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ON CLASSIFICATION OF DYNAMICAL r-MATRICES

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Abstract. Using the gauge transformations of the Classical Dynamical Yang-Baxter Equation introduced by P. Etingof and A. Varchenko in [EV], we reduce the classification of dynamical r-matrices r on a commutative subalgebra $\mathfrak{l}$ of a Lie algebra $\mathfrak{g}$ to a purely algebraic problem, under some assumption on the symmetric part of r. We then describe, for a simple complex Lie algebra $\mathfrak{g}$, all non skew-symmetric dynamical r-matrices on a commutative subalgebra $\mathfrak{l} \subset \mathfrak{g}$ which contains a regular semisimple element. This interpolates results of P. Etingof and A. Varchenko ([EV], when $\mathfrak{l}$ is a Cartan subalgebra) and results of A. Belavin and V. Drinfeld for constant r-matrices ([BD]). This classification is similar, and in some sense simpler than the Belavin-Drinfeld classification.

1. The classical Yang-Baxter equation

Let $\mathfrak{g}$ be a Lie algebra. The CYBE is the following algebraic equation for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \tag{1}$$

Solutions of this equation are called r-matrices. In the theory of quantum groups, one is mainly interested in r-matrices satisfying

$$r + r^{21} \in (S^2\mathfrak{g})^{\mathfrak{g}}. \tag{2}$$

See [CP] for the links with the theory of quantum groups, and [Che] for links with Conformal Field Theory and the Wess-Zumino-Witten model on $\mathbb{P}^1$. The geometric interpretation of the CYBE was given by Drinfeld in terms of Poisson-Lie groups ([Dr1]).

2. The Belavin-Drinfeld classification

Notations. Let $\mathfrak{g}$ be a simple complex Lie algebra with a nondegenerate invariant form $(\, , \,)$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and Δ the root system. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ denote the root subspace associated to α. Let W be the Weyl group and s_α, $\alpha \in \Delta$ the reflection with respect to $\alpha^\perp$. Finally, let $\Omega \in S^2\mathfrak{g}$ and $\Omega_{\mathfrak{h}} \in S^2\mathfrak{h}$ be the inverse elements to the form $(\, , \,)$. Notice that $(S^2\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}\Omega$.

For any polarization $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, we denote by Π or $\Pi(\mathfrak{n}_+)$ the corresponding set of simple positive roots, by Δ_+ the set of positive roots and by $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h}$ the Borel subalgebras. For $\Gamma \subset \Pi$, set $\langle\Gamma\rangle = \mathbb{Z}\Gamma \cap \Delta$, and let $\mathfrak{g}_\Gamma$ be the subalgebra generated by $\mathfrak{g}_\alpha$, $\alpha \in \langle\Gamma\rangle$. We will write $\mathfrak{g}_\Gamma = \mathfrak{n}_+(\Gamma) \oplus \mathfrak{h}(\Gamma) \oplus \mathfrak{n}_-(\Gamma)$

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for the induced polarization and $W(\Gamma)$ for the subgroup of W generated by s_α, $\alpha \in \Gamma$. Let us fix a polarization of $\mathfrak{g}$.

Definition. A *Belavin-Drinfeld triple* is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$ and $\tau : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ is a norm-preserving bijection satisfying the following "nilpotency" condition:

"For any $\gamma_1 \in \Gamma_1$, there exists $n > 0$ such that $\tau^n(\gamma_1) \in \Gamma_2 \backslash \Gamma_1$".

Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin-Drinfeld triple. For each choice of Chevalley generators $(e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_i}$, $i = 1, 2$, the isomorphism τ induces a Lie algebra isomorphism $\mathfrak{g}_{\Gamma_1} \xrightarrow{\sim} \mathfrak{g}_{\Gamma_2}$ (by $e_\alpha \mapsto e_{\tau(\alpha)}$, $f_\alpha \mapsto f_{\tau(\alpha)}$, $h_\alpha \mapsto h_{\tau(\alpha)}$). Define a partial order on Δ_+ by setting $\alpha < \beta$ if there exists $n > 0$ such that $\tau^n(\alpha) = \beta$ (in particular, $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$).

Definition. A basis $(x_\alpha)_{\alpha \in \Delta}$ of $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is called *admissible* if $(x_\alpha, x_{-\alpha}) = 1$ and $\tau(x_\alpha) = x_{\tau(\alpha)}$ for $\alpha \in \langle \Gamma_1 \rangle$.

Theorem 1 (Belavin-Drinfeld). *Let $\mathfrak{g}$ be a simple complex Lie algebra.*
1. *Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin-Drinfeld triple, (x_α) an admissible basis, and let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ be such that*

$$r_0 + r_0^{21} = \Omega_{\mathfrak{h}}, \tag{3}$$

$$(\tau(\alpha) \otimes 1)r + (1 \otimes \alpha)r = 0 \qquad \text{for } \alpha \in \Gamma_1. \tag{4}$$

Then

$$r = r_0 + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_\alpha + \sum_{\alpha, \beta \in \Delta_+, \alpha < \beta} x_{-\alpha} \wedge x_\beta \tag{5}$$

is an r-matrix satisfying $r + r^{21} = \Omega$.
2. *Any r-matrix satisfying $r + r^{21} = \Omega$ is of the above type for a suitable polarization of $\mathfrak{g}$.*

This theorem is proved in [BD]. For instance, the standard r-matrix for a fixed polarization $r = \frac{\Omega_{\mathfrak{h}}}{2} + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_\alpha$ corresponds to $\Gamma_1 = \Gamma_2 = \emptyset$.

Remark. Skew-symmetric r-matrices admit a well known interpretation in terms of nondegenerate 2-cocycles on Lie subalgebras of $\mathfrak{g}$ ([Dr1]), but their classification is unavailable since it requires a classification of Lie subalgebras in $\mathfrak{g}$.

3. The dynamical Yang-Baxter equation

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{l} \subset \mathfrak{g}$ a subalgebra. An element $x \in \mathfrak{g} \otimes \mathfrak{g}$ will be called $\mathfrak{l}$-invariant if

$$[k \otimes 1 + 1 \otimes k, x] = 0 \qquad (\forall\, k \in \mathfrak{l}). \tag{6}$$

For $x \in \mathfrak{g}^{\otimes 3}$, we let $\mathrm{Alt}(x) = x^{123} + x^{231} + x^{312}$. Let $D \subset \mathfrak{l}^*$ be any open region.

The CDYBE is the following differential equation for a holomorphic $\mathfrak{l}$-invariant function $r: \quad D \to \mathfrak{g} \otimes \mathfrak{g}$:

$$\mathrm{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{7}$$

where the differential of r is considered as a holomorphic function

$$dr : D \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}, \qquad \lambda \mapsto \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda), \qquad (\lambda \in \mathfrak{l}^*),$$

for any basis (x_i) of $\mathfrak{l}$. In this case,

$$\mathrm{Alt}(dr) = \sum_i x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} + \sum_i x_i^{(2)} \frac{\partial r^{31}}{\partial x_i} + \sum_i x_i^{(3)} \frac{\partial r^{12}}{\partial x_i}.$$

The solutions to this equation are called *dynamical* r-matrices. Dynamical r-matrices which are relevant to the theory of quantum groups are those satisfying the following condition, analogous to (2):

$$\text{Generalized unitarity: } r(\lambda) + r^{21}(\lambda) \in (S^2\mathfrak{g})^{\mathfrak{g}}. \tag{8}$$

Remark. The CDYBE was first written down by G. Felder and C. Wiezcerkowski in connection with the Wess-Zumino-Witten model on elliptic curves ([FW]). The relation with elliptic quantum groups is explained in [Fe]. A geometric interpretation of the CDYBE analogous to the theory of Poisson-Lie groups for the CYBE is given in [EV].

4. Gauge transformations

We recall some results from [EV]. We suppose here that $\mathfrak{l}$ is commutative and we let D be the formal polydisc centered at the origin. Let G be a complex Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$, and let L be the connected subgroup of G such that $\mathrm{Lie}(L) = \mathfrak{l}$. Let G^L be the centralizer of L in G and $\mathfrak{g}^{\mathfrak{l}}$ its Lie algebra.We will denote by $(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{l}}$ the space of all $\mathfrak{l}$-invariant elements in $\mathfrak{g} \otimes \mathfrak{g}$.

Let $g : D \to G^L$ be any holomorphic function; the 1-form $\eta = g^{-1}dg$ gives rise to a function $\overline{\eta} : D \to \mathfrak{l} \otimes \mathfrak{g}^{\mathfrak{l}}$. If $r : D \to (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{l}}$ is an $\mathfrak{l}$-invariant function satisfying (8), we set

$$r^g = (g \otimes g)(r - \overline{\eta} + \overline{\eta}^{21})(g^{-1} \otimes g^{-1}).$$

Proposition 1. *The function r is a dynamical r-matrix if and only if the function r^g is.*

Thus the group $\mathrm{Map}(D, G^L)$ is a gauge transformation group for the CDYBE. Notice that this group is not commutative if G^L isn't.

Theorem 2. *Let $\rho, r : D \to \mathfrak{g}^{\otimes 2}$ be two dynamical r-matrices satisfying* (8) *such that $r(0) = \rho(0)$. There exists $g \in \mathrm{Map}(D, G^L)$ such that $\rho = r^g$.*

This shows that the space of dynamical r-matrices is, up to gauge equivalence, finite dimensional. Proofs of the above results can be found in [EV].

We will now prove a converse of Theorem 2 which reduces the CDYBE to a purely algebraic equation under some assumption on the symmetric part $\frac{\Omega}{2}$ of r: let $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$, let $\mathfrak{g}_\Omega$ be the ideal in $\mathfrak{g}$ generated by the components of Ω and denote by $\mathfrak{g}_\Omega = \bigoplus_\lambda \mathfrak{g}_\Omega(\lambda)$ the generalized weight space decomposition of $\mathfrak{g}_\Omega$ with respect to the adjoint action of $\mathfrak{l}$. The condition we will need is the following:

$$(*) \qquad \mathfrak{g}^{\mathfrak{l}} \text{ acts semisimply on } \mathfrak{g}_\Omega(0).$$

Suppose that (*) is fulfilled and let $z(\mathfrak{g}^{\mathfrak{l}})$ denote the center of $\mathfrak{g}^{\mathfrak{l}}$. Then we have a decomposition $\mathfrak{g}_\Omega(0) = z_0(\mathfrak{g}^{\mathfrak{l}}) \oplus V$ where $z_0(\mathfrak{g}^{\mathfrak{l}}) = z(\mathfrak{g}^{\mathfrak{l}}) \cap \mathfrak{g}_\Omega(0)$ and V is the sum of all non-trivial irreducible $\mathfrak{g}^{\mathfrak{l}}$-modules in $\mathfrak{g}_\Omega(0)$. It is clear that $\mathfrak{l} \cap V = \{0\}$. We will say that a complement $\mathfrak{l}'$ of $\mathfrak{l}$ in $\mathfrak{g}$ is admissible if $V \subset \mathfrak{l}'$, and write $\pi : \mathfrak{g} \to \mathfrak{l}$ for the projection along $\mathfrak{l}'$. Notice that by $\mathfrak{g}^{\mathfrak{l}}$-invariance of Ω,

$$\Omega \in S^2 z_0(\mathfrak{g}^{\mathfrak{l}}) \oplus S^2 V \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_\Omega(\lambda) \otimes \mathfrak{g}_\Omega(-\lambda). \tag{9}$$

We will denote by $CYB : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 3}$ the map:

$$r \mapsto [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

It is more convenient to work with the skew-symmetric part of r. If $r(\lambda) + r^{21}(\lambda) = \Omega \in (S^2(\mathfrak{g}))^{\mathfrak{g}}$, we set $s(\lambda) = r(\lambda) - \frac{\Omega}{2}$. It is easy to see that the CDYBE for r is equivalent to the following equation for s:

$$\mathrm{Alt}(ds) + CYB(s) + \frac{1}{4} CYB(\Omega) = 0. \tag{10}$$

Recall that as Ω is symmetric and invariant, $CYB(\Omega) = [\Omega_{13}, \Omega_{23}]$.

Theorem 3. *Let G be a complex Lie group and $L \subset G$ a connected commutative subgroup. Let $\mathfrak{g}, \mathfrak{l}, \mathfrak{g}^{\mathfrak{l}}$ denote the Lie algebras of G, L and G^L. Let $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$. Then*

1. *Let $\mathfrak{l}'$ be any complement of $\mathfrak{l}$ in $\mathfrak{g}$. Any dynamical r-matrix $r(\lambda)$ on $\mathfrak{l}$ such that $r(\lambda) + r^{21}(\lambda) = \Omega$ is gauge equivalent to a dynamical r-matrix $\tilde{r}(\lambda)$ such that $\tilde{r}(0) \in \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$.*
2. *Suppose that condition $(*)$ is true and let $\mathfrak{l}'$ be any admissible complement of $\mathfrak{l}$ in $\mathfrak{g}$. Let $r_0 \in \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$ satisfy*

$$CYB(r_0) \in \mathrm{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g}), \tag{11}$$

such that $s_0 = r_0 - \frac{\Omega}{2}$ *is a regular point of the algebraic manifold*

$$M_\Omega = \{s \in (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}} \,|\, CYB(s + \frac{\Omega}{2}) \in \mathrm{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})\}.$$

Then there exists a dynamical r-matrix $r(\lambda) : D \to \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$ *such that* $r(0) = r_0$.

The condition (*) is satisfied in the following two interesting special cases: when $\Omega = 0$ (triangular case) or when $\mathfrak{g}^{\mathfrak{l}}$ acts semisimply on $\mathfrak{g}$ (for instance, G is reductive and L is contained in a maximal torus of G or more generally, if G^L is reductive).

The proof of this theorem will occupy the rest of this section. Let us first prove part 1:

Lemma 1. *Any dynamical r-matrix such that* $r(\lambda) + r^{21}(\lambda) = \Omega$ *is gauge-equivalent to a dynamical r-matrix* $\tilde{r}(\lambda)$ *such that* $\tilde{r}(0) \in \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}')^{\mathfrak{l}}$.

Proof. Let $\overline{\eta} \in \mathfrak{l} \otimes \mathfrak{g}^{\mathfrak{l}}$ be such that $r(0) - \overline{\eta} + \overline{\eta}^{21} \in \frac{\Omega}{2} + \Lambda^2(\mathfrak{l}')$. There exists a function $g : D \to G^L$ such that $g^{-1}dg(0) = \eta$ (see [EV], Lemma 1.3). It is easy to see that $\tilde{r} = r^g$ satisfies the desired conditions. □

Let us now prove part 2. We will interpret the CDYBE (10) as a consistent system of differential equations defined on M_Ω.

For $s \in M_\Omega$, (10) is equivalent to

$$(\pi \otimes 1 \otimes 1)\,\mathrm{Alt}(ds) = -(\pi \otimes 1 \otimes 1)(CYB(s) + \frac{1}{4}CYB(\Omega)).$$

This reduces to

$$ds = -(\pi \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega)), \tag{12}$$

or, in coordinates (x_i), where (x_i) is a basis of $\mathfrak{l}$,

$$\frac{\partial s}{\partial x_i} = -(x_i \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega)).$$

Lemma 2. *The system* (12) *is consistent.*

Proof. Set $X : M_\Omega \to \mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g},\ s \mapsto (\pi \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega))$. By definition, the curvature of (12) is given by

$$\begin{aligned}
&\sum_{i,j} x_i \otimes x_j \otimes \big(\frac{\partial^2 s}{\partial x_i \partial x_j} - \frac{\partial^2 s}{\partial x_j \partial x_i}\big) \\
&\quad = (\pi \otimes \pi \otimes 1 \otimes 1)\big(\big\{[s^{23}, [s^{12}, s^{14}]] + [s^{23}, \frac{1}{4}CYB(\Omega)^{124}] \\
&\quad + [[s^{12}, s^{13}], s^{24}] + [\frac{1}{4}CYB(\Omega)^{123}, s^{24}]\big\} \\
&\quad - \big\{[s^{13}, [s^{21}, s^{24}]] + [s^{13}, \frac{1}{4}CYB(\Omega)^{214}] \\
&\quad + [[s^{21}, s^{23}], s^{14}] + [\frac{1}{4}CYB(\Omega)^{213}, s^{14}]\big\}\big) \\
&\quad = (\pi \otimes \pi \otimes 1 \otimes 1)\big(\big\{[s^{23}, [s^{12}, s^{14}]] \\
&\quad + [[s^{12}, s^{13}], s^{24}] - [s^{13}, [s^{21}, s^{24}]] - [[s^{21}, s^{23}], s^{14}]\big\} \\
&\quad + \frac{1}{4}\big\{[s^{13} + s^{23}, CYB(\Omega)^{124}] - [s^{14} + s^{24}, CYB(\Omega)^{123}]\big\}\big).
\end{aligned}$$

By the Jacobi identity,

$$[s^{23}, [s^{12}, s^{14}]] = [[s^{21}, s^{23}], s^{14}], \qquad [[s^{12}, s^{13}], s^{24}] = [s^{13}, [s^{21}, s^{24}]].$$

By $\mathfrak{g}$-invariance of $CYB(\Omega)$, we have

$$\begin{aligned}
[s^{13} + s^{23}, CYB(\Omega)^{124}] &= [s^{34}, CYB(\Omega)^{124}], \\
[s^{14} + s^{24}, CYB(\Omega)^{123}] &= -[s^{34}, CYB(\Omega)^{123}].
\end{aligned}$$

Overall, we have the following expression for the curvature of (12):

$$\frac{1}{4}(\pi \otimes \pi \otimes 1 \otimes 1)([CYB(\Omega)^{123} + CYB(\Omega)^{124}, s^{34}] = \frac{1}{4}[(\pi \otimes \pi \otimes 1)CYB(\Omega), s]$$

But (9) and the fact that $\mathfrak{l}'$ is admissible imply that $(\pi \otimes \pi \otimes 1)CYB(\Omega) = 0$. Thus, (12) is consistent. □

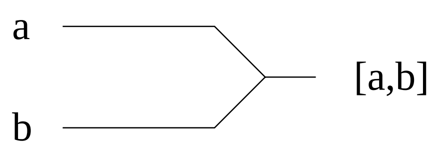
Lemma 3. *The system* (12) *is defined on* M_Ω*, i.e the vector fields defined by* (12) *are tangent to* M_Ω*.*

Proof. Let $x^* \in \mathfrak{l}^* \overset{\pi^*}{\hookrightarrow} \mathfrak{g}^*$, and set $h = (x^* \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega))$. Since $s \in \Lambda^2(\mathfrak{l}')$ we have $(x^* \otimes 1 \otimes 1)[s^{12}, s^{13}] \in \Lambda^2(\mathfrak{l}')$. Moreover, the admissibility of $\mathfrak{l}'$ and (9) together imply that $(x^* \otimes 1 \otimes 1)(CYB(\Omega)) \in (\Lambda^2\mathfrak{l}')^{\mathfrak{l}}$ since $[\mathfrak{l} \otimes 1, S^2 z_0(\mathfrak{g}^{\mathfrak{l}})] = 0$. Thus $h \in \Lambda^2\mathfrak{l}'$.

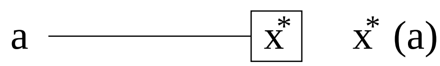
To conclude the proof of Lemma 3 and Theorem 3, we now show that

$$\begin{aligned}
&[s^{12}, h^{13}] + [s^{12}, h^{23}] + [s^{13}, h^{23}] \\
&\quad + [h^{12}, s^{13}] + [h^{12}, s^{23}] + [h^{13}, s^{23}] \in \mathrm{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g}).
\end{aligned} \tag{13}$$

To make the presentation more clear, we will use the pictorial technique to represent expressions and make computations: we associate to each morphism from a n-tensor to a m-tensor a diagram in the following way: the operation of taking the commutator is represented by

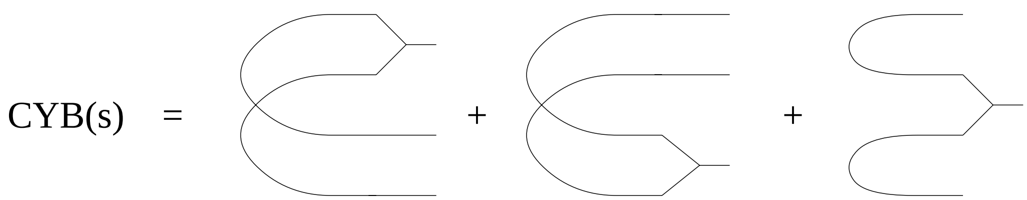


Applying a linear form x^* will be denoted by

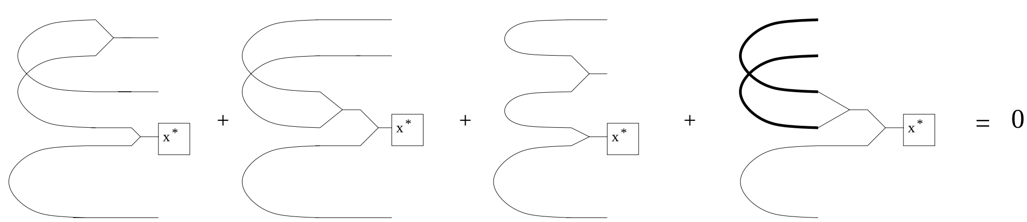


Finally, we will represent s and $\frac{\Omega}{2}$, which can be thought of as maps from a 0-tensor to a 2-tensor, by

For instance,



Lemma 4. *We have* $x^{*(3)}[CYB(s+\frac{\Omega}{2})^{123}, s^{34}] \in \mathrm{Alt}(\mathfrak{l}\otimes\mathfrak{g}\otimes\mathfrak{g})$ *or, in pictures* (*modulo* $\mathrm{Alt}(\mathfrak{l}\otimes\mathfrak{g}\otimes\mathfrak{g})$)



Proof. Recall that $CYB(s+\frac{\Omega}{2}) \in \mathrm{Alt}(\mathfrak{l}\otimes\mathfrak{g}\otimes\mathfrak{g})$. Thus the only part of the above expression which can lie outside of $\mathrm{Alt}(\mathfrak{l}\otimes\mathfrak{g}\otimes\mathfrak{g})$ is obtained from the $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{l}$-part of $CYB(s)$. But if $y\in\mathfrak{l}$,

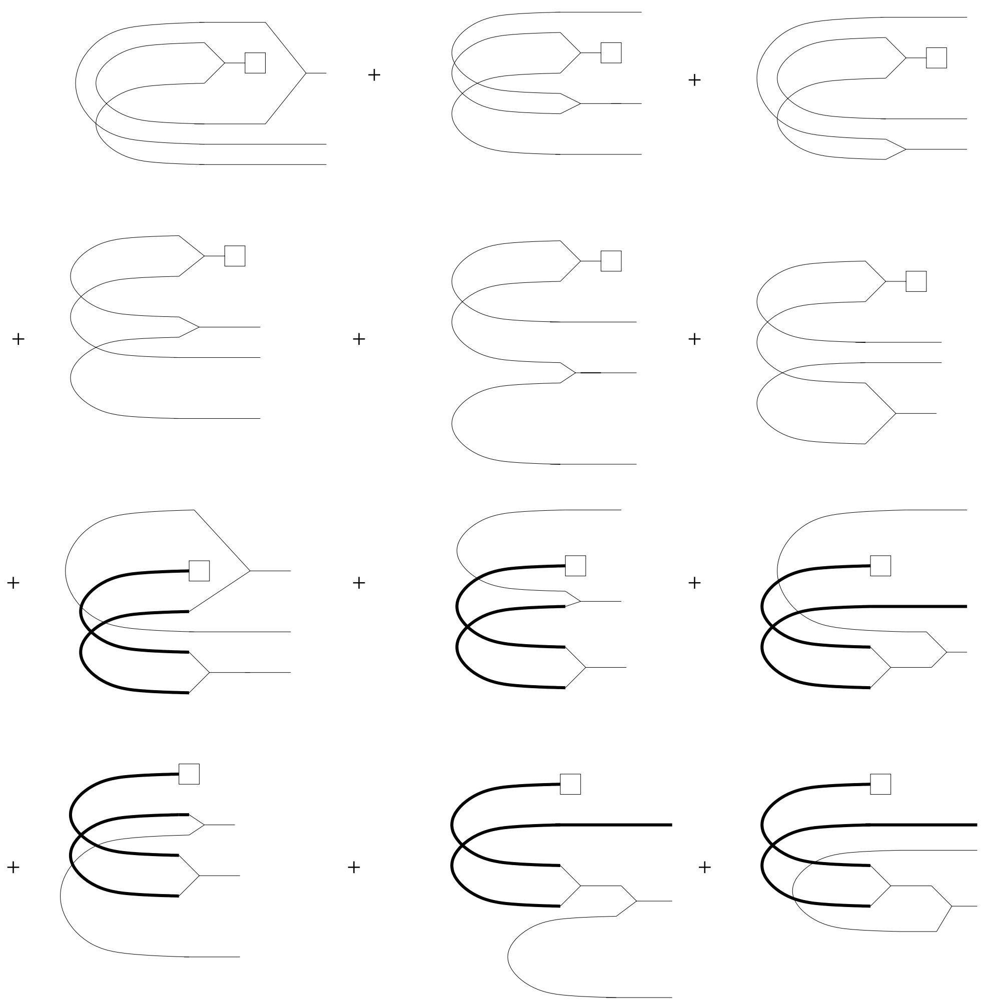
$$(x^*\otimes 1)[y\otimes 1, s] = -(x^*\otimes 1)[1\otimes y, s]$$

by $\mathfrak{l}$-invariance of s. This last expression is zero since $s \in (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$. Lemma 4 is proved. $\square$

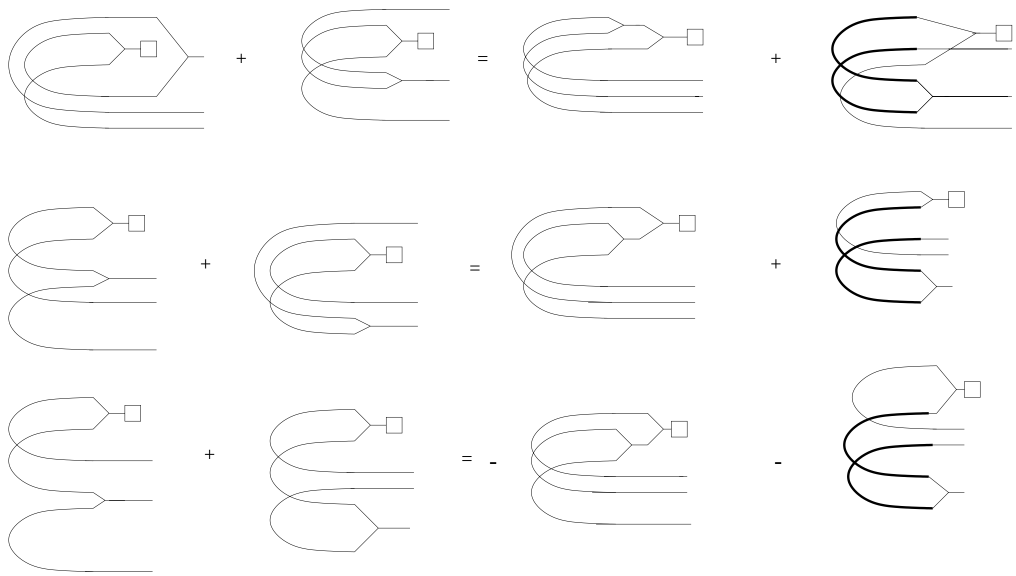
It is clear how to generalize Lemma 4 to other expressions of the form

$$x^{*(k)}[CYB(s+\frac{\Omega}{2})^{123}, s^{k4}].$$

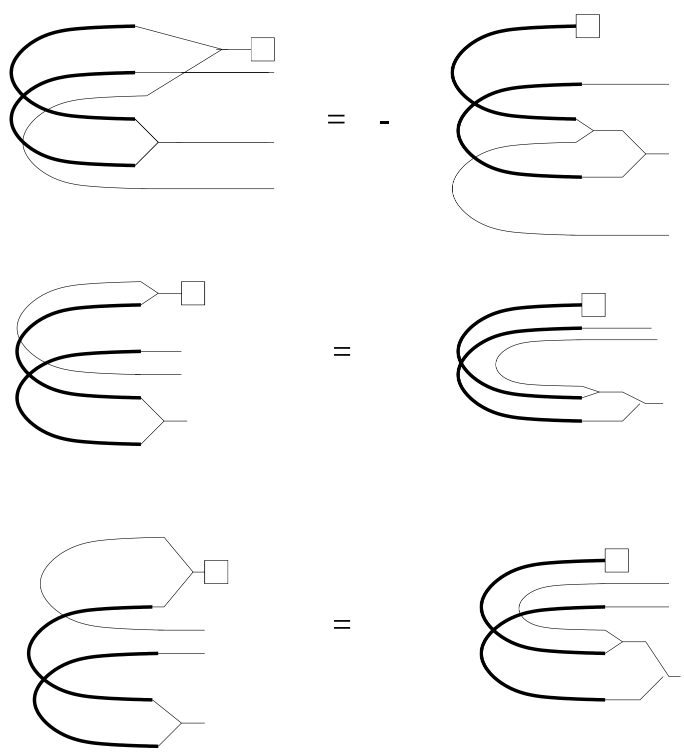
Now, (13) can be drawn as



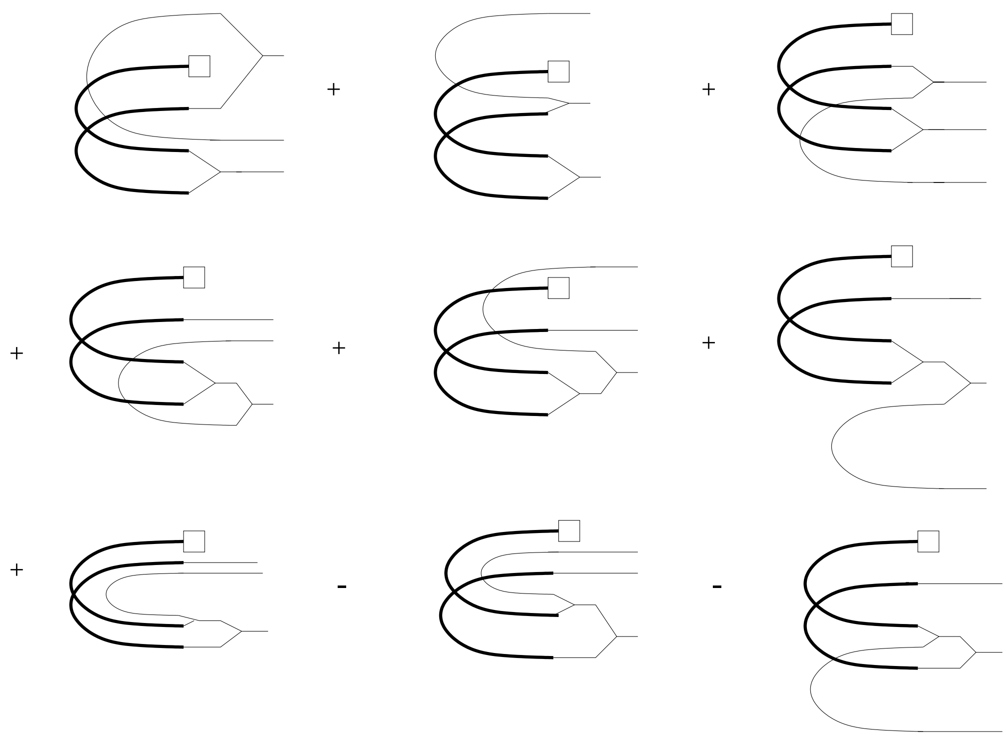
but by Lemma (4) we have, modulo $\mathrm{Alt}(\mathfrak{l}\otimes\mathfrak{g}\otimes\mathfrak{g})$,



It is easy to check that the sum of the terms of type $[CYB(s), s]$ in this last expression is zero by the Jacobi identity. Moreover, by $\mathfrak{g}$-invariance of Ω, we have



Thus, modulo $\mathrm{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$, (13) reduces to



The sums of terms in each column is zero by Jacobi Identity. This concludes the proof of Theorem 3. $\square$

5. Classification of dynamical r-matrices

Let $\mathfrak{g}$ be a simple Lie algebra and let $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ be the Casimir element. In that case, (8) becomes

$$r(\lambda) + r^{21}(\lambda) = \epsilon\Omega. \tag{14}$$

We will classify all solutions of equations (6,7,14) when $\epsilon \neq 0$ and when $\mathfrak{l}$ contains a semisimple regular element. In particular, in this case, the centralizer $\mathfrak{h}$ of $\mathfrak{l}$ is the unique Cartan subalgebra containing $\mathfrak{l}$. Notice that we can assume that $\epsilon = 1$ (since the assignement $r(\lambda) \to \epsilon r(\epsilon\lambda)$ is a gauge transformation of (7)). We can also assume that the restriction of $(\, , \,)$ to $\mathfrak{l}$ is nondegenerate. Indeed, for any dynamical r-matrix, we can replace $\mathfrak{l}$ by the largest subspace of $\mathfrak{h}$ for which r is invariant, and such a subspace is real. Let $\mathfrak{h}_0$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{h}$ and let $i : \mathfrak{l} \hookrightarrow \mathfrak{h}$ be the inclusion map. We will also write $(\, , \,)$ for the induced bracket on $\mathfrak{l}^*$. Let $\Omega_{\mathfrak{h}_0}$ denote the Casimir element of the restriction of $(\, , \,)$ to $\mathfrak{h}_0$.

5.1. Statement of the theorem. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a polarization of $\mathfrak{g}$.

Definition. A generalized Belavin-Drinfeld triple is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$, and $\tau : \Gamma_1 \tilde{\to} \Gamma_2$ is a norm-preserving bijection.

In other terms, in a generalized Belavin-Drinfeld triple, we drop the nilpotency condition. We will say that a generalized Belavin-Drinfeld triple is $\mathfrak{l}$-graded if τ preserves the decomposition of $\mathfrak{g}$ in $\mathfrak{l}$-weight spaces. If $(\Gamma_1, \Gamma_2, \tau)$ is a generalized Belavin-Drinfeld triple, we will denote by Γ_3 the largest subset of $\Gamma_1 \cap \Gamma_2$ which is stable under τ, and $\tilde{\Gamma}_1 = \Gamma_1 \backslash \Gamma_3$, $\tilde{\Gamma}_2 = \Gamma_2 \backslash \Gamma_3$. It is clear that $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tau)$ is a Belavin-Drinfeld triple. As before, for each choice of Chevalley generators $(e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_i}$, the map τ induces isomorphisms $\mathfrak{g}_{\tilde{\Gamma}_1} \to \mathfrak{g}_{\tilde{\Gamma}_2}$ and $\tau : \mathfrak{g}_{\Gamma_3} \to \mathfrak{g}_{\Gamma_3}$.

For $\lambda \in \mathfrak{l}^*$, consider the map:

$$\begin{aligned} K(\lambda) : \mathfrak{n}_+(\Gamma_1) &\to \mathfrak{n}_+(\Gamma_2) \\ e_\alpha &\mapsto \frac{1}{2} e_\alpha + e^{-(\alpha,\lambda)} \frac{\tau}{1 - e^{-(\alpha,\lambda)}\tau}(e_\alpha). \end{aligned}$$

Notice that we have

$$K(\lambda)(e_\alpha) = \frac{1}{2} e_\alpha + \sum_{n>0} e^{-n(\alpha,\lambda)} \tau^n(e_\alpha).$$

This sum is finite for $\alpha \notin \langle \Gamma_3 \rangle$.

Theorem 4. *Let $\mathfrak{g}$ be a simple Lie algebra with nondegenerate invariant bilinear form $(\,,\,)$, $\mathfrak{l} \subset \mathfrak{g}$ a commutative subalgebra containing a regular semisimple element on which $(\,,\,)$ is nondegenerate, $\mathfrak{h}$ the Cartan subalgebra containing $\mathfrak{l}$ and $\mathfrak{h}_0$ the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{h}$. Then*
1. *Any dynamical r-matrix is gauge-equivalent to a dynamical r-matrix $\tilde{r}$ such that*

$$\tilde{r}(\lambda) - \tilde{r}(\lambda)^{21} \in (\mathfrak{l}^\perp)^{\otimes 2} = (\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \oplus \mathfrak{h}_0)^{\otimes 2}. \tag{15}$$

2. *Let $(\Gamma_1, \Gamma_2, \tau)$ be an $\mathfrak{l}$-graded generalized Belavin-Drinfeld triple and let $(e_\alpha, f_\alpha, h_\alpha)_{\Gamma_i}$ be a choice of Chevalley generators. Let $r_{\mathfrak{h}_0,\mathfrak{h}_0} \in \mathfrak{h}_0 \otimes \mathfrak{h}_0$ satisfy the equation*

$$(\tau(\alpha) \otimes 1) r_{\mathfrak{h}_0,\mathfrak{h}_0} + (1 \otimes \alpha) r_{\mathfrak{h}_0,\mathfrak{h}_0} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{\mathfrak{h}_0}. \tag{16}$$

Then

$$r(\lambda) = \frac{1}{2}\Omega + r_{\mathfrak{h}_0,\mathfrak{h}_0} + \sum_{\alpha \in \langle \Gamma_1 \rangle \cap \Delta_+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta_+,\, \alpha \notin \langle \Gamma_1 \rangle} \frac{1}{2} e_\alpha \wedge e_{-\alpha}$$

is a solution the CDYBE satisfying (15).
3. *Any solution of the CDYBE satisfying* (15) *is of the above type for a suitable polarization of $\mathfrak{g}$.*

The proof of this theorem will occupy the rest of this section. Our methods are greatly inspired by the paper [BD]. Notice that 1. follows from Theorem 3, but we will describe the gauge transformations explicitely in this case.

Notations. Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and set $\Delta_{\mathfrak{l}} = i^*(\Delta) \subset \mathfrak{l}^*$. We will denote by $\mathfrak{g}_{\bar{\alpha}}$ the weight subspace associated to $\bar{\alpha} = i^*(\alpha) \in \Delta_{\mathfrak{l}}$, and we set $\mathfrak{g}_{\overline{0}} = \mathfrak{h}_0$. It is clear that

$$\mathfrak{g}_{\overline{\alpha}} = \bigoplus_{\beta \in \Delta,\, i^*(\beta)=\overline{\alpha}} \mathfrak{g}_\beta$$

In particular, $(\,,\,)$ is a pairing $\mathfrak{g}_{\overline{\alpha}} \times \mathfrak{g}_{-\overline{\alpha}} \to \mathbb{C}$.

A vector space $V \subset \mathfrak{g}$ will be called $\mathfrak{h}$-graded (resp. $\mathfrak{l}$-graded) if it is an $\mathfrak{h}$-submodule (resp. $\mathfrak{l}$-submodule) of $\mathfrak{g}$. Finally, let $\Omega' \in (\mathfrak{l}^\perp)^{\otimes 2}$ denote the Casimir (inverse element) of the restriction of $(\,,\,)$ to $\mathfrak{l}^\perp = \mathfrak{h}_0 \bigoplus \mathfrak{g}_{\overline{\alpha}}$.

Now let $r : \mathfrak{l}^* \supset D \to (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{l}}$ be a formal power series satisfying (14) (with $\epsilon = 1$). By (6), we can write

$$r(\lambda) = \frac{1}{2}\Omega + r_{\mathfrak{l},\mathfrak{l}}(\lambda) + r_{\mathfrak{l},\mathfrak{h}_0}(\lambda) + r_{\mathfrak{h}_0,\mathfrak{l}}(\lambda) + (\varphi(\lambda) \otimes 1)\Omega', \tag{17}$$

where $r_{\mathfrak{l},\mathfrak{l}}(\lambda) \in \mathfrak{l} \otimes \mathfrak{l}$, $r_{\mathfrak{l},\mathfrak{h}_0}(\lambda) \in \mathfrak{l} \otimes \mathfrak{h}_0$, $r_{\mathfrak{h}_0,\mathfrak{l}}(\lambda) \in \mathfrak{h}_0 \otimes \mathfrak{l}$ and where $\varphi(\lambda) \in \mathrm{End}\left(\mathfrak{h}_0 \bigoplus \mathfrak{g}_{\overline{\alpha}}\right)$ is a sum of maps $\varphi_{\overline{\alpha}}(\lambda) \in \mathrm{End}\,(\mathfrak{g}_{\overline{\alpha}})$. By the unitarity condition, $r_{\mathfrak{l},\mathfrak{l}}(\lambda) \in \Lambda^2\mathfrak{l}$, $r_{\mathfrak{l},\mathfrak{h}_0}(\lambda) = -r^{21}_{\mathfrak{h}_0,\mathfrak{l}}(\lambda)$ and $\varphi_{-\overline{\alpha}}(\lambda) = -\varphi^*_{\overline{\alpha}}(\lambda)$.

With these notations, the CDYBE splits into 4 components: the $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}$-part, the $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{h}_0$-part, the $\mathfrak{l} \otimes \mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{-\bar{\alpha}}$-part and the $\mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{\bar{\beta}} \otimes \mathfrak{g}_{\bar{\gamma}}$-part where $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$.

- The $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}$-part: let us set $r_{\mathfrak{l},\mathfrak{l}} = \sum_{i,j} C_{i,j}(\lambda) x_i \otimes x_j$. This part of the CDYBE can then be written:

$$\frac{\partial C_{j,k}}{\partial x_i} + \frac{\partial C_{k,i}}{\partial x_j} + \frac{\partial C_{i,j}}{\partial x_k} = 0 \qquad \forall\, i,j,k \tag{18}$$

  and says that $\sum_{i,j} C_{i,j} dx_i \wedge dx_j$ is a closed 2-form.
- The $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{h}_0$-part: let us set $r_{\mathfrak{l},\mathfrak{h}_0} = \sum_{i,j} D_{i,j}(\lambda) x_i \otimes y_j$ for some basis (y_j) of $\mathfrak{h}_0$. This part of the CDYBE is

$$\frac{\partial D_{i,j}}{\partial x_k} = \frac{\partial D_{k,j}}{\partial x_i} \qquad \forall\, i,k,j \tag{19}$$

  and says that for any j, $\sum_i D_{i,j}(\lambda) dx_i$ is a closed 1-form.

Since r is defined on a polydisc, the above forms are exact. Let $f : D \to \mathfrak{h}_0$ be such that $df(\lambda) = \sum_i D_{i,j}(\lambda) dx_i \otimes y_j$ and let ξ be a 1-form on D such that $d\xi = \sum_{i,j} C_{i,j} dx_i \wedge dx_j$. Then ξ defines a function $\overline{\xi} : D \to \mathfrak{l}$. The gauge transformation which should be applied to r to make it satisfy (15) is easily seen to be the following: $r(\lambda) \mapsto r(\lambda)^g = \frac{1}{2}\Omega + (e^{-ad\, f(\lambda)}\varphi(\lambda)e^{ad\, f(\lambda)} \otimes 1)\Omega'$ where $g(\lambda) = e^{f(\lambda)} e^{-\overline{\xi}(\lambda)}$.

Thus, we can assume that $r_{\mathfrak{l},\mathfrak{l}} = r_{\mathfrak{l},\mathfrak{h}_0} = 0$, in which case the remaining components of the CDYBE can be written in the following way:

- The $\mathfrak{l} \otimes \mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{-\bar{\alpha}}$-part:

$$d\varphi_{\bar{\alpha}} + (\varphi_{\bar{\alpha}}^2 - \frac{1}{4})dh_{\bar{\alpha}} = 0. \tag{20}$$

  In particular, $r_{\mathfrak{h}_0,\mathfrak{h}_0} \in \Lambda^2\mathfrak{h}_0$ is constant.
- The $\mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{\bar{\beta}} \otimes \mathfrak{g}_{\bar{\gamma}}$-part where $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$:

$$\Lambda\big(\varphi_{\bar{\alpha}} \otimes \varphi_{\bar{\beta}} \otimes 1 + \varphi_{\bar{\alpha}} \otimes 1 \otimes \varphi_{\bar{\gamma}} + 1 \otimes \varphi_{\bar{\beta}} \otimes \varphi_{\bar{\gamma}} + \frac{1}{4}Id\big) = 0 \tag{21}$$

  where $\Lambda : \ \mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{\bar{\beta}} \otimes \mathfrak{g}_{\bar{\gamma}} \to \mathbb{C}$, $x \otimes y \otimes z \mapsto ([x, y], z)$.

This set of equations is sufficient by skew-symmetry of the CDYBE.

5.2. The Cayley transform. Let us set $A_\pm = \mathrm{Im}\,(\varphi(\lambda) \pm \frac{1}{2})$, $I_\pm = \mathrm{Ker}\,(\varphi(\lambda) \mp \frac{1}{2})$. Notice that, by (20), $A_\pm$ and $I_\pm$ are indeed independent of λ. Furthermore, $A_\pm$, $I_\pm$ are $\mathfrak{l}$-graded by the weight-zero condition, $I_\pm \subset A_\pm$ and $A_\pm = I_\pm^\perp$ by the unitarity condition. Notice also that $A_+ + A_- \oplus \mathfrak{l} = \mathfrak{g}$. Now consider

$$\psi(\lambda) = \frac{\varphi - \frac{1}{2}}{\varphi + \frac{1}{2}} : \ A_+/I_+ \to A_-/I_-.$$

Extend $\psi(\lambda)$ to $\psi(\lambda) : \mathfrak{l} \oplus A_+/I_+ \to \mathfrak{l} \oplus A_-/I_-$ by setting $\psi_{|\mathfrak{l}} = Id$. It is clear that ψ is a well-defined linear isomorphism. The following proposition is crucial:

Proposition 2. *The maps $\varphi_{\bar{\alpha}}$ satisfy* $(20, 21)$ *if and only if the following hold*:

(i) *$A_\pm \oplus \mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ and $I_\pm \oplus \mathfrak{l}$ is an ideal of $A_\pm \oplus \mathfrak{l}$.*
(ii) *there exists a* (*constant*) *map $\psi_0 : \ \mathfrak{l} \oplus A_+/I_+ \to \mathfrak{l} \oplus A_-/I_-$ such that $\psi(\lambda)_{|\mathfrak{g}_{\bar{\alpha}}} = e^{-(\bar{\alpha},\lambda)}\psi_{0|\mathfrak{g}_{\bar{\alpha}}}$.*
(iii) *The map ψ_0 is a Lie algebra map*:

$$[\psi_0(x), \psi_0(y)] = \psi_0[x, y]. \tag{22}$$

Proof. Assume that φ satisfies (20,21) and let $a \in \mathfrak{g}_{\bar{\alpha}}$, $b \in \mathfrak{g}_{\bar{\beta}}$, $c \in \mathfrak{g}_{\bar{\gamma}}$ with $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$. From (21), we have

$$\big([(\varphi_{\bar{\alpha}} + \frac{1}{2})a, (\varphi_{\bar{\beta}} + \frac{1}{2})b], c\big) + \big([a, (\varphi_{\bar{\beta}} + \frac{1}{2})b], (\varphi_{\bar{\gamma}} - \frac{1}{2})c\big) + \big([(\varphi_{\bar{\alpha}} - \frac{1}{2})a, b], (\varphi_{\bar{\gamma}} - \frac{1}{2})c\big) = 0.$$

Since $\varphi_{\bar{\gamma}} = -\varphi^*_{-\bar{\gamma}}$, and $(\ ,\ )$ is a nondegenerate pairing $\mathfrak{g}_{\bar{\gamma}} \otimes \mathfrak{g}_{-\bar{\gamma}} \to \mathbb{C}$, this implies that $A_+ \oplus \mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}$. Note that the term in $\mathfrak{l}$ is necessary here since $[\mathfrak{g}_{\bar\alpha},\mathfrak{g}_{-\bar\alpha}]\not\subset\mathfrak{g}_{\overline{0}}=\mathfrak{h}_0$, but is not consequential as A_+ is $\mathfrak{l}$-graded. The second claim of (i) follows from the relation

$$\big([(\varphi_{\bar\alpha}-\frac{1}{2})a,(\varphi_{\bar\beta}-\frac{1}{2})b],c\big)+\big([a,(\varphi_{\bar\beta}+\frac{1}{2})b],(\varphi_{\bar\gamma}+\frac{1}{2})c\big)$$
$$+\big([(\varphi_{\bar\alpha}-\frac{1}{2})a,b],(\varphi_{\bar\gamma}+\frac{1}{2})c\big)=0.$$

The proof is the same for A_- and I_-. The equivalence of (ii) and (20) follows from the equality

$$\begin{aligned}d\psi_{|\mathfrak{g}_{\bar\alpha}}&=\frac{d\varphi_\alpha(\varphi_\alpha+\frac{1}{2})-(\varphi_\alpha-\frac{1}{2})d\varphi_\alpha}{(\varphi_\alpha+\frac{1}{2})^2}\\&=-\frac{(\varphi_\alpha^2-\frac{1}{4})}{(\varphi_\alpha+\frac{1}{2})^2}dh_{\bar\alpha}\\&=-(\bar\alpha,\lambda)\psi_{|\mathfrak{g}_{\bar\alpha}}.\end{aligned}$$

where we used (20). Finally it follows from (21) that

$$(\varphi_{\overline{\alpha+\beta}}-\frac{1}{2})\big([(\varphi_{\bar\alpha}+\frac{1}{2})a,(\varphi_{\bar\beta}+\frac{1}{2})b]\big)=(\varphi_{\overline{\alpha+\beta}}+\frac{1}{2})\big([(\varphi_{\bar\alpha}-\frac{1}{2})a,(\varphi_{\bar\beta}-\frac{1}{2})b]\big).$$

This implies (iii).

Conversely, if (i-iii) are satisfied then for any $x\in\mathfrak{g}_{\bar\alpha}$, $y\in\mathfrak{g}_{\bar\beta}$ $(\bar\alpha+\bar\beta\neq 0)$ there exist $z\in\mathfrak{g}_{\overline{\alpha+\beta}}$ such that $[(\varphi_{\bar\alpha}-\frac{1}{2})x,(\varphi_{\bar\beta}-\frac{1}{2})y]=(\varphi_{\overline{\alpha+\beta}}-\frac{1}{2})z$. Since ψ is a Lie algebra map, $[(\varphi_{\bar\alpha}+\frac{1}{2})x,(\varphi_{\bar\beta}+\frac{1}{2})y]-(\varphi_{\overline{\alpha+\beta}}+\frac{1}{2})z\in\mathrm{Ker}\,(\varphi_{\overline{\alpha+\beta}}-\frac{1}{2})$. Subtracting, we obtain $[(\varphi_{\bar\alpha}+\frac{1}{2})x,y]+[x,(\varphi_{\bar\beta}+\frac{1}{2})y]-[x,y]-z\in\mathrm{Ker}\,(\varphi_{\overline{\alpha+\beta}}-\frac{1}{2})$. Applying $(\varphi-\frac{1}{2})$ and dropping the indices, we have

$$(\varphi-\frac{1}{2})\Big([(\varphi+\frac{1}{2})x,y]+[x,(\varphi+\frac{1}{2})y]-[x,y]\Big)=[(\varphi-\frac{1}{2})x,(\varphi-\frac{1}{2})y].$$

Thus,

$$[(\varphi+\frac{1}{2})x,(\varphi+\frac{1}{2})y]-(\varphi+\frac{1}{2})\Big([(\varphi-\frac{1}{2})x,y]+[x,(\varphi+\frac{1}{2})y]\Big)=0.$$

which is equivalent to (21). □

We will call the triple (A_+,A_-,ψ_0) the Cayley transform of φ. We are now reduced to the classification of all triples satisfying (i-iii) and which arise as a Cayley transform (Cayley triples).

5.3. Classification of Cayley triples. Let (A_+,A_-,ψ_0) be a Cayley triple. If $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$ is a polarization of $\mathfrak{g}$ and $\Gamma\subset\Pi(\mathfrak{n}_+)$ we will denote by $\mathfrak{q}_\Gamma^+$ (resp. $\mathfrak{q}_\Gamma^-$) the subalgebra generated by $\mathfrak{n}_+$ and $\mathfrak{g}_{-\alpha}$, $\alpha\in\Gamma$ (resp. generated by $\mathfrak{n}_-$ and $\mathfrak{g}_\alpha$, $\alpha\in\Gamma$). We denote by $\mathfrak{p}_\Gamma^\pm=\mathfrak{h}+\mathfrak{q}_\Gamma^\pm$ the parabolic subalgebras associated to Γ.

Proposition 3. *There exists a polarization* $\mathfrak{g} = \mathfrak{n}^1_+ \oplus \mathfrak{h} \oplus \mathfrak{n}^1_-$*, two subsets* $\Gamma_+, \Gamma_- \subset \Pi(\mathfrak{n}^1_+)$ *and two vector spaces* $V_+, V_- \subset \mathfrak{h}$ *with* $V_\pm^\perp \subset V_\pm$ *such that*

$$\mathfrak{l} \oplus A_+ = \mathfrak{q}^+_{\Gamma_+} \oplus V_+, \qquad \mathfrak{l} \oplus A_- = \mathfrak{q}^-_{\Gamma_-} \oplus V_-.$$

Proof. Notice that $(\mathfrak{l}\oplus A_+)^\perp = I_+ \subset \mathfrak{l}\oplus A_+$. It is known, (c.f [Bou, chap.VIII,§10, Thm. 1] or [BD]), that this implies that $\mathfrak{l} \oplus A_+ = \tilde{\mathfrak{q}}^+_\Gamma \oplus \tilde{V}_+$ for *some* polarization $\mathfrak{g} = \mathfrak{n}'_+ \oplus \mathfrak{h}' \oplus \mathfrak{n}'_-$. Similarly, $\mathfrak{l} \oplus A_- = \tilde{\mathfrak{q}}^-_{\Gamma'} \oplus \tilde{V}_-$ for some polarization $\mathfrak{g} = \mathfrak{n}''_+ \oplus \mathfrak{h}'' \oplus \mathfrak{n}''_-$. Moreover, $\mathfrak{l}$ acts semisimply on $A_\pm$ so $\mathfrak{l} \subset \mathfrak{h}'$, $\mathfrak{l} \subset \mathfrak{h}''$. But $\mathfrak{l}$ contains a regular element, thus $\mathfrak{l} = \mathfrak{h}' = \mathfrak{h}''$. Proposition 3 is now an easy consequence of the following lemma:

Lemma 5. *Let* $\mathfrak{g}$ *be a simple Lie algebra and* $\mathfrak{h}$ *a Cartan subalgebra. Let* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *be two parabolic subalgebras containing* $\mathfrak{h}$ *such that* $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{g}$*. Then there exists a polarization* $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ *and* $\Gamma_+, \Gamma_- \subset \Pi$ *such that* $\mathfrak{a}_1 = \mathfrak{p}^+_{\Gamma_+}$ *and* $\mathfrak{a}_2 = \mathfrak{p}^-_{\Gamma_-}$.

Proof. Let $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a polarization of $\mathfrak{g}$ such that $\mathfrak{b}_+ \subset \mathfrak{a}_1$ and for which $\dim(\mathfrak{n}_+ \cap \mathfrak{a}_2)$ is minimal. We claim that $\mathfrak{b}_- \subset \mathfrak{a}_2$. Suppose on the contrary that there exists a simple root $\alpha \in \Pi$ such that $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{a}_2$. Then $\mathfrak{g}_{-\alpha} \subset \mathfrak{a}_1$ since $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{g}$ and $\mathfrak{g}_\alpha \subset \mathfrak{a}_2$ since $\mathfrak{a}_2$ is parabolic. But then $s_\alpha\mathfrak{n}_+ \oplus \mathfrak{h} \oplus s_\alpha\mathfrak{n}_-$ is a polarization of $\mathfrak{g}$ for which $s_\alpha\mathfrak{b}_+ \subset \mathfrak{a}_1$ and $\dim(s_\alpha\mathfrak{n}_+ \cap \mathfrak{a}_2) < \dim(\mathfrak{n}_+ \cap \mathfrak{a}_2)$. Contradiction. □

In particular, $A_\pm$, $I_\pm$ are all $\mathfrak{h}$-graded and

$$I_+ = (\mathfrak{q}^+_{\Gamma_+} \oplus V_+)^\perp = \bigoplus_{\alpha\in\Delta_+\setminus\langle\Gamma_+\rangle} \mathfrak{g}_\alpha \oplus (V_+^\perp \cap \mathfrak{h}_0),$$
$$I_- = (\mathfrak{q}^-_{\Gamma_-} \oplus V_-)^\perp = \bigoplus_{\alpha\in\Delta_-\setminus\langle\Gamma_-\rangle} \mathfrak{g}_\alpha \oplus (V_-^\perp \cap \mathfrak{h}_0).$$

Thus $A_+/I_+ = \mathfrak{g}_{\Gamma_+} \oplus V_1$ and $A_-/I_- = \mathfrak{g}_{\Gamma_-} \oplus V_2$ for some suitable $V_1, V_2 \subset \mathfrak{h}_0$.

Let $L_{\pm\frac{1}{2}}(\lambda)$ be the generalized eigenspace of $\varphi(\lambda)$ associated to $\pm\frac{1}{2}$. Since φ is a solution of an ordinary differential equation with stationary points at $\frac{1}{2}, -\frac{1}{2}$, $L_{\pm\frac{1}{2}}(\lambda)$ is independent of λ and we will simply denote it by $L_{\pm\frac{1}{2}}$. Similarly, let L' be the sum of all other generalized eigenspaces so that $\mathfrak{g} = \mathfrak{l} \oplus L_{\frac{1}{2}} \oplus L' \oplus L_{-\frac{1}{2}}$.

Proposition 4. *There exists a polarization* $\mathfrak{g} = \overline{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \overline{\mathfrak{n}}_-$ *and a subset* $\Gamma_3 \subset \Pi(\overline{\mathfrak{n}}_+)$ *such that* $L_{\pm\frac{1}{2}} \subset \overline{\mathfrak{b}}_\pm$, $L' \subset \mathfrak{g}_{\Gamma_3} + \mathfrak{h}$ *and* $\varphi(\overline{\mathfrak{n}}_+) \subset \overline{\mathfrak{n}}_+$.

Proof. We will construct a polarization satisfying the above conditions in several steps.

Lemma 6. *We have*:

(i) $\mathfrak{l} \oplus L_{\pm\frac{1}{2}}$ *is an* $\mathfrak{h}$*-graded solvable subalgebra,*
(ii) $\mathfrak{l} \oplus L'$ *is an* $\mathfrak{h}$*-graded subalgebra,*
(iii) *we have* $[L_{\pm\frac{1}{2}}, L'] \subset \mathfrak{l} \oplus L_{\pm\frac{1}{2}}$.

Proof. This follows from the proofs of Lemma 12.3 and Theorem 12.6 in [BD].

Notice that $L_{\pm\frac{1}{2}} \not\subset \mathfrak{b}^1_{\pm}$ in general. We first construct a polarization $\mathfrak{g} = \mathfrak{n}^2_+ \oplus \mathfrak{h} \oplus \mathfrak{n}^2_-$ such that $L_{\pm\frac{1}{2}} \subset \mathfrak{b}^2_{\pm}$. We have $I_{\pm} \subset L_{\pm\frac{1}{2}}$. Notice that $L_{\frac{1}{2}} \cap \mathfrak{n}^1_- \subset \mathfrak{g}_{\Gamma_+} \cap \mathfrak{g}_{\Gamma_-} = \mathfrak{g}_{\Gamma_+\cap\Gamma_-}$ since $\mathfrak{n}^1_- \subset (\mathfrak{g}_{\Gamma_-} \oplus I_-)$ and $L_{\frac{1}{2}}$ is solvable. Similarly, $L_{-\frac{1}{2}} \cap \mathfrak{n}^1_+ \subset \mathfrak{g}_{\Gamma_+\cap\Gamma_-}$. Moreover, by Lemma 6, $\mathfrak{l} \oplus (L_{\frac{1}{2}} \cap \mathfrak{g}_{\Gamma_+\cap\Gamma_-})$ and $\mathfrak{l} \oplus (L_{-\frac{1}{2}} \cap \mathfrak{g}_{\Gamma_+\cap\Gamma_-})$ are disjoint, solvable, $\mathfrak{h}$-graded subalgebras. By lemma 5 it follows that there exists an element s of the group $W_{\Gamma_+\cap\Gamma_-}$ such that $\mathfrak{l} \oplus (L_{\pm\frac{1}{2}} \cap \mathfrak{g}_{\Gamma_+\cap\Gamma_-}) \subset s\mathfrak{b}^1_{\pm}$. Notice that s permutes elements of $\Delta^+ \backslash \langle \Gamma_+ \cap \Gamma_- \rangle$, leaving it globally unchanged. Thus, $\mathfrak{l} \oplus L_{\pm\frac{1}{2}} \subset s\mathfrak{b}^1_{\pm}$. Set $\mathfrak{n}^2_{\pm} = s\mathfrak{n}^1_{\pm}$.

Now we construct a polarization of $\mathfrak{g}$ satisfying the other conditions of proposition 4. Recall that $\mathfrak{l} \oplus L \subset \mathfrak{g}_{\Gamma_+\cap\Gamma_-} + (V_1 \cap V_2)$. Thus $\big(L' \cap \mathfrak{n}^2_+\big) \oplus \big(L_{\frac{1}{2}} \cap \mathfrak{n}^2_+(\Gamma_+ \cap \Gamma_-)\big) = \mathfrak{n}^2_+(\Gamma_+ \cap \Gamma_-)$.

Since $[L', L_{\frac{1}{2}}] \subset \mathfrak{l} \oplus L_{\frac{1}{2}}$ by Lemma 6,(iii), $L_{\frac{1}{2}} \cap \mathfrak{n}^2_+(\Gamma_+ \cap \Gamma_-)$ is an ideal of $\mathfrak{n}^2_+(\Gamma_+ \cap \Gamma_-)$. But $L' \cap \mathfrak{n}^2_+$ is a subalgebra. It is easy to see that this implies that $L' \cap \mathfrak{n}^2_+$ is generated by a set of simple root subspaces of $\mathfrak{n}^2_+(\Gamma_+ \cap \Gamma_-)$, i.e $L' \cap \mathfrak{n}^2_+ = \mathfrak{n}^2_+(\Gamma)$ for some $\Gamma \subset \Pi(\mathfrak{n}^2_+)$. Moreover, the restriction of $(\,,\,)$ to L' is nondegenerate, hence $L' \cap \mathfrak{n}^2_- = \mathfrak{n}^2_-(-\Gamma)$. Thus $\mathfrak{l} \oplus \mathfrak{g}_{\Gamma} \subset \mathfrak{l} \oplus L' \subset \mathfrak{l} \oplus \mathfrak{g}_{\Gamma} + (V_1 \cap V_2)$.

Since $\varphi(\lambda) + \frac{1}{2}$ is invertible in L', $\psi(\lambda)$ is a well-defined operator $L' \to L'$, satisfying (22), and $\psi(\lambda)(\mathfrak{h}_0 \cap L') \subset \mathfrak{h}_0 \cap L'$. Now, $\mathfrak{l}$ contains a regular element. Thus there exists a polarization of $\mathfrak{g}$ compatible with the $\mathfrak{l}$-weight decomposition. This induces a polarization of $\mathfrak{g}_{\Gamma}$, compatible with the $\mathfrak{l}$-weight decomposition of $\mathfrak{g}_{\Gamma}$. Hence, there exists $s' \in W_{\Gamma} \subset W$ such that $\psi_{0|\mathfrak{g}_{\Gamma}}$ is compatible with the polarization $s'\mathfrak{n}^2_+ \oplus \mathfrak{h} \oplus s'\mathfrak{n}^2_-$. Since s' leaves $\Delta_+ \backslash \langle \Gamma \rangle$ globally unchanged, the polarization $\mathfrak{g} = \overline{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \overline{\mathfrak{n}}_-$ with $\overline{\mathfrak{n}}_{\pm} = s'\mathfrak{n}^2_{\pm}$ and $\Gamma_3 = s'\Gamma$ satisfies the requirements of proposition 4. □

To sum up, we have shown that there exists a polarization $\mathfrak{g} = \overline{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \overline{\mathfrak{n}}_-$, compatible with φ, subsets $\Gamma_1 = s's\Gamma_+$, $\Gamma_2 = s's\Gamma_-$ and $\Gamma_3 \subset \Pi(\overline{\mathfrak{n}}_+)$ such that $(A_+/I_+) \cap \mathfrak{n}_+ = \overline{\mathfrak{n}}_+(\Gamma_1)$, $A_- \cap \mathfrak{n}_+ = \overline{\mathfrak{n}}_+(\Gamma_2)$ and $L' \cap \mathfrak{n}_+ = \overline{\mathfrak{n}}_+(\overline{\Gamma_3})$.

The map ψ_0 now restricts to a Lie algebra isomorphism $\overline{\mathfrak{n}}_+(\Gamma_1) \to \overline{\mathfrak{n}}_+(\Gamma_2)$. This isomorphism maps weight spaces to weight spaces as ψ_0 preserves $\mathfrak{h}_0$ and φ is $\mathfrak{l}$-invariant. Define $\tau : \Gamma_1 \to \Gamma_2$ by $\psi_0(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{\tau(\alpha)}$. It is a norm-preserving bijection. Thus $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a generalized Belavin-Drinfeld triple. It is clear that Γ_3 is the largest subset of $\Gamma_1 \cap \Gamma_2$ stable under τ, and that $\psi_0 : \overline{\mathfrak{n}}_+(\Gamma_3) \to \overline{\mathfrak{n}}_+(\Gamma_3)$ is a Lie algebra isomorphism. Finally, it is easy to see that the map φ is obtained from this data by formulas

$$\varphi(\lambda)(e_{\alpha}) = \frac{1}{2}e_{\alpha} \qquad (\alpha \notin \langle \Gamma_1 \rangle)$$

$$\varphi(\lambda)(e_{\alpha}) = \frac{1}{2}e_{\alpha} + \frac{\psi_0}{1 - e^{(\alpha,\lambda)}\psi_0}(e_{\alpha}) \qquad (\alpha \in \langle \Gamma_1 \rangle)$$

Conversely, it is clear how to construct from a generalized Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ the subalgebras $\mathfrak{n}_+(\Gamma_1)$, $\mathfrak{n}_+(\Gamma_2)$, $\mathfrak{n}_+(\Gamma_3)$ and, for each choice of

Chevalley generators, a Lie algebra isomorphism $\psi_0 : \mathfrak{n}_+(\Gamma_1) \to \mathfrak{n}_+(\Gamma_2)$, and the map $\varphi(\lambda)$. Condition (16) on the $\mathfrak{h}_0 \otimes \mathfrak{h}_0$-part comes from (21)-see [BD].

6. Examples

6.1. Constant r-matrices. Our results imply the following:

Corollary 1. *A dynamical r-matrix associated to a generalized Belavin-Drinfeld triple* $(\Gamma_1, \Gamma_2, \tau)$ *is gauge equivalent to a constant r-matrix if and only if* $\Gamma_3 = \emptyset$.

6.2. $\mathfrak{h}$-invariant dynamical r-matrices. When $\mathfrak{l} = \mathfrak{h}$, our classification coincides with that given in [EV]: the only $\mathfrak{h}$-graded generalized Belavin-Drinfeld triple is of the form $(\Gamma, \Gamma, \tau = Id)$. The dynamical r-matrices obtained are then (up to gauge transformations and choice of Chevalley generators):

$$r(\lambda) = \frac{\Omega}{2} + \sum_{\alpha \in \Delta_+,\, \alpha \notin \langle \Gamma \rangle} \frac{1}{2} e_\alpha \wedge e_{-\alpha} + \sum_{\alpha \in \langle \Gamma \rangle \cap \Delta_+} \frac{1}{2} \coth(\frac{1}{2}(\alpha, \lambda) e_\alpha \wedge e_{-\alpha}.$$

6.3. Example for $\mathfrak{sl}_3$ and $\mathfrak{sl}_n$. The first nontrivial example is for $\mathfrak{g} = \mathfrak{sl}_3$: fix a polarization $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Delta} \mathfrak{g}_\gamma$ where $\Delta^+ = \{\alpha, \beta, \alpha + \beta\}$ and set $\mathfrak{l} = \mathbb{C} h_\rho$. Consider the generalized Belavin-Drinfeld triple with $\Gamma_1 = \Gamma_2 = \{\alpha, \beta\}$ and $\tau : \alpha \mapsto \beta$, $\beta \mapsto \alpha$. In this case, we can choose the map ψ_0 to be the following

$$\begin{array}{ccc} e_\alpha \mapsto e_\beta, & h_\alpha \mapsto h_\beta, & e_{-\alpha} \mapsto e_{-\beta} \\ e_\beta \mapsto e_\alpha, & h_\beta \mapsto h_\alpha, & e_{-\beta} \mapsto e_{-\alpha} \\ e_{\alpha+\beta} \mapsto -e_{\alpha+\beta}, & & e_{-\alpha-\beta} \mapsto -e_{-\alpha-\beta}. \end{array}$$

The corresponding dynamical r-matrix is given by:

$$\begin{aligned} r(\lambda) = \frac{\Omega}{2} &+ r_{\mathfrak{h}_0, \mathfrak{h}_0} + \frac{1}{2}\coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} \\ &+ \frac{1}{2} \operatorname{th}(\alpha+\beta, \lambda) e_{\alpha+\beta} \wedge e_{-\alpha-\beta} + \frac{1}{2\sinh(\alpha, \lambda)} e_\beta \wedge e_{-\alpha} \\ &+ \frac{1}{2\sinh(\alpha, \lambda)} e_\alpha \wedge e_{-\beta}. \end{aligned} \tag{23}$$

This dynamical r-matrix is gauge-equivalent to the dynamical r-matrix

$$\begin{aligned} \tilde{r}(\lambda) = &\frac{\Omega}{2} + r_{\mathfrak{h}_0, \mathfrak{h}_0} + r_{\mathfrak{l}, \mathfrak{h}_0} - r^{21}_{\mathfrak{l}, \mathfrak{h}_0} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} \\ &+ \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} + \frac{1}{2} \operatorname{th}(\alpha+\beta, \lambda) e_{\alpha+\beta} \wedge e_{-\alpha-\beta} \\ &+ \frac{e^{(\alpha,\lambda)}}{2\sinh(\alpha, \lambda)} e_\beta \wedge e_{-\alpha} + \frac{e^{-(\alpha,\lambda)}}{2\sinh(\alpha, \lambda)} e_\alpha \wedge e_{-\beta}. \end{aligned} \tag{24}$$

when

$$(\alpha \otimes 1 + 1 \otimes \tau(\alpha))\big(r_{\mathfrak{h}_0, \mathfrak{h}_0} + r_{\mathfrak{l}, \mathfrak{h}_0} - r^{21}_{\mathfrak{l}, \mathfrak{h}_0}\big) = \frac{1}{2}(\alpha + \tau(\alpha))\Omega_{\mathfrak{h}}.$$

In particular, $\tilde{r}(\lambda)$ interpolates the constant r-matrix obtained from the Belavin-Drinfeld triple $(\Gamma_1 = \alpha, \Gamma_2 = \beta, \tau : \alpha \mapsto \beta)$ at $(\alpha, \lambda) \to \infty$ and the r-matrix obtained from $(\Gamma_1 = \beta, \Gamma_2 = \alpha, \tau : \beta \mapsto \alpha)$ at $(\alpha, \lambda) \to -\infty$.

Remark. The generalization of this example to $\mathfrak{g} = \mathfrak{sl}_{2n+1}$ is the following. Fix a polarization and let $\mathfrak{l} = \mathbb{C}h_\rho$. Denote by Δ the root system and by $\Pi = (\alpha_1, \dots \alpha_{2n})$ the set of positive simple roots. Let $i : \alpha_k \mapsto \alpha_{2n+1-k}$ be the involution of the Dynkin diagram. The dynamical r-matrix obtained from the generalized Belavin-Drinfeld triple $(\Gamma_1 = \Gamma_2 = \Pi, \tau = i)$ interpolates the constant r-matrices obtained from the Belavin-Drinfeld triples $(\Gamma_1 = (\alpha_1, \dots \alpha_n), \Gamma_2 = (\alpha_{n+1}, \dots \alpha_{2n}),\ \tau = i)$ and $(\Gamma_1 = (\alpha_{n+1}, \dots \alpha_{2n}), \Gamma_2 = (\alpha_1, \dots \alpha_n),\ \tau = i^{-1})$.

6.4. Permutation dynamical r-matrices. Consider $\mathfrak{g} = \mathfrak{sl}_{2n}$, and let $\Pi = (\alpha_1, \dots \alpha_{2n-1})$ denote a system of simple roots. For any $\sigma \in S_n$, we can construct a generalized Belavin-Drinfeld triple by setting $\Gamma_1 = \Gamma_2 = (\alpha_1, \alpha_3, \dots \alpha_{2n-1})$ and $\tau : \alpha_{2k-1} \mapsto \alpha_{2\sigma(k)-1}$.

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