n(k-n)}(t^n-s^n)(t^{k-n}-s^{k-n}).$$

The proof is complete.

4. The proof of Theorem 2

Let $\{\mathcal{Q}_j\}_{j=1}^{\infty}$ be a sequence of polynomials approximating ψ in $C^2(K)$. Given $\varepsilon > 0$ we choose k_0 such that that $\|\psi - \mathcal{Q}_k\|_{C^2(K)} \leqslant \varepsilon$ for all $k \geqslant k_0$. Obviously

$$\mathrm{Tr}\Big(P_\lambda\psi(B)P_\lambda - P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda\Big) = T_1 + T_2,$$

where

$$T_1(\lambda, A, B) := \mathrm{Tr}\Big(P_\lambda\mathcal{Q}_k(B)P_\lambda - P_\lambda\mathcal{Q}_k(P_\lambda BP_\lambda)P_\lambda\Big) \tag{11}$$

and

$$T_2(\lambda, A, B) := \mathrm{Tr}\Big(P_\lambda(\psi-\mathcal{Q}_k)(B)P_\lambda - P_\lambda(\psi-\mathcal{Q}_k)(P_\lambda BP_\lambda)P_\lambda\Big).$$

From Proposition 4 we obtain

$$|T_2(\lambda, A, B)| \leqslant \frac{1}{2}\|\psi-\mathcal{Q}_k\|_{C^2(K)}\|P_\lambda B(I-P_\lambda)\|^2_{\mathfrak{S}_2} \leqslant \frac{\varepsilon}{2}\,\|P_\lambda B(I-P_\lambda)\|^2_{\mathfrak{S}_2}.$$

The well known asymptotic properties of the spectrum of the operator A (see for example [LS2, Section 2]) imply the estimate

$$\|P_\lambda B(I-P_\lambda)\|^2_{\mathfrak{S}_2} = O(\lambda^{d-1})$$

and, therefore, there exists a constant C independent of ε, such that

$$\limsup_{\lambda\to\infty}\,\lambda^{1-d}\,|T_2(\lambda, A, B)| \leqslant \varepsilon\, C.$$

Applying Theorem 1 to the trace (11) and taking into account Proposition 5 we obtain

$$\lim_{\lambda\to\infty}\,\lambda^{1-d}\,T_1(\lambda, A, B) = (2\pi)^{-d}\gamma_{\mathcal{Q}_k}(A,B)$$

and thus

$$\limsup_{\lambda\to\infty}|\lambda^{1-d}\ \mathrm{Tr}\Big(P_\lambda\psi(B)P_\lambda - P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda\Big) - (2\pi)^{-d}\gamma_{\mathcal{Q}_k}(A,B)| \leqslant \varepsilon\, C\,. \tag{12}$$

Since $\gamma_\psi(A,B)$ is a continuous linear functional on $C^2(K)$, (12) implies that

$$\limsup_{\lambda\to\infty}|\lambda^{1-d}\ \mathrm{Tr}\Big(P_\lambda\psi(B)P_\lambda - P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda\Big) - (2\pi)^{-d}\gamma_\psi(A,B)| \leqslant 2\varepsilon\, C$$

for sufficiently large k. Since ε can be chosen arbitrarily small, this completes the proof of Theorem 2.

References

[DG] J. Duistermaat and V. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.

[GO] V. Guillemin and K. Okikiolu, *Subprincipal terms in Szegö estimates*, Math. Res. Lett. **4** (1997), 173–179.

[H] L. Hörmander, *The analysis of linear partial differential operators.* IV, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1985.

[LS1] A. Laptev and Yu. Safarov, *A generalization of the Berezin–Lieb inequality*, Amer. Math. Soc. Transl. (2) **175** (1996), 67–79.

[LS2] ———, *Szegö type limit theorems*, J. Funct. Anal. **138** (1996), 544–559.

[SV] Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, Amer. Math. Soc., Providence, Rhode Island, 1996.

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