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SURFACES IN 4-MANIFOLDS

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Abstract. In this paper we introduce a technique, called *rim surgery*, which can change a smooth embedding of an orientable surface Σ of positive genus and nonnegative self-intersection in a smooth 4-manifold X while leaving the topological embedding unchanged. This is accomplished by replacing the tubular neighborhood of a particular nullhomologous torus in X with $S^1 \times E(K)$, where $E(K)$ is the exterior of a knot $K \subset S^3$. The smooth change can be detected easily for certain pairs (X, Σ) called *SW-pairs*. For example, (X, Σ) is an SW-pair if Σ is a symplectically and primitively embedded surface with positive genus and nonnegative self-intersection in a simply connected symplectic 4-manifold X. We prove the following theorem:

Theorem. *Consider any SW-pair (X, Σ). For each knot $K \subset S^3$ there is a surface $\Sigma_K \subset X$ such that the pairs (X, Σ_K) and (X, Σ) are homeomorphic. However, if K_1 and K_2 are two knots for which there is a diffeomorphism of pairs $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then their Alexander polynomials are equal: $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.*

1. Introduction

We say that a surface Σ is *primitively embedded* in a simply connected smooth 4-manifold X if Σ is smoothly embedded with $\pi_1(X \setminus \Sigma) = 0$. In particular, by Alexander duality, Σ must represent a primitive homology class $[\Sigma] \in H_2(X;\mathbb{Z})$. In general, any smoothly embedded (connected) surface S in a simply connected smooth 4-manifold X with $[S] \neq 0$ has the property that the surface Σ which represents the homology class $[S] - [E]$ in $X \# \overline{\mathbb{CP}}^2$ and which is obtained by tubing together the surface S with the exceptional sphere E of $\overline{\mathbb{CP}}^2$ is primitively embedded (since the surface Σ transversally intersects the sphere E in one point).

Given a primitively embedded positive genus surface Σ in X, in the first part of this paper we shall construct for each knot K in S^3 a smoothly embedded surface Σ_K in X which is Σ-*compatible*; i.e. $[\Sigma] = [\Sigma_K]$ and there is a homeomorphism $(X, \Sigma) \to (X, \Sigma_K)$. This construction will have two properties. The first is that $(X, \Sigma_{\text{unknot}}) = (X, \Sigma)$. The main result of this paper is the second property: under suitable hypotheses on the pair (X, Σ), if K_1 and K_2 are two knots in S^3 and if there is a diffeomorphism $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then K_1 and K_2 have the same symmetric Alexander polynomial, i.e. $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. As a special case we show:

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Theorem 1.1. *Let X be a simply connected symplectic 4-manifold and Σ a symplectically and primitively embedded surface with positive genus and nonnegative self-intersection. If K_1 and K_2 are knots in S^3 and if there is a diffeomorphism of pairs $(X,\Sigma_{K_1}) \to (X,\Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. Furthermore, if $\Delta_K(t) \neq 1$, then Σ_K is not smoothly ambient isotopic to a symplectic submanifold of X.*

For example, Theorem 1.1 applies to the $K3$ surface where Σ is a generic elliptic fiber. It also applies to surfaces of the form $S - E$ in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, where S is any positive genus symplectically embedded surface in $\mathbb{CP}^2$.

The outline of this paper is as follows. In §2 we shall construct the surfaces Σ_K with $[\Sigma_K] = [\Sigma]$ and show that if $\pi_1(X) = \pi_1(X \setminus \Sigma) = 0$, there is a homeomorphism of (X,Σ) with (X,Σ_K), i.e. Σ_K is Σ-compatible. We give two descriptions of Σ_K. One is explicit, while the other describes how to obtain Σ_K by removing a tubular neighborhood $T^2 \times D^2$ of a homologically trivial torus in a tubular neighborhood of Σ and replacing it with $S^1 \times E(K)$, where $E(K)$ is the exterior of the knot K in S^3. This is reminiscent of our construction in [FS] where we performed the same operation on homologically essential tori. There, the Alexander polynomial $\Delta_K(t)$ of K detected a change in the diffeomorphism type of the ambient manifold X. Here, we shall show that $\Delta_K(t)$ detects a change in the diffeomorphism type of the embedding of Σ in X.

If the self-intersection of Σ is $n \geq 0$, then in $X_n = X \# n\overline{\mathbb{CP}}^2$ consider the surface $\Sigma_n = \Sigma - \sum_{j=1}^n E_j$ (resp. $\Sigma_{n,K} = \Sigma_K - \sum_{j=1}^n E_j$) obtained from Σ (resp. Σ_K) by tubing together with the exceptional spheres E_j, $j = 1, \dots, n$, of the copies of $\overline{\mathbb{CP}}^2$ in X_n. If there is a diffeomorphism $H : (X,\Sigma_{K_1}) \to (X,\Sigma_{K_2})$, then there is a diffeomorphism $H_n : (X_n,\Sigma_{n,K_1}) \to (X_n,\Sigma_{n,K_2})$. For each genus $g \geq 1$ we construct in §3 a standard pair (Y_g, S_g), with the properties that Y_g is a Kahler surface, S_g is a primitively embedded genus g Riemann surface in Y_g, and the torus used to construct $S_{g,K} = (S_g)_K$ is contained in a cusp neighborhood. Then in §4 we will study *SW-pairs*, i.e. pairs (X,Σ) where X is a smooth simply connected 4-manifold, Σ is a primitively embedded genus g surface with self-intersection $n \geq 0$, and the fiber sum of X_n and Y_g along the surfaces Σ_n and S_g has a nontrivial Seiberg-Witten invariant $\mathcal{SW}_{X_n \#_{\Sigma_n = S_g} Y_g} \neq 0$. The point here is that the nullhomologous torus used to construct the surface Σ_K in X still resides in $X_n \#_{\Sigma_n = S_g} Y_g$ and is now homologically essential and is contained in a cusp neighborhood. It will also follow that if X is a symplectic 4-manifold and Σ is a symplectically and primitively embedded surface with nonnegative self-intersection, then (X,Σ) is a SW-pair.

In §5 we use in a straightforward fashion the results of [FS] to show that the Alexander polynomial of K distinguishes the Σ_K for SW-pairs, and we complete the proof our main theorem:

Theorem 1.2. *Consider any SW-pair (X,Σ). If K_1 and K_2 are two knots in S^3 and if there is a diffeomorphism of pairs $(X,\Sigma_{K_1}) \to (X,\Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.*

Finally, in §6 we complete the proof of Theorem 1.1 by showing that in the case that Σ is sympletically embedded in X and $\Delta_K(t) \neq 1$, then Σ_K is not smoothly ambient isotopic to a symplectic submanifold of X.

We conclude this introduction with two conjectures. The first conjecture is that, under the hypothesis of Theorem 1.2, there is a diffeomorphism $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$ if and only if the knots K_1 and K_2 are isotopic. In particular, this conjecture would imply that the study of the equivalence clases of Σ-compatible surfaces under diffeomorphism is at least as complicated as classical knot theory. The second conjecture is a finiteness conjecture: given a symplectic 4-manifold X and a symplectic submanifold Σ, we conjecture that there are only finitely many distinct smooth isotopy classes of symplectic submanifolds Σ' which are topologically isotopic to Σ.

2. The construction of Σ_K

Let X be a smooth 4-manifold which contains a smoothly embedded surface Σ with genus $g > 0$. Then there is a diffeomorphism

$$h : \Sigma \to T^2 \# \cdots \# T^2 = (T^2 \setminus D^2) \cup (T^2 \setminus (D^2 \amalg D^2)) \cup \cdots \cup (T^2 \setminus D^2).$$

Let $C \subset \Sigma$ be a curve whose image under h is the curve $S^1 \times \{\text{pt}\} \subset T^2 \setminus D^2 = (S^1 \times S^1) \setminus D^2$ in the first $T^2 \setminus D^2$ summand of $h(\Sigma)$. Keep in mind that, since there are many such diffeomorphisms h, there are many such curves C. Given a knot K in S^3 we shall give two different constructions of a surface $\Sigma_{K,C}$. The first is an explicit construction, while the second shows how to obtain $\Sigma_{K,C}$ by what we call a *rim surgery*, a surgical operation on a particular homologically trivial torus in a neighborhood of Σ. It is this second construction that will allow us to compute appropriate invariants to distinguish the surfaces $\Sigma_{K,C}$.

2.1. An explicit description of $\Sigma_{K,C}$. Viewing S^1 as the union of two arcs A_1 and A_2, we have

$$\begin{aligned} T^2 \setminus D^2 &= (S^1 \times S^1) \setminus D^2 \\ &= ((A_1 \cup A_2) \times (A_1 \cup A_2)) \setminus (A_1 \times A_1) \\ &= (A_2 \times S^1) \cup (A_1 \times A_2) \end{aligned}$$

with $h(C) = A_2 \times \{\text{pt}\} \cup A_1 \times \{\text{pt}\}$. Now the normal bundle of Σ in X when restricted to $T^2 \setminus D^2 \subset \Sigma$ is trivial, hence it is diffeomorphic to

$$((A_2 \times S^1) \cup (A_1 \times A_2)) \times D^2 = ((A_2 \times D^2) \times S^1) \cup ((A_1 \times D^2) \times A_2)).$$

Furthermore, under this diffeomorphism, the inclusion

$$(T^2 \setminus D^2) \times \{0\} \subset (T^2 \setminus D^2) \times D^2$$

becomes

$$(A_2 \times \{0\}) \times S^1) \cup ((A_1 \times \{0\}) \times A_2) \subset ((A_2 \times D^2) \times S^1) \cup ((A_1 \times D^2) \times A_2).$$

Now tie a knot K in the arc $(A_2 \times \{0\}) \subset (A_2 \times D^2)$ to obtain a knotted arc A_K and to obtain a new punctured torus

$$\begin{aligned} T_K \setminus D^2 &= (A_K \times S^1) \cup ((A_1 \times \{0\}) \times A_2) \\ &\subset ((A_2 \times D^2) \times S^1) \cup ((A_1 \times D^2) \times A_2) \end{aligned}$$

with

$$\partial(T_K \setminus D^2) = \partial(T \setminus D^2).$$

Then let

$$\Sigma_{K,C} = (T_K \setminus D^2) \cup (T^2 \setminus (D^2 \amalg D^2)) \cup \cdots \cup (T^2 \setminus D^2) \subset N(\Sigma) \subset X.$$

2.2. A description of $\Sigma_{K,C}$ via rim surgery. Keeping the notation above, we first recall how, via a 3-manifold surgery, we can tie a knot K in the arc $(A_2 \times \{0\}) \subset (A_2 \times D^2)$. In short, we just remove a small tubular neighborhood in $A_2 \times D^2$ of a pushed-in copy γ of the meridional circle $\{0\} \times S^1 \subset A_2 \times D^2$ and sew in the exterior of the knot K in S^3 so that the meridian of K is identified with γ. This has the effect of tying a knot in the arc $A_2 \times \{0\} \subset A_2 \times D^2$. More specifically, consider the standard embedding of the solid torus $A = (A_1 \cup A_2) \times D^2 = S^1 \times D^2$ in S^3 with complementary solid torus $B = D^2 \times S^1$ with core $C' = \{0\} \times S^1 \subset D^2 \times S^1$. In $A \setminus C = (S^1 \times D^2) \setminus C = S^1 \times S^1 \times (0, 1] = (A_1 \cup A_2) \times S^1 \times (0, 1]$, consider the circle $\gamma = \{t\} \times S^1 \times \{\frac{1}{2}\}$, with $t \in A_2$, and with tubular neighborhood $N(\gamma) \subset A \setminus C$. The curve γ is isotopic in $S^3 \setminus C$ to the core C' of B. We denote by γ' the curve γ pushed off into $\partial N(\gamma)$ so that the linking number in S^3 of γ and γ' is zero. For later reference, note that $D = (A \setminus N(\gamma)) \cup B$ is again diffeomorphic to a solid torus. (It is the exterior of the unknot $\gamma \subset A \subset S^3$.) The core of D is isotopic (in D) to C.

Let M_K be the 3-manifold obtained by performing 0-framed surgery on K. Then the meridian m of K is a circle in M_K and has a canonical framing in M_K; we denote a tubular neighborhood of m in M_K by $m \times D^2$. Let S_K denote the 3-manifold

$$S_K = (A \setminus N(\gamma)) \cup (M_K \setminus (m \times D^2)).$$

The two pieces are glued together so as to identify γ' with m. In other words, we remove $N(\gamma)$ and sew in the exterior $E(K)$ of the knot K in S^3. Note that the core C of the solid torus A is untouched by this operation, so $C \subset S_K$. Also, the boundary ∂A of A and the set $G = A_1 \times D^2 \subset (A_1 \cup A_2) \times D^2 \subset A$ remain untouched and thus can be viewed as subsets of S_K.

Lemma 2.1. *There is a diffeomorphism $h : S_K \to A$ which is the identity on G and on the boundary. Furthermore, $h(C)$ is the knotted core $K \subset A$.*

Proof. In $S^3 = A \cup B$, the above operation replaces a tubular neighborhood of the unknot $\gamma \subset A \subset S^3$ with the exterior $E(K)$ of the knot K in S^3. Thus there is a diffeomorphism $h : E(K) \cup D \to A \cup B = S^3$ sending the core circle of D to the knot K. Now $C' \subset B \subset E(K) \cup D$ is unknotted, since in D, the curve C' is isotopic to γ', which bounds a disk. Thus S_K, which is the complement of a tubular neighborhood of C', is an unknotted solid torus in $S^3 = E(K) \cup D$. Furthermore, as we have noted above, C is isotopic to the core of D; so $C \subset S_K$ is the knot K. Thus there is a diffeomorphism $h : S_K \to S^1 \times D^2$ which is the identity on the boundary. After an isotopy rel boundary we can arrange that $h(G) = G$. □

To obtain $\Sigma_{K,C}$ we cross everything with S^1; i.e. remove the neighborhood $N(\gamma) \times S^1 \subset (A_2 \times D^2) \times S^1 \subset N(\Sigma)$ of the (nullhomologous) torus $\gamma \times S^1 \subset (A_2 \times D^2) \times S^1 \subset N(\Sigma)$ and sew in $E(K) \times S^1$ as above on the $E(K)$ factor and the identity on the S^1 factor. We refer to this as a *rim surgery* on Σ. Notice that this construction does not change the ambient manifold X. Except where it is absolutely necessary to keep track of the curve C, we shall suppress it from our notation and abbreviate $\Sigma_{K,C}$ as Σ_K.

2.3. The complement of Σ_K. From the construction, it is clear that if the complement of Σ in X is simply connected, then so is the complement of Σ_K in X, since the meridian of the knot (which is identified with the boundary of the normal fiber to Σ) normally generates the fundamental group of the exterior of K. Now there is a map $f : E(K) \to B \cong D^2 \times S^1$ which induces isomorphisms on homology and restricts to a homeomorphism $\partial E(K) \to \partial B$ taking the class of a meridian to $[\{\text{pt}\} \times S^1]$ and the class of a longitude to $[\partial D^2 \times \{\text{pt}\}]$. The map $f \times \text{id}_{S^1}$ on $E(K) \times S^1$ extends via the identity to a homotopy equivalence $X \backslash N(\Sigma_K) \to X \backslash N(\Sigma)$ which restricts to a homeomorphism $\partial N(\Sigma_K) \to \partial N(\Sigma)$. Then topological surgery [F, B] guarantees the existence of a homeomorphism $h : (X, \Sigma) \to (X, \Sigma_K)$.

If $\pi_1(X \setminus \Sigma) \neq 0$, it is not clear when $X \setminus \Sigma_K$ is homeomorphic (or even homotopy equivalent) to $X \setminus \Sigma_K$. We avoid such issues in this paper and only deal with the case where $\pi_1(X \setminus \Sigma) = 0$. However, as already noted; the surface $\Sigma - E$ in $X \# \overline{\mathbb{CP}}^2$ obtained by tubing together the surface Σ with the exceptional sphere E of $\overline{\mathbb{CP}}^2$ is primitively embedded; so there is a homeomorphism $h : (X \# \overline{\mathbb{CP}}^2, \Sigma - E) \to (X \# \overline{\mathbb{CP}}^2, \Sigma_K - E)$. In summary:

Theorem 2.2. *Let X be a simply connected smooth 4-manifold with a primitively embedded surface Σ. Then for each knot K in S^3, the above construction produces a Σ-compatible surface Σ_K.*

3. The standard pair (Y_g, S_g)

Let $g > 0$. In this section we shall construct a simply connected smooth 4-manifold Y_g and a primitive embedding of S_g, the surface of genus g, in Y_g such that the torus used in the previous section to construct the S_g-compatible embedding $(S_g)_K = S_{g,K}$ is contained in a cusp neighborhood.

To this end, consider the $(2, 2g+1)$-torus knot $T(2, 2g+1)$. It is a fibered knot whose fiber is a punctured genus g surface and whose monodromy t' is periodic of order $4g+2$. If we attach a 2-handle to ∂B^4 along $T(2, 2g+1)$ with framing

0, we obtain a manifold $C(g)$ which fibers over the 2-disk with generic fiber a Riemann surface S_g of genus g and whose monodromy map t, induced from t', is a periodic holomorphic map $t : S_g \to S_g$ of order $4g + 2$. The singular fiber is the topologically (non-locally flatly) embedded sphere obtained from the cone in B^4 on the torus knot $T(2, 2g + 1)$ union the core of the 2-handle. Now consider the fibration over the punctured 2-sphere obtained from gluing together $4g + 2$ such neighborhoods $C(g)$ along a neighborhood of a fiber in the boundary of $C(g)$. This is a complex surface, and the monodromy is trivial around a loop which contains in its interior the images of all the singular fibers. Thus we may compactify this manifold to obtain a complex surface Y_g which is holomorphically fibered over S^2. For example, Y_1 is the rational elliptic surface $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. In fact, Y_g is just the Milnor fiber of the Brieskorn singularity $\Sigma(2, 2g + 1, 4g + 1)$ union a generalized nucleus consisting of the 4-manifold obtained as the trace of the 0-framed surgery on $T(2, 2g + 1)$ and a -1 surgery on a meridian [Fu]. The fibration $\pi : Y_g \to S^2$ has a holomorphic section which is a sphere Λ of self-intersection -1 (the sphere obtained by the -1-surgery above (cf. [Fu]). This proves that $\pi_1(Y_g \setminus S_g) = 0$; so S_g is a primitively embedded surface with self-intersection 0.

Let T denote the torus in $S_g \times D^2$ on which we perform a rim surgery in order to obtain the surface $S_{g,K}$. We wish to see that T lies in a cusp neighborhood. A cusp neighborhood is nothing more than the regular neighborhood of a torus together with two vanishing cycles, one for each generating circle in the torus. The torus T has the form $T = \gamma \times \tau$ where τ is a closed curve on S_g and $\gamma = \{\mathrm{pt}\} \times (\{\frac{1}{2}\} \times \partial D^2)$. The curve τ is one of the generating circles for $H_1(S_g; \mathbb{Z})$ with a dual circle σ. The curve γ spans a -1-disk contained in Λ. The curve τ degenerates to a point on the singular fiber in $C(g)$. Thus we see both required vanishing cycles.

4. SW-pairs

Recall that the Seiberg-Witten invariant SW_X of a smooth closed oriented 4-manifold X with $b^+ > 1$ is an integer valued function which is defined on the set of *spin*^{c} structures over X, (cf. [W]). In case $H_1(X; \mathbb{Z})$ has no 2-torsion, there is a natural identification of the *spin*^{c} structures of X with the characteristic elements of $H^2(X; \mathbb{Z})$. In this case we view the Seiberg-Witten invariant as

$$\mathrm{SW}_X : \{k \in H^2(X, \mathbb{Z}) | k \equiv w_2(TX) \pmod 2)\} \to \mathbb{Z}.$$

The Seiberg-Witten invariant SW_X is a smooth invariant whose sign depends on an orientation of $H^0(X; \mathbb{R}) \otimes \det H^2_+(X; \mathbb{R}) \otimes \det H^1(X; \mathbb{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then we call β a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. If β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$$

where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X.

As in [FS] we need to view the Seiberg-Witten invariant as a Laurent polynomial. To do this, let $\{\pm\beta_1, \dots, \pm\beta_n\}$ be the set of nonzero basic classes for X. We may then view the Seiberg-Witten invariant of X as the 'symmetric' Laurent polynomial

$$\mathcal{SW}_X = b_0 + \sum_{j=1}^{n} b_j (t_j + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_j^{-1})$$

where $b_0 = \mathrm{SW}_X(0)$, $b_j = \mathrm{SW}_X(\beta_j)$ and $t_j = \exp(\beta_j)$.

Now let Σ be genus $g > 0$ primitively embedded surface in the simply connected 4-manifold X. If the self-intersection of Σ is $n \geq 0$, then in $X_n = X \# n\overline{\mathbb{CP}}^2$, consider the surface $\Sigma_n = \Sigma - \sum_{j=1}^{n} E_j$ (resp. $\Sigma_{n,K} = \Sigma_K - \sum_{j=1}^{n} E_j$) obtained from Σ (resp. Σ_K) by tubing together with the exceptional spheres E_j, $j = 1, \dots, n$, of the $\overline{\mathbb{CP}}^2$ in X_n. Note that the fiber sum $X_n \#_{\Sigma_n = S_g} Y_g$ of X_n and Y_g along the surfaces Σ_n and S_g has $b^+ > 1$. An *SW-pair* is such a pair (X, Σ) which satisfies the property that the Seiberg-Witten invariant $\mathcal{SW}_{X_n \#_{\Sigma_n = S_g} Y_g} \neq 0$.

As we have pointed out earlier, there are several curves C that can be used to construct the surfaces $\Sigma_{K,C}$, and there are potentially several different fiber sums that can be performed in the construction of $X_n \#_{\Sigma_n = S_g} Y_g$. We pin down our choice of C by declaring it to be the image of the curve σ from §3 under the diffeomorphism used in the construction of the fiber sum. A simple Mayer-Vietoris argument shows that in $X_n \#_{\Sigma_n = S_g} Y_g$ the rim torus (equivalently $\gamma \times \tau$) becomes homologically essential and is contained in a cusp neighborhood. Thus our results from [FS] apply.

5. SW-pairs and the Alexander polynomial

We are now in a position to prove our main theorem:

Theorem 1.2. *Consider any SW-pair (X, Σ). If K_1 and K_2 are two knots in S^3 and if there is a diffeomorphism of pairs $(X, \Sigma_{K_1}) \to (X, \Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.*

Proof. With notation as above, we have a diffeomorphism $(X_n, \Sigma_{n,K_1}) \to (X_n, \Sigma_{n,K_2})$. Then there is a diffeomorphism

$$Z_1 = X_n \#_{\Sigma_{n,K_1} = S_g} Y_g \to Z_2 = X_n \#_{\Sigma_{n,K_2} = S_g} Y_g.$$

It follows from [FS] that $\mathcal{SW}_{Z_i} = \mathcal{SW}_{X_n \#_{\Sigma_n = S_g} Y_g} \cdot \Delta_{K_i}(t)$ for $t = \exp(2[T])$, where T denotes the rim torus. Since (X, Σ) is a SW-pair, and since $[T] \neq 0$ in $H_2(Z_i; \mathbb{Z})$ we must have $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. □

6. Rim surgery on symplectically embedded surfaces

We conclude with a proof of our claim of the introduction.

Theorem 1.1. *Let X be a simply connected symplectic 4-manifold and Σ a symplectically and primitively embedded surface with positive genus and nonnegative* *self-intersection. If K_1 and K_2 are knots in S^3 and if there is a diffeomorphism of pairs $(X,\Sigma_{K_1}) \to (X,\Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. Furthermore, if $\Delta_K(t) \neq 1$, then Σ_K is not smoothly ambient isotopic to a symplectic submanifold of X.*

Proof. Since Σ and S_g are symplectic submanifolds of X and Y_g, the fiber sum $X_n\#_{\Sigma_n=S_g}Y_g$ is also a symplectic manifold [G]. Thus $\mathcal{SW}_{X_n\#_{\Sigma_n=S_g}Y_g} \neq 0$ [T1]; so (X,Σ) forms an SW-pair. This proves the first statement of the theorem.

Next, suppose that Σ_K is smoothly ambient isotopic to a symplectic submanifold Σ' of X. This isotopy carries the rim torus T to a rim torus T' of Σ'. We have

$$\mathcal{SW}_{X_n\#_{\Sigma'_n=S_g}Y_g} = \mathcal{SW}_{X_n\#_{\Sigma_{n,K}=S_g}Y_g} = \mathcal{SW}_{X_n\#_{\Sigma_n=S_g}Y_g} \cdot \Delta_K(t) \tag{1}$$

with $t = \exp(2[T'])$ when this expression is viewed as $\mathcal{SW}_{X_n\#_{\Sigma'_n=S_g}Y_g}$. As above, $[T'] \neq 0$ in $H_2(X_n\#_{\Sigma'_n=S_g}Y_g;\mathbb{Z})$.

Symplectic forms ω_X on X_n (with respect to which Σ'_n is symplectic) and ω_Y on Y_g induce a symplectic form ω on the symplectic fiber sum $X_n\#_{\Sigma'_n=S_g}Y_g$ which agrees with ω_X and ω_Y away from the region where the manifolds are glued together. In particular, since T' is nullhomologous in X_n, we have $\langle\omega,T'\rangle = \langle\omega_X,T'\rangle = 0$. Now (1) implies that the basic classes of $X_n\#_{\Sigma'_n=S_g}Y_g$ are exactly the classes $b+2mT'$ where b is a basic class of $X_n\#_{\Sigma_n=S_g}Y_g$ and t^m has a nonzero coefficient in $\Delta_K(t)$. Thus the basic classes of $X_n\#_{\Sigma'_n=S_g}Y_g$ can be grouped into collections $\mathcal{C}_b = \{b+2mT'\}$, and if $\Delta_K(t)\neq 1$ then each $\mathcal{C}_b$ contains more than one basic class. Note, however, that $\langle\omega, b+2mT'\rangle = \langle\omega,b\rangle$. Now Taubes has shown [T2] that the canonical class κ of a symplectic manifold with $b^+>1$ is the basic class which is characterized by the condition $\langle\omega,\kappa\rangle > \langle\omega,b'\rangle$ for any other basic class b'. But this is impossible for $X_n\#_{\Sigma_n=S_g}Y_g$ since each $\mathcal{C}_b$ contains more than one class. □

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