GERMS OF LOCAL AUTOMORPHISMS OF REAL-ANALYTIC CR STRUCTURES AND ANALYTIC DEPENDENCE ON k-JETS.

DMITRI ZAITSEV

ABSTRACT. The topic of the paper is the study of germs of local holomorphisms f between $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ such that $f(M) \subset M'$ and $df(T^c M) = T^c M'$ for $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n'}$ generic real-analytic CR submanifolds of arbitrary codimensions. It is proved that for M minimal and M' finitely nondegenerate, such germs depend analytically on their jets. As a corollary, an analytic structure on the set of all germs of this type is obtained.

1. Introduction

Let $M \subset \mathbb{C}^n, M' \subset \mathbb{C}^{n'}$ be connected locally closed real-analytic submanifolds, $x \in M, x' \in M'$ be arbitrary points. The complex tangent subspace $T_x M \cap i T_x M$ will be denoted by $T_x^c M$. M is a *CR manifold*, if $\dim T_x^c M$ is constant. In this case $\dim_{\mathrm{CR}} M := \dim_{\mathbb{C}} T^c M$ is called the *CR dimension* and $\operatorname{codim}_{\mathrm{CR}} M := \dim_{\mathbb{R}} T M - \dim_{\mathbb{R}} T^c M$ the *CR codimension*. A CR submanifold $M \subset \mathbb{C}^n$ is called *generic*, if $TM + iTM = \mathbb{C}^n$.

Suppose for the moment that $M, M' \subset \mathbb{C}^n$ are generic real-analytic CR submanifolds of the same CR dimension and the same CR codimension. Baouendi, Ebenfelt and Rothschild found optimal nondegeneracy conditions on M and M' such that a germ at x of a local biholomorphism f (between some neighborhoods in $\mathbb{C}^n$) with $f(M) \subset M'$, $f(x) = x'$, is uniquely determined by its k-jet $j_x^k f$, where k is an integer which depends only on M and M' (see [2], Theorem 1, Proposition 2.3). A similar statement for the case of hypersurfaces M and M' is implicitly contained in [8]. See also [5] for the case of Levi-nondegenerate higher codimensional CR manifolds. These results should be compared with the following theorem of H. Cartan ([6, 11]):

Let $D \subset \mathbb{C}^n$ be a bounded domain. Then the group of all biholomorphic automorphisms $\operatorname{Aut}(D)$ equipped with the compact-open topology is a Lie group. Moreover, if $x \in D$ is arbitrary, an automorphism $f \in \operatorname{Aut}(D)$ is uniquely determined by its 1-jet $j_x^1 f$ and depends analytically on it.

In the above CR setting it is not clear how a map (germ) f depends on its k-jet $j_x^k f$. Is it somehow continuous, analytic etc.?

The special case when both M and M' are Levi-nondegenerate hypersurfaces was previously considered by Tanaka [14] and Chern-Moser [7], where the determinacy of automorphisms by their 2-jets was shown. Even in this special context an analytic dependence on the jets is of interest. An algebraic dependence of the automorphisms of bounded domains on their 1-jets is studied in [18], which can be seen as an algebraic version of the above theorem of Cartan.

The present paper has a different point of view. Instead of global automorphisms we consider locally defined germs of holomorphic maps f sending M into M' with no assumptions on their domains of definition. If either M or M' is locally biholomorphic to $\mathbb{R}^s \times \mathbb{C}^l$, the space of such germs is infinite-dimensional (see also [2], Theorem 3) and hence it cannot be parametrized by a finite dimensional k-jet space.

Therefore both M and M' have to satisfy certain nondegeneracy conditions. We start with some stronger conditions formulated in terms of the Levi form. Recall that in the case of arbitrary CR codimension the Levi form is a sesquilinear map (see §2):

$$(1) \qquad\qquad L \colon T_x^{1,0}M \times T_x^{1,0}M \to (T_x M / T_x^c M) \otimes_{\mathbb{R}} \mathbb{C}.$$

Definition 1.1. We say that the Levi form at x is *nondegenerate* if for $X_0 \in T_x^{1,0}M$,

$$(2) \qquad\qquad \big(L(X_0, Y) = 0 \ \ \forall Y \in T_x^{1,0}M\big) \quad \Longrightarrow \quad X_0 = 0.$$

The Levi form is said to be *surjective* at x, if the map (1) is surjective. The Levi form is said to be nondegenerate (resp. surjective) if it is nondegenerate (resp. surjective) for all $x \in M$.

Now we introduce the notion of *analytic dependency on k-jets*. Let S be a subset of the set of germs at $x \in \mathbb{C}^n$ of all local holomorphic maps f between $\mathbb{C}^n$ and $\mathbb{C}^{n'}$. By $J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$ denote the k-jet space at x of such maps and by $j_x^k f \in J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$ the k-jet of f at x. We write also $J^k(\mathbb{C}^n, \mathbb{C}^{n'})$ for the (trivial) bundle of all k-jets at all $x \in \mathbb{C}^n$.

Definition 1.2. We say that *the germs in S depend analytically on their k-jets at x* if the following conditions are satisfied:

(1) $f \in S$ is uniquely determined within S by $j_x^k f$, i.e. for $f_1, f_2 \in S$, $j_x^k f_1 = j_x^k f_2 \implies f_1 = f_2$;

(2) For every $f_0 \in S$, there exist neighborhoods $U(x) \subset \mathbb{C}^n$, $U(j_x^k f_0) \subset J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$ such that every $f \in S$ with $j_x^k f \in U(j_x^k f_0)$ extends holomorphically to $U(x)$;

(3) In addition there exists a holomorphic map $F \colon U(j_x^k f_0) \times U(x) \to \mathbb{C}^{n'}$ such that for all $f \in S$ with $j_x^k f \in U(j_x^k f_0)$,

$$(3) \qquad\qquad f(z) = F(j_x^k f, z), \ z \in U(x).$$

For $x \in \mathbb{C}^n$, we equip S with the inductive limit topology, i.e. a sequence (f_n) in S converges to $f_0 \in S$ if and only if all f_n extend to some neighborhood of x and converge there to f_0 uniformly.

Theorem 1.1. *Suppose that the Levi form of M is nondegenerate and surjective. Then there exists an integer $k > 0$ such that for all $x \in M$, the germs at x of local biholomorphisms f with $f(M) \subset M$ depend analytically on their k-jets. One can take $k = 2(1 + \mathrm{codim}_{\mathrm{CR}} M)$.*

As an application we obtain a Lie group structure on the set of germs fixing a point $x \in M$ (see the end of the section for the proof).

Corollary 1.1. *Let $x \in M$ be arbitrary and the Levi form of M be nondegenerate and surjective at x. Then the group of all germs of local biholomorphisms f with $f(M) \subset M$, $f(x) = x$, is a Lie group.*

Notice that the domain of definition of a germ f can vary as $j_x^k f$ changes. It is not even clear a priori whether a 1-parameter family $f_t, t \in \mathbb{R}$, of such automorphisms yields a germ of a vector field. The analytic dependence on the k-jets guarantees, in particular, the following extension result for the germs f with $j_x^k f$ sufficiently close to $j_x^k f_0$ for f_0 given.

Corollary 1.2. *Under the assumptions of Theorem 1.1 suppose that f_0 is a germ of a local biholomorphism at x with $f_0(M) \subset M$. Then there exists a neighborhood $U(x) \in \mathbb{C}^n$ such that all germs of local biholomorphisms f with $f(M) \subset M$ and $j_x^k f$ sufficiently close to $j_x^k f_0$ extend biholomorphically to $U(x)$.*

Following Tanaka [14] we call a map $\varphi \colon M \to M'$ *pseudo-conformal*, if it extends to a holomorphic map between some neighborhoods of M and M' respectively. One obtains the following global version of Corollary 1.1 (see the end of the section for the proof).

Corollary 1.3. *Let M be a* compact *CR submanifold of a complex manifold X which in addition satisfies the assumptions of Theorem 1.1. Then the group of all pseudo-conformal automorphisms of M is a Lie group.*

In fact, all of these corollaries are proved here in the more general situation, where $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n'}$ are generic real-analytic CR manifolds of arbitrary CR dimension and codimension. In particular, the unique determinacy by k-jets, i.e. the injectivity of the k-jet evaluation $f \mapsto j_x^k f$, is also shown.

In this paper f always denotes a germ of a holomorphic map between open subsets of $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ respectively. The condition "f is biholomorphic" (see [1],[2]) is relaxed to

$$(4) \qquad\qquad df_x(T_x^c M) = T_{x'}^c M',$$

where $f(x) = x'$ and $T_x^c M := T_x M \cap i T_x M \subset T_x \mathbb{C}^n$ is the complex tangent space.

The following simple example shows that even if both CR dimensions and codimensions of M and M' are equal, the case when (4) is satisfied but f is not biholomorphic is also of interest.

Example 1.1. *Let φ be strongly plurisubharmonic function in a neighborhood of $0 \in \mathbb{C}^3$ satisfying*

$$(5) \qquad\qquad \varphi(0) = 0, \quad \frac{\partial\varphi}{\partial z_3}(0) \neq 0.$$

Define M and M' in $\mathbb{C}^4$:

$$(6) \qquad M := \{(z_1, z_2, z_3, z_4) : \varphi(z_1, z_2, z_3) = \varphi(2z_1, z_2, z_4) = 0\},$$

$$(7) \qquad M' := \{(z_1, z_2, z_3, z_4) : \varphi(z_1, z_2, z_3) = Re\, z_4 = 0\}.$$

One has

$$(8) \qquad \dim_{CR} M = \dim_{CR} M' = 2, \quad \operatorname{codim}_{CR} M = \operatorname{codim}_{CR} M' = 2.$$

The ($\mathbb{C}^2$-valued) Levi forms are given by the second order derivatives $\partial^2\varphi/\partial z_i \partial\bar{z}_j$. Simple calculation shows that the Levi form of M (resp. M') is surjective (resp. nondegenerate). By Proposition 2.1 and Lemma 2.1, M and M' satisfy conditions of Theorem 1.2 below. The map f given by

$$(9) \qquad\qquad f(z_1, z_2, z_3, z_4) := (z_1, z_2, z_3, 0),$$

satisfies $f(M) \subset M'$ and condition (4) but is not a diffeomorphism between M and M'.

On the other hand, the condition (4) cannot be removed as the following elementary example shows.

Example 1.2. *Define*

$$(10) \qquad\qquad M := \{|z_1|^2 + |z_2|^2 - |z_3|^2 = 1\} \subset \mathbb{C}^3.$$

Then M contains the complex line

$$(11) \qquad\qquad L = \{z_1 = 1, z_2 = z_3\}.$$

Therefore every holomorphic map $f \colon \mathbb{C}^3 \to L$ satisfies $f(M) \subset M$. Although the Levi form of M is nondegenerate and surjective, the k-jet evaluation $f \mapsto j_x^k f$ is not injective for no integer k.

As noted above, the nondegeneracy and surjectivity of the Levi form are only sufficient conditions. The optimal conditions which are necessary in many cases are given in [2] for the case f is biholomorphic. Here we reformulate them in a form suitable for our purposes.

We first recall the notion of the Segre varieties associated to a generic real-analytic CR submanifold $M \subset \mathbb{C}^n$. Let $x \in M$ be arbitrary and let M be defined near x by the real-analytic equations

$$(12) \qquad \rho_1(z, \bar{z}) = \cdots = \rho_d(z, \bar{z}) = 0, \quad \partial \rho_1 \wedge \cdots \wedge \partial \rho_d \neq 0.$$

The complexification $\mathcal{M} \subset \mathbb{C}^n \times \overline{\mathbb{C}^n}$ is defined by

$$(13) \qquad \mathcal{M} := \{(z, \bar{w}) \in U(x) \times U(\bar{x}) : \rho_1(z, \bar{w}) = \cdots = \rho_d(z, \bar{w}) = 0\},$$

where $U(x) \subset \mathbb{C}^n$ and $U(\bar{x}) \subset \overline{\mathbb{C}^n}$ are sufficiently small neighborhoods. As a germ at $(x, \bar{x})$ of a complex-analytic subset, $\mathcal{M}$ is independent of the choice of the defining equations (12). By fixing the coordinate $\bar{w}$, we obtain the Segre variety

$$(14) \qquad Q_w := \{z \in U(x) : (z, \bar{w}) \in \mathcal{M}\}.$$

Segre varieties were introduced by Segre [13] and play an important role in the reflection principle (see e.g. [12, 10, 9, 16, 17]). For z, w close to x, it follows that Q_w is a complex manifold of dimension $n - d = \dim_{CR} M$. The following symmetry property is a direct corollary of the invariance of $\mathcal{M}$ under the involution $(z, \bar{w}) \mapsto (w, \bar{z})$:

$$(15) \qquad z \in Q_w \iff w \in Q_z.$$

For arbitrary $x \in \mathbb{C}^n$, denote by $J_x^{k,d}(\mathbb{C}^n)$ the space of k-jets at x of d-codimensional complex submanifolds $V \subset \mathbb{C}^n$ with $x \in V$ and by $j_x^k(V) \in J_x^{k,d}(\mathbb{C}^n)$ the k-jet of V. The following is another form of the definition of a k-nondegenerate manifold given in [1].

Definition 1.3. A generic real-analytic CR manifold $M \subset \mathbb{C}^n$ is called k-*nondegenerate* at $x \in M$ ($k \geq 1$), if the (antiholomorphic) map

$$(16) \qquad \varphi^k \colon Q_x \to J_x^{k,d}(\mathbb{C}^n), \; w \mapsto j_x^k(Q_w),$$

is of the rank $n - d = \dim Q_x$ at x. M is called *finitely nondegenerate*, if it is k-nondegenerate for some k.

A manifold M is called *essentially finite* at x if the germ of Q_x at x is different from germs of Q_w at x for all $w \in Q_x \setminus \{x\}$ sufficiently close to x. This notion was introduced in [4] and earlier implicitly discussed in [8]. By Proposition 1.3.1 in [1], if M is essentially finite at x, there exists the so-called Levi number $l(M)$. Then M is $l(M)$-nondegenerate at all points $p \in M \setminus W$, where $W \subset M$

is a proper real-analytic subset. Clearly the k-nondegeneracy guarantees the k'-nondegeneracy for all $k' \geq k$.

The second important notion is the minimality condition introduced by Tumanov in [15]:

Definition 1.4. A CR manifold M is called *minimal* at $x \in M$, if there does not exist a proper CR submanifold $N \subset M$ with $x \in N$ such that $\dim_{CR} N = \dim_{CR} M$.

Tumanov [15] shows that, if M is minimal at x, all CR functions on M extend holomorphically to a wedge with the edge M.

Definition 1.5. We call a germ f *admissible* if it satisfies the following conditions:

$$\text{(17)} \qquad f(M) \subset M', \quad df(T_x^c M) = T_{f(x)}^c M'.$$

Theorem 1.2. *Suppose that M' is r-nondegenerate, M is minimal at x and $k := 2r(1 + \operatorname{codim}_{CR} M)$. Let S be the set of all admissible germs f at x. Then the germs in S depend analytically on their k-jets at x.*

A different proof of Theorem 1.2 in the case of hypersurfaces ($\operatorname{codim}_{CR} M = \operatorname{codim}_{CR} M' = 1$) with a sharper estimate for the jet order ($2r$-jets instead of $4r$-jets) has been obtained recently by Baouendi, Ebenfelt and Rothschild [3].

Remark. The case, where one of the manifolds M, M' is not generic, can be reduced to the generic case. Every real-analytic CR submanifold $M \subset \mathbb{C}^n$ is generic in the so-called *intrinsic complexification* $V \subset \mathbb{C}^n$ defined to be the minimal (in the sense of germs) complex-analytic subvariety which contains M. If M is CR, V is smooth. After a change of local coordinates V becomes open in $\mathbb{C}^m \subset \mathbb{C}^n$ near $x \in M$. The condition $f(M) \subset M'$ automatically implies $f(V) \subset V'$. If M' is not generic, we can therefore replace $\mathbb{C}^{n'}$ with V'. If M is not generic, Theorem 1.2 yields a parametrization (3) for the restricted admissible germs $f : V \to \mathbb{C}^{n'}$, whereas outside V the germs can be chosen arbitrary and cannot be determined by their s-jets even for s arbitrary large.

As a corollary, we obtain the following description of the space of admissible germs as a (locally closed) real-analytic subset of $J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$.

Corollary 1.4. *Let k, r, x, M, M', S satisfy the conditions of Theorem* 1.2.
(1) Then the set $A \subset J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$ of the k-jets of all $f \in S$ is a locally closed real-analytic subset.
(2) There exists a neighborhood U of $A \times \{x\} \subset A \times \mathbb{C}^n$ and a (unique) real-analytic map

$$\text{(18)} \qquad G : U \to \mathbb{C}^{n'},$$

which is holomorphic in the $\mathbb{C}^n$-factor and is such that every $f \in S$ satisfies

$$(19) \qquad\qquad f(w) = G(j_x^k f, w).$$

Proof. By Theorem 1.2, the germs in S depend analytically on their k-jets. Let M, M' be closed in some open subsets

$$(20) \qquad\qquad U(M) \subset \mathbb{C}^n, \quad U(M') \subset \mathbb{C}^{n'}$$

respectively. For every $f_0 \in S$, Theorem 1.2 yields the local parametrization (3). Without loss of generality, $U(x) \subset U(M)$ and

$$(21) \qquad\qquad F(U(j_x^k f_0) \times U(x)) \subset U(M').$$

Then the above set $A \subset J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$ is locally defined by

$$(22) \quad A \cap U(j_x^k f_0) := \{j \in U(j_x^k f_0) :$$
$$F(j, w) \subset M' \text{ for all } w \in M \cap U(x), \quad j = j_x^k F(j, \cdot)\}.$$

It follows from the real-analyticity of M and M' that A is locally defined by finitely many real-analytic equations, i.e. it is a real-analytic subset. The restrictions of F's to $A \times U(x)$ for different f_0's glue together to a well-defined real-analytic map G satisfying the required properties.

Proof.[Corollary 1.1] By Corollary 1.4, the set $A \subset J_x^k(\mathbb{C}^n, \mathbb{C}^{n'})$ of the k-jets $(k := 2(d+1)r)$ of all admissible germs is real-analytic. Denote by $A_0 \subset A$ the subset of the k-jets fixing x. Clearly A_0 is also real-analytic and locally closed. Since A_0 is also closed under the composition of the jets, it is a closed Lie subgroup of the Lie group of all k-jets fixing x.

Proof.[Corollary 1.3] By Theorem 1.2 and the definition of analytic dependence on k-jets, there exists a finite open covering of M with coordinate neighborhoods $U(x_i)$ with parametrizations

$$(23) \qquad\qquad F_i \colon U(j_{x_i}^k f_0) \times U(x_i) \to \mathbb{C}^n,$$

where $f_0 := \mathrm{id} \in \mathrm{Aut}(M)$. An automorphism $f \in \mathrm{Aut}(M)$ in a neighborhood of id is therefore uniquely determined by finitely many parameters

$$(24) \qquad\qquad p_i = j_{x_i}^k f, \quad q_i = j_{x_i}^k f^{-1}.$$

Set $f_i := F_i(p_i, \cdot)$, $g_i := F_i(q_i, \cdot)$. A tuple $(p_1, \ldots, p_m, q_1, \ldots, q_m)$ determines an automorphism of M if and only if the following is satisfied:

(1) $f_i = f_j$, $g_i = g_j$ on $U(x_i) \cap U(x_j)$ for all i, j;
(2) $f_i(M) \subset M$, $g_i(M) \subset M$ for all i;
(3) The glued maps f_p and g_q satisfy $f_p \circ g_q = g_q \circ f_p = \mathrm{id}$, $j_{x_i}^k f_p = p_i$, $j_{x_i}^k g_q = q_i$.

Conditions (1.),(2.),(3.) define an analytic subset of the parameter space of p_i's and q_i's. In these coordinates the group operation is given by

$$((p,q),(p',q')) \mapsto (j^k_{x_i}(f_{p'} \circ f_p), j^k_{x_i}(g_q \circ g_{q'}))$$

and is therefore real-analytic. Hence $\mathrm{Aut}(M)$ is a Lie group.

2. The Levi form and nondegeneracy conditions

Let $M \subset \mathbb{C}^n$ be a CR submanifold and $x \in M$ an arbitrary point. Recall that a $(1,0)$-vector field on M is a vector field X in $T^c M \otimes_{\mathbb{R}} \mathbb{C}$ such that $JX = iX$, where $J \colon T^c M \otimes_{\mathbb{R}} \mathbb{C} \to T^c M \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the CR structure $J \colon T^c M \to T^c M$ and iX is the multiplication by i in the component $\mathbb{C}$ of the tensor product.

The Levi form of M at x is the hermitian (vector-valued) form

(25) $$L \colon T^{1,0}_x M \times T^{1,0}_x M \to (T_x M / T^c_x M) \otimes_{\mathbb{R}} \mathbb{C},$$

given by

(26) $$L(X,Y) := \frac{1}{2i} \pi([X, \bar{Y}]),$$

where X, Y are $(1,0)$-vector fields on M and

(27) $$\pi \colon T_x M \otimes_{\mathbb{R}} \mathbb{C} \to (T_x M / T^c_x M) \otimes_{\mathbb{R}} \mathbb{C}$$

is the canonical projection. Notice that $L(X,Y)(x)$ depends only on $X(x), Y(x)$.

Remark. If $M \subset \mathbb{C}^n$ is a hypersurface defined by $M = \{\varphi = 0\}$ with $d\varphi \neq 0$, the standard Levi form of φ coincides with the evaluation of $\partial \varphi$ on $L(X,Y)$.

The Levi form is a first order holomorphic invariant of M whereas the conditions in Theorem 1.2 are of possibly higher order. However, we obtain the following

Proposition 2.1. *A generic real-analytic CR manifold $M \subset \mathbb{C}^n$ is 1-nondegenerate at x if and only if the Levi form at x is nondegenerate.*

Proof. Let M be locally defined near x by the real-analytic equations

(28) $$\rho_1(z, \bar{z}) = \cdots = \rho_d(z, \bar{z}) = 0,$$

where $\partial \rho_1 \wedge \cdots \wedge \partial \rho_d \neq 0$.

The differentials $\partial \rho_j$ vanish on $T^c M \otimes_{\mathbb{R}} \mathbb{C}$. Hence, for a $(1,0)$-vector field $X \subset T^{1,0} M$,

(29) $$\partial \rho_j(X) = \partial \rho_j(\bar{X}) = 0.$$

Denote by $X\psi$ the derivative of ψ along X.

Applying the formula

(30) $$d\omega(X_1, X_2) = X_1 \omega(X_2) - X_2 \omega(X_1) - \omega([X_1, X_2])$$

for $\omega = \partial\rho_j$, $X_1 = X, X_2 = \bar{Y}$, we obtain

(31) $$\bar{\partial}\partial\rho_j(X,\bar{Y}) = -\partial\rho_j([X,\bar{Y}]).$$

The Segre varieties Q_w are given by

(32) $$\rho_1(z,\bar{w}) = \cdots = \rho_d(z,\bar{w}) = 0$$

and their 1-jets $j_x^1 Q_w$ by the linear equations

(33) $$\partial\rho_1(z,\bar{w}) = \cdots = \partial\rho_s(z,\bar{w}) = 0.$$

Let $e_1(z,\bar{w}), \ldots, e_d(z,\bar{w})$ be a collection of $(1,0)$-vector fields on the complexification $\mathcal{M}$ which is a basis of $T_z Q_w$ at every point $(z,\bar{w}) \in \mathcal{M}$ close to $(x,\bar{x})$. Let $e_1', \ldots, e_d'$ be a similar collection of $(1,0)$-vector fields in the w-direction, i.e. pointwise a basis of $T_{\bar{w}}\overline{Q_{\bar{z}}}$. The rank of the map φ^1 in Definition 1.3 is the same as the rank of the matrix with the rows

(34) $$\bar{\partial}\partial\rho_j(e_k, \bar{e}_l') \text{ with } l \text{ fixed}.$$

By (31), this matrix has the rows

(35) $$-\partial\rho_j([e_k, \bar{e}_l']) \text{ with } l \text{ fixed}.$$

Since $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d \neq 0$, the tuple $(\partial\rho_1, \ldots, \partial\rho_d)$ defines an isomorphism between $(TM/T^cM) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C}^d$. Hence the matrix with the rows (35) has the same rank as the matrix with the rows

(36) $$L(e_k, e_l) \text{ with } l \text{ fixed}.$$

This rank equals to $n - d$ if and only if the Levi form of M is nondegenerate.

The minimality condition (Definition 1.4) involves high order commutators and therefore cannot be formulated in terms of the Levi form. However, one obviously has the following sufficient condition.

Lemma 2.1. *Suppose that the Levi form of M at x is surjective onto $(T_x M/T_x^c M) \otimes_{\mathbb{R}} \mathbb{C}$. Then M is minimal at x.*

Now we can formulate special cases of Theorem 1.2 and Corollary 1.4 under Levi form conditions.

Theorem 2.1. *Suppose that the Levi form of M' is everywhere nondegenerate, the Levi form of M is surjective at x and $d = \operatorname{codim}_{\mathrm{CR}} M$. Then the conclusions of Theorem 1.2 and Corollary 1.4 hold.*

3. Local parametrization of jets of holomorphic maps

One of the technical tools for proving Theorem 1.2 is the following connection between the k-jets at z and the $(k + r)$-jets at $\bar{w}$. For simplicity, we write for a subset $U(j) \subset J^{k+r}(\mathbb{C}^n, \mathbb{C}^{n'})$,

$$(37) \qquad U_z(j) := U(j) \cap J_z^{k+r}(\mathbb{C}^n, \mathbb{C}^{n'}).$$

Proposition 3.1. *Suppose that M' is r-nondegenerate at x' and f_0 is an admissible germ at $x \in M$, $x' = f_0(x)$. Then, for every integer k there exist neighborhoods $U(j_x^{k+r} f_0) \subset J^{k+r}(\mathbb{C}^n, \mathbb{C}^{n'})$, $U(x, \bar{x}) \subset \mathcal{M}$ and for every $(z, \bar{w}) \in U(x, \bar{x})$ a holomorphic map $F_{(z, \bar{w})}^k \colon U_z(j_x^{k+r} f_0) \to J_{\bar{w}}^k(\overline{\mathbb{C}^n}, \overline{\mathbb{C}^{n'}})$, such that for all admissible germs f at x with $j_x^{k+r} f \in U(j_x^{k+r} f_0)$,*

$$(38) \qquad j_{\bar{w}}^k \bar{f} = F_{(z, \bar{w})}^k(j_z^{k+r} f),$$

where $(z, \bar{w}) \subset \mathcal{M}$ is sufficiently close to $(x, \bar{x})$. Moreover, the map $F_{(z, \bar{w})}^k$ depends holomorphically on $(z, \bar{w}) \in U(x, \bar{x})$.

A different proof of Proposition 3.1 is given in [1] (Assertion 3.3.1) and [2] (Proposition 2.2).

Proof. Let $(z, \bar{w}) \in \mathcal{M}$ be close to $(x, \bar{x})$. By the construction (see (12),(13),(14)), the Segre variety Q_w is smooth at z and $d := \dim_z Q_w$ is constant. We choose local coordinates $z = (z_1, z_2) \in \mathbb{C}^d \times \mathbb{C}^{n-d}$ near x such that the Segre variety Q_w has the form of the graph:

$$(39) \qquad Q_w = \{(z_1, \varphi_w(z_1)) : z_1 \in U(x_1)\}, \quad x = (x_1, x_2),$$

where $U(x_1) \subset \mathbb{C}^n$ is an open neighborhood and $\varphi_w \colon U(x_1) \to \mathbb{C}^{n-d}$ is holomorphic. The map φ_w is uniquely defined and depends holomorphically on $\bar{w}$ in some neighborhood $U(\bar{x}) \subset \overline{\mathbb{C}^n}$. Denote by $j_z^r Q$ the holomorphic r-jet evaluation

$$(40) \qquad j_z^r Q \colon U(\bar{x}) \to J_{z_1}^r(\mathbb{C}^d, \mathbb{C}^{n-d}), \quad \bar{w} \mapsto j_{z_1}^r \varphi_w.$$

A similar evaluation is obtained for $(z', \bar{w}') \in \mathcal{M}'$ sufficiently close to $(x', \bar{x}')$:

$$(41) \qquad j_{z'}^r Q' \colon U(\bar{x}') \to J_{z_1'}^r(\mathbb{C}^{d'}, \mathbb{C}^{n'-d'}), \quad \bar{w}' \mapsto j_{z_1'}^r \varphi_{w'}'.$$

We claim that (41) is an immersion. Since M' is r-nondegenerate at $x' = (x_1', x_2')$, $j_x^r Q'$ restricted to the Segre variety $Q_{x'}'$ is an immersion. By the construction, the 0-jet evaluation $j_{x'}^0 Q' = x_2'$ is constant on $Q_{x'}'$. On the other hand, the restriction of $j_{x'}^0 Q'$ to the transversal direction, i.e. $\{x_1'\} \times \mathbb{C}^{n'-d'}$, is also an immersion. Indeed, since $M' \cap (\{x_1'\} \times \mathbb{C}^{n'-d'})$ is totally real in $\{x_1'\} \times \mathbb{C}^{n'-d'}$, it is locally biholomorphically equivalent to $\mathbb{R}^{n'-d'}$. For $\mathbb{R}^s \subset \mathbb{C}^s$, the immersion property of $j_{x'}^0 Q'$ can be directly verified.

Hence the total map (41) splits into $j_{x'}^0 Q'$ which is an immersion on $\{x_1'\} \times \mathbb{C}^{n'-d'}$ and constant on $Q_{x'}'$ and the remainder which is an immersion on $Q_{x'}'$.

Therefore (41) is an immersion for z close to x and locally there exists a left inverse map $(j_z^r Q')^{-1}$ which satisfies $(j_z^r Q')^{-1} \circ (j_z^r Q') = \mathrm{id}$.

Let f_0 be an admissible germ. Complexifying the condition $f_0(M) \subset M'$ we obtain $(f_0, \bar{f}_0)(\mathcal{M}) \subset \mathcal{M}'$, which means

$$(42) \qquad\qquad f_0(Q_w) \subset Q'_{f_0(w)}.$$

Since $\dim Q_w$ is constant, the second condition $df(T_x^c M) = T_{f(x)}^c M'$ implies

$$(43) \quad \dim f_0(Q_w) = \dim df_0(T_x Q_x) = \dim df_0(T^c M)$$
$$= \dim T^c M' = \dim Q_{x'} = \dim Q'_{f_0(w)}.$$

Along with (42) this yields

$$(44) \qquad\qquad f_0(Q_w) = Q'_{f_0(w)}, \quad df_0(T_z Q_w) = T_{z'} Q'_{w'}.$$

After going to the k-jet evaluations we obtain the following commutative diagram

$$(45)$$
$$
\begin{array}{ccc}
U(\bar{x}) & \xrightarrow{\;j_z^r Q\;} & J_{z_1}^r(\mathbb{C}^d, U(x_2)) \\[4pt]
{\scriptstyle \bar{f}_0}\big\downarrow & & \big\downarrow{\scriptstyle f_{0*}} \\[4pt]
U(\bar{x}') & \xrightarrow{\;j_{z'}^r Q'\;} & J_{z_1'}^r(\mathbb{C}^{d'}, U(x_2')),
\end{array}
$$

where f_{0*} is the corresponding map on the jet level defined as follows.

By (44), there exists a local splitting of the source space $\mathbb{C}^d = \mathbb{C}^{d'} \times \mathbb{C}^{d-d'}$ near $x_1 = (x_{11}, x_{12})$ such that the restriction

$$(46) \qquad\qquad f_0 \colon Q_w \cap \{z_{12} = x_{12}\} \to Q'_{w'}$$

is locally biholomorphic for all $(z, \bar{w}) \in \mathcal{M}$ close to $(x, \bar{x})$.

We write $f_{01} \colon U(x) \to \mathbb{C}^{d'}$, $f_{02} \colon U(x) \to \mathbb{C}^{n'-d'}$ for the components of f_0 (here $\mathbb{C}^{d'}$ and $\mathbb{C}^{n'-d'}$ are in the target space). Since (46) is locally biholomorphic, the map

$$(47) \qquad\qquad \tilde{f}_0 \colon \mathbb{C}^{d'} \to \mathbb{C}^{d'}, \quad z_{11} \mapsto f_{01}(z_{11}, x_{12}, \varphi_{\bar{w}}(z_{11}, x_{12}))$$

is also locally biholomorphic.

Let $\tilde{f}_0^{-1}$ be the local inverse. By (39) and (44)

$$(48) \qquad\qquad \varphi'_{\bar{w}'}(z_1') = f_{02}(\tilde{f}_0^{-1}(z_1'), x_{12}, \varphi_{\bar{w}}(\tilde{f}_0^{-1}(z_1'), x_{12}))$$

and hence, passing to the r-jets,

$$(49) \qquad j_{z_1'}^r \varphi'_{\bar{w}'} = (j_z^r f_{02})((j_{z_{11}}^r \tilde{f}_0)^{-1}, x_{12}, (j_{z_1}^r \varphi_{\bar{w}})((j_{z_{11}}^r \tilde{f}_0)^{-1}, x_{12})),$$

where by (47),

$$(50) \qquad\qquad j_{z_{11}}^r \tilde{f}_0 = (j_z^r f_{01})(z_{11}, x_{12}, (j_{z_1}^r \varphi_{\bar{w}})(z_{11}, x_{12}))$$

with obvious notation.

The formulae (49,50) can be written together in the form

$$(51) \qquad j^r_{z'_1} \varphi'_{\bar{w}'} = \Phi(j^r_z f_0, j^r_{z_1} \varphi_{\bar{w}}),$$

where

$$(52) \qquad \Phi \colon U_z(j^r_x f_0) \times U_{z_1}(j^r_{x_1} \varphi_{\bar{x}}) \to J^r_{z'_1}(\mathbb{C}^{d'}, U(x'_2))$$

is a holomorphic family of maps and $U(j^r_x f_0) \subset J^r(\mathbb{C}^n, \mathbb{C}^{n'})$, $U(j^r_{x_1} \varphi_{\bar{x}}) \subset J^r(\mathbb{C}^d, \mathbb{C}^{n-d})$ are sufficiently small neighborhoods.

Define f_{0*} as in (45) by

$$(53) \qquad f_{0*} := \Phi(j^r_z f_0, \cdot).$$

By (51), the diagram (45) is commutative.
The commutativity means that

$$(54) \qquad (j^r_{z'} Q') \circ \bar{f}_0 = f_{0*} \circ (j^r_z Q).$$

Applying the left inverse $(j^r_{z'} Q')^{-1}$ to both sides we obtain

$$(55) \qquad \bar{f}_0 = (j^r_{z'} Q')^{-1} \circ f_{0*} \circ (j^r_z Q)$$

and, passing to the k-jets,

$$(56) \qquad j^k_{\bar{w}} \bar{f}_0 = j^k_{g'} (j^r_{z'} Q')^{-1} \circ j^k_g f_{0*} \circ j^k_{\bar{w}} (j^r_z Q),$$

where $g := (j^r_z Q)(\bar{w})$ and $g' := j^k_g f_{0*}(g)$.

The k-jet $j^k_g f_{0*}$ can be calculated from (53):

$$(57) \qquad j^k_g f_{0*} := (j^k_s \Phi)(j^{k+r}_z f_0, \cdot), \quad s := (j^r_z f_0, g).$$

We write (56) and (57) together in the form

$$(58) \qquad j^k_{\bar{w}} \bar{f}_0 = F^k_{z,\bar{w}}(j^{k+r}_z f_0),$$

where

$$(59) \qquad F^k_{z,\bar{w}} \colon U_z(j^{k+r}_x f_0) \to J^k_{\bar{w}}(\overline{\mathbb{C}^n}, \overline{\mathbb{C}^{n'}})$$

is a holomorphic family of maps and $U(j^{k+r}_x f_0) \subset J^{k+r}(\mathbb{C}^n, \mathbb{C}^{n'})$ a sufficiently small open neighborhood.

Now we notice that f_0 was an arbitrary admissible germ. Let f be another one with $j^{k+r}_x f \in U(j^{k+r}_x f_0)$. Then for $(z, \bar{w}) \in \mathcal{M}$ sufficiently close to $(x, \bar{x})$, (48)-(58) remain valid if we replace f_0 with f. In particular, the proposition follows from (58) with f_0 replaced by f.

4. Proof of Theorem 1.2

Given j_z^{k+r} for $z \in \mathbb{C}^n$ fixed, (38) yields the k-jet $j_w^k f$, in particular f itself ($f(w) = j_w^0 f$). However, the point w is not arbitrary but must satisfy

$$(60) \qquad (z, \bar{w}) \in \mathcal{M}, \text{ i.e. } w \in Q_z.$$

Our next goal will be to obtain a similar formula which yields f at least on an open subset of $\mathbb{C}^n$. The idea here is to iterate the construction of the previous paragraph.

4.1. Iterated complexification. The iterated complexification $\mathcal{M}^s$ of $\mathcal{M}$ is defined by

$$(61) \quad \mathcal{M}^{2l-1} := \{(z_1, \bar{w}_1, \dots, z_l, \bar{w}_l) \in \mathbb{C}^n \times \overline{\mathbb{C}^n} \times \cdots \times \mathbb{C}^n \times \overline{\mathbb{C}^n} :$$
$$(z_1, \bar{w}_1) \in \mathcal{M}, (w_1, \bar{z}_2) \in \mathcal{M}, \dots, (w_{l-1}, \bar{z}_l) \in \mathcal{M}, (z_l, \bar{w}_l) \in \mathcal{M}\},$$

if $s = 2l - 1$ is odd and by

$$(62) \quad \mathcal{M}^{2l} := \{(z_1, \bar{w}_1, \dots, z_l, \bar{w}_l, z_{l+1}) \in \mathbb{C}^n \times \overline{\mathbb{C}^n} \times \cdots \times \mathbb{C}^n \times \overline{\mathbb{C}^n} \times \mathbb{C}^n :$$
$$(z_1, \bar{w}_1) \in \mathcal{M}, (w_1, \bar{z}_2) \in \mathcal{M}, \dots, (z_l, \bar{w}_l) \in \mathcal{M}, (w_l, \bar{z}_{l+1}) \in \mathcal{M}\},$$

if $s = 2l$ is even.

For simplicity we restrict the proof of Theorem 1.2 in the sequel to the case (61) of odd iterates. The case (62) of even iterates is completely analogous. We write

$$(63) \qquad \mathcal{M}_z^s := \mathcal{M}^s \cap \{z_1 = z\}.$$

Lemma 4.1. *The analytic subset $\mathcal{M}^s \subset \mathbb{C}^N$ is smooth at $(x, \dots, \bar{x})$ and the projection on the first copy of $\mathbb{C}^n$ is of rank n at this point. $\mathcal{M}_x^s$ is also smooth at $(x, \dots, \bar{x})$.*

Proof. By the construction of the Segre varieties (see (12),(13),(14)), there exists a local splitting $\overline{\mathbb{C}^n(w)} = \overline{\mathbb{C}^d(w^1)} \times \overline{\mathbb{C}^{n-d}(w^2)}$ and a germ of a local holomorphic map $\varphi \colon \mathbb{C}^n \times \overline{\mathbb{C}^d} \to \overline{\mathbb{C}^{n-d}}$ such that the germ

$$(64) \qquad \mathbb{C}^n \times \overline{\mathbb{C}^d} \to \mathcal{M}, \quad (z, \bar{w}^1) \mapsto (z, \bar{w}^1, \varphi(z, \bar{w}^1)),$$

is biholomorphic. Then the germ

$$(65) \quad \Phi \colon \mathbb{C}^n \times \overline{\mathbb{C}^d} \times \mathbb{C}^d \times \cdots \times \overline{\mathbb{C}^d} \to \mathcal{M}^s,$$
$$(z_1, \bar{w}_1^1, \dots, z_l^1, \bar{w}_l^1) \mapsto$$
$$(z_1, \bar{w}_1^1, \varphi(z_1, \bar{w}_1^1)), z_2^1, \bar{\varphi}(\bar{w}_1^1, \varphi(z_1, \bar{w}_1^1), z_2^1), \dots, \bar{w}_l^1, \varphi(\dots, \bar{w}_l^1)),$$

is also biholomorphic, where $s = 2l - 1$. Furthermore, the subset $\mathcal{M}_x^s$ is equal to $\Phi(\{z_1 = x_1\})$. This proves the lemma.

Lemma 4.2. *Under the assumptions of Proposition* 3.1, *for every integers* $s \geq 1$, $l \geq 0$, $m := sr + l$, *there exist open neighborhoods*

$$(66) \qquad U(j_x^m f_0) \subset J^m(\mathbb{C}^n, \mathbb{C}^{n'}), \quad U(x, \bar{x}, \dots, \bar{x}) \subset \mathcal{M}^s$$

and for every $\nu = (z, \bar{w}_1, \dots, z_l, \bar{w}) \in \mathcal{M}^s$, *a family of holomorphic maps*

$$(67) \qquad F_\nu^l : U_z(j_x^m f_0) \to J_{\bar{w}}^l(\overline{\mathbb{C}^n}, \overline{\mathbb{C}^{n'}})$$

such that for all admissible germs f *at* x *with* $j_x^m f \in U(j_x^m f_0)$,

$$(68) \qquad j_{\bar{w}}^l \bar{f} = F_\nu^l(j_z^m f),$$

where z, w *and all* z_j, w_j *are sufficiently close to* x. *Moreover,* F_ν *depends holomorphically on* $\nu \in \mathcal{M}^s$.

Proof. The required maps $F_\nu^l, \nu \in \mathcal{M}^s$, are obtained as iterates of (38):

$$(69) \qquad F_\nu(j) := \overline{F_{(w_l, \bar{z}_{l+1})}^l} \circ F_{(z_l, \bar{w}_l)}^{r+l} \circ \cdots \circ F_{(z_1, \bar{w}_1)}^{(s-1)r+l}(j).$$

Here we iterate step by step the map in (38) and its conjugate. Notice that the conjugate is taken exactly for z_j's and w_j's conjugated to the coordinates of ν. Hence F_ν depends holomorphically on $\nu \in \mathcal{M}^s$. The required formula (68) for admissible germs is obtained by iterating (38).

4.2. Segre sets. The right-hand side in (68) depends on ν which contains several coordinates z_j, w_j other than z and w. To avoid this ambiguity we project out the auxiliary coordinates. The corresponding projection of $\mathcal{M}^s$ is a family of the so-called *Segre sets* introduced in [1]. The family of Segre sets

$$(70) \qquad Q^{2l+1} \subset \mathbb{C}^n \times \overline{\mathbb{C}^n}, \ Q^{2l} \subset \mathbb{C}^n \times \mathbb{C}^n$$

is defined as the projection on the product of the fist space $\mathbb{C}^n$ and the last one (which is either $\overline{\mathbb{C}^n}$ or $\mathbb{C}^n$) of $\mathcal{M}^s$ for $s = 2l + 1$ or $s = 2l$ respectively.

Notice that, on the contrary to the family of Segre varieties $Q_w = Q_w^1$, the family of Segre sets Q_w^s for $s > 1$ is not analytic in general and depends on the neighborhood, where $\mathcal{M}$ is taken. A smaller neighborhood of $(x, \bar{x}) \in \mathcal{M}$ induces a family of smaller Segre sets in *each* neighborhood of $(x, \bar{x})$.

Let

$$(71) \qquad Q_w^s := \{z \in \mathbb{C}^n : (z, \bar{w}) \in Q^s\}$$

be the s-th Segre set associated with $w \in \mathbb{C}^n$. We make use of the following basic properties of the Segre sets (see [1], §2.2, in particular Corollary 2.2.2):

Theorem 4.1 (Baouendi, Ebenfelt, Rothschild). *Let* $M \subset \mathbb{C}^n$ *be a real-analytic CR submanifold. Then the dimension of* Q_z^s *increases strictly until it stabilizes. If* M *is minimal at* $x \in M$, *then* Q_x^s *contains an open subset of* $\mathbb{C}^n$ *for* s *sufficiently large.*

Since

$$\dim Q_w^1 = \dim Q_w = \dim_{\mathrm{CR}} M, \tag{72}$$

we can take $s = d + 1$, where $d = \mathrm{codim}_{\mathrm{CR}} M$. By Theorem 4.1, the generic rank of the projection

$$\mathcal{M}_x^{d+1} \to \overline{\mathbb{C}^n} \tag{73}$$

is n. Unfortunately this is true only generically whereas the rank of (73) at $(x, \bar{x}, \dots, \bar{x})$ is usually not maximal. Hence we cannot lift Q_x^s to M in a neighborhood of x but we shall do this in an open set of $\overline{\mathbb{C}^n}$ with $\bar{x}$ on the boundary.

4.3. Liftings of the Segre sets to the iterated complexification. Our next goal will be to obtain a formula similar to (68) where the family of holomorphic maps is parametrized by $w \in Q_z^s$ instead of $\nu \in \mathcal{M}^s$. For this we choose carefully open subsets of the Segre sets Q_z^s (see (71)) and lift them simultaneously to $\mathcal{M}_z^s$. The pullbacks of (67) under these liftings will yield the required formulae.

We start with some elementary lemmata which we prove here for convenience of the reader.

Lemma 4.3. *Let X, Y be complex manifolds, $x_0 \in X$, $\dim X = \dim Y = n \geq 1$, $Y' \subset Y$ a submanifold and $f \colon X \to Y$ a holomorphic map with $y_0 := f(x_0) \in Y'$. Suppose that f is of the maximal rank n on a dense subset of $f^{-1}(Y')$. Then there exists an open subset $N \subset Y$ and a holomorphic lifting $\psi \colon N \to X$ (i.e. $f \circ \psi = \mathrm{id}_N$) such that the following is satisfied:*

    (1) $N \cap Y'$ is connected;
    (2) y_0 is in the closure of $N \cap Y'$;
    (3) ψ extends continuously to y_0 with $\psi(y_0) = x_0$.

Proof. Without loss of generality, X is an open subset in $\mathbb{C}^n$. Set

$$D := \{x \in X : \mathrm{rank}_x f < n\}. \tag{74}$$

Since $D \subset X$ is an analytic subset and $D \cap f^{-1}(Y')$ is nowhere dense, there exists a local holomorphic curve $C \subset f^{-1}(Y')$ such that $C \cap C \subset \{y_0\} \subset C$. Then the restriction of f to C is a ramified covering given by $z \mapsto z^k$ with respect to some local coordinates. Therefore there exists a real curve

$$\gamma \colon [0, 1] \to C, \quad \gamma(0) = x_0, \quad f \text{ is injective on } \gamma[0, 1]. \tag{75}$$

Let $t_0 \in [0, 1)$ be minimal with the property that there exist N and ψ satisfying (1.) and (3.) in this lemma and such that

$$f(\gamma(t)) \subset N \text{ for } t \in (t_0, 1]. \tag{76}$$

By the construction of γ, t_0 exists. It is sufficient to show that $t_0 = 0$.

Suppose that $t_0 > 0$. Since $\mathrm{rank}_{\gamma(t_0)} f = n$, f is locally biholomorphic at $\gamma(t_0)$. Therefore the lifting ψ can be uniquely extended to a neighborhood U of $f(\gamma(t_0))$. Set $N_1 := N \cup U$ and further

$$(77) \qquad N_2 := \{y \in N_1 : d(\psi(y), \gamma[0,1]) < d(\psi(y), f^{-1}(y_0))\},$$

where d is the euclidean distance with respect to the ambient coordinates of $\mathbb{C}^n$. Then N_2 and ψ satisfy (3.). By passing to a smaller neighborhood of $f(\gamma[t_0, 1])$ we also obtain (1.). Hence t_0 is not minimal and we have a contradiction.

The following lemma is an elementary fact from linear algebra.

Lemma 4.4. *Let* $A_1 \colon V \to W_1$, $A_2 \colon V \to W_2$ *be linear maps between vector spaces such that*

(1) A_1 *is surjective;*
(2) $A_2 | \mathrm{Ker}\, A_1$ *is surjective.*

Then the direct sum $A_1 \oplus A_2 \colon V \to W_1 \oplus W_2$ *is also surjective.*

In the following lemma let Y_1, Y_2, Y_3 be complex manifolds, $X \subset Z := Y_1 \times Y_2 \times Y_3$ a submanifold and $a = (a_1, a_2, a_3) \in X$. Denote by π_k the natural projection from X to Y_k, $k = 1, 2, 3$. Further we use the notation

$$(78) \qquad X_a^1 := \{z \in X : z_1 = a_1\}, \quad X_a^2 := \{z \in X : z_2 = a_2\}$$

and similarly N_a^1, N_a^2 for $N \subset Y_1 \times Y_2$.

Lemma 4.5. *Suppose that* X_a^1 *is a submanifold and for all z from a dense subset of* X_a^1 $\mathrm{rank}_z \pi_1 = \dim Y_1$ *and* $\mathrm{rank}_z(\pi_2 | X_a^1) = \dim Y_2$. *Then there exist an open subset* $N \subset Y_1 \times Y_2$ *and a holomorphic lifting* $\psi \colon N \to X$ *(i.e.* $(\pi_1 \times \pi_2) \circ \psi = \mathrm{id}_N$*) such that the following is satisfied:*

(1) N_a^1 *is connected;*
(2) (a_1, a_2) *is in the closure of* N_a^1;
(3) ψ *extends continuously to* (a_1, a_2) *with* $\psi(a_1, a_2) = a$.

Proof. Fix some local coordinates in Z near a (independently of the product structure) such that X and X_a^1 equal to the unit balls in corresponding linear subspaces. By the assumptions, there exists $z \in X_a^1$, $z_2 \neq a_2$, such that

$$(79) \qquad \mathrm{rank}_z \pi_1 = \dim Y_1, \quad \mathrm{rank}_z(\pi_2 | X_a^1) = \dim Y_2.$$

By Lemma 4.4, $\mathrm{rank}_z(\pi_1 \times \pi_2) = \dim Y_1 + \dim Y_2$ and therefore the fibers

$$(80) \qquad (\pi_1 \times \pi_2)^{-1}(w_1, w_2)$$

through w are smooth and of constant dimension for $w \in X$ close to z.

We claim that z with this property can be chosen such that a does not lie in the tangent subspace at z to the fiber $(\pi_1 \times \pi_2)^{-1}(z_1, z_2)$ (the tangent subspace is understood as an affine subspace of the ambient coordinate space). Otherwise a would lie in every fiber which contradicts the assumptions.

Therefore there exists a linear subspace $L \subset Y_1 \times Y_2 \times Y_3$ through a and z which is transverse and of complementary dimension to the fiber $(\pi_1 \times \pi_2)^{-1}(z_1, z_2)$. Then $\mathrm{rank}_z(\pi_1 \times \pi_2) = \dim(Y_1 \times Y_2)$.

In addition, we can choose L transverse to X and X_a^1 at a. Since X_a^1 is a ball, $L \cap X_a^1$ is connected. Hence $\mathrm{rank}_w(\pi_1 \times \pi_2) = \dim(Y_1 \times Y_2)$ for all w from a dense subset of $L \cap X_a^1$.

Then $f := (\pi_1 \times \pi_2)|(L \cap X)$ together with $Y' := \{a_1\} \times Y_2 \subset Y := Y_1 \times Y_2$ satisfies the assumptions of Lemma 4.3. The lemma follows now directly from Lemma 4.3.

4.4. Applications of liftings. As the next step we apply the above lemmata to our situation. Again we assume that $s := d+1$ is odd. Set $b := (x, \bar{x}) \in \mathbb{C}^n \times \overline{\mathbb{C}^n}$.

Lemma 4.6. *Under the assumptions of Theorem 1.2 for all integers $s \geq 1$, $l \geq 0$, $m := sr+l$, there exist an open subset $N \subset \mathbb{C}^n \times \overline{\mathbb{C}^n}$, an open neighborhood $U(j_x^m f_0) \subset J^m(\mathbb{C}^n, \mathbb{C}^{n'})$ and for $(x, \bar{w}) \in N$, holomorphic maps*

$$(81) \quad \varphi^l \colon U_x(j_x^m f_0) \times N_b^1 \to J^l(\overline{\mathbb{C}^n}, \overline{\mathbb{C}^{n'}}), \quad \Phi_{\bar{w}}^l \colon \overline{U_w(j_x^m f_0)} \times N_{(x,\bar{w})}^2 \to \mathbb{C}^{n'}$$

such that the following is satisfied:

(1) *N_b^1 is connected;*
(2) *b is in the closure of N_b^1;*
(3) *for all admissible germs f at x with $j_x^m f \in U(j_x^m f_0)$,*

$$(82) \qquad j_{\bar{w}}^l \bar{f} = \varphi^l(j_x^m f, \bar{w}), \quad j_z^l f = \Phi_{\bar{w}}^l(j_{\bar{w}}^m \bar{f}, z),$$

for $(z, \bar{w}) \in N$ sufficiently close to $(x, \bar{x})$.

Moreover, the map $\Phi_{\bar{w}}^l$ depends holomorphically on $\bar{w}$.

Proof. Put in Lemma 4.5

$$(83) \quad X := M^s \subset \mathbb{C}^n \times \mathbb{C}^N \times \overline{\mathbb{C}^n}, \quad Y_1 := \mathbb{C}^n, \quad Y_2 := \overline{\mathbb{C}^n}, \quad Y_3 := \mathbb{C}^N$$

and fix the point $a := (x, \dots, \bar{x}) \in X$. By Lemma 4.1, X and X_a^1 are manifolds and the first condition $\mathrm{rank}_z \pi_1 = \dim Y_1$ in Lemma 4.5 is satisfied. The second condition $\mathrm{rank}_z(\pi_2|X_x^1) = \dim Y_2$ is satisfied by the choice of s which we made by Theorem 4.1. Let

$$(84) \qquad N \subset \mathbb{C}^n \times \overline{\mathbb{C}^n}, \quad \psi \colon N \to M^s$$

be given by Lemma 4.5. Statements (1.) and (2.) in this lemma follow directly from (1.) and (2.) in Lemma 4.5 respectively. It remains to satisfy (3.).

Let

$$(85) \qquad F_\nu^l \colon U_z(j_x^m f_0) \to J_{\bar{w}}^l(\overline{\mathbb{C}^n}, \overline{\mathbb{C}^{n'}})$$

be as in Lemma 4.2. Passing if necessary to a smaller subset N define

$$(86) \qquad \varphi^l(j, \bar{w}) := F_{\psi(x,\bar{w})}^l(j)$$

for all $j \in U_x(j_x^m f_0)$, $(x, \bar{w}) \in N_x^1$. By (3.) in Lemma 4.5, φ^l satisfies the required properties. Similarly define

$$(87) \qquad \Phi_{\bar{w}}^l(\bar{j}, z) := \bar{F}_{\bar{\psi}(\bar{w}, z)}^l(\bar{j}).$$

4.5. The end of the proof. The last step is based on combining the equations in (82). Let f_0, f be as in Theorem 1.2 and set

$$(88) \qquad m := sr = (d+1)r, \quad k := 2m.$$

We further use the notation of Lemma 4.6. The set N_b^1 can be seen in the canonical way as an open subset of $\mathbb{C}^n$. In general it may not contain $\bar{x}$. Combining the equations in (82) we obtain

$$(89) \qquad f(z) = \Phi_{\bar{w}}^m(\varphi^m(j_x^k f, \bar{w}), z),$$

where $(z, \bar{w}) \in N$ is sufficiently close to $(x, \bar{x})$.

Recall that we denoted by S the set of all admissible germs at x (see Definition 1.5). For the proof of Theorem 1.2 we have to show conditions (1.),(2.),(3.) in Definition 1.2. Suppose that $j_x^k f = j_x^k f_0$. Then (1.) is followed by (89).

Denote by Φ the map in the right-hand side of (89):

$$(90) \qquad \Phi \colon U_x(j_x^k f_0) \times N \to \mathbb{C}^{n'}, \quad \Phi(j, z, \bar{w}) := \Phi_{\bar{w}}^m(\varphi^m(j, \bar{w}), z).$$

Then (89) can be rewritten as

$$(91) \qquad f(z) = \Phi(j_x^k f, z, \bar{w}).$$

The left-hand side in (91) is defined for z close to x. The right-hand side is defined for $j_x^k f \in U(j_x^k f_0)$ and $(z, \bar{w}) \in N$. To show (2.) and (3.) in Definition 1.2 we fix some $\bar{w}_0 \in N_b^1$.

Lemma 4.7. *For every admissible f with $j_x^k f \in U(j_x^k f_0)$, there exists a neighborhood $U(x) \subset \mathbb{C}^n$ such that (91) is valid for all $z \in U(x)$ and for $w = w_0$.*

Proof. Since (91) is valid for all $(z, \bar{w}) \in N$ close to $(x, \bar{x})$, we can choose it in the form $(x, \bar{w}_1)$ because of (2.) in Lemma 4.6. Since N_b^1 is connected, there exists a real curve $\gamma \subset N_b^1$ which connects $\bar{w}_1$ with $\bar{w}_0$. Since γ is compact, there exists a connected product neighborhood

$$(92) \qquad \{x\} \times \gamma \quad \subset \quad U(x) \times U(\gamma) \quad \subset \quad N.$$

Without loss of generality, $f(z)$ is defined for all $z \in U(x)$.

Hence both sides of (91) are defined for $z \in U(x)$, $\bar{w} \in U(\gamma)$. By the choice of $\bar{w}_1$, they are equal in a neighborhood of $(x, \bar{w}_1)$. The conclusion of the lemma follows from the identity principle.

By Lemma 4.7, for f fixed and z close to x

$$(93) \qquad f(z) = \Phi(j_x^k f, z, \bar{w}_0), \quad z \in U(x).$$

The right-hand side is defined for

$$(94) \qquad\qquad j_x^k f \in U(j_x^k f_0), \quad z \in N_{(x,\bar{w}_0)}^2$$

independently of f. Hence all such admissible germs f extend to the open set $N_{(x,\bar{w}_0)}^2$ which proves (2.).

Finally define

$$(95) \qquad\qquad F(j,z) := \Phi(j,z,\bar{w}_0), \quad F\colon U_x(j_x^k f_0) \times N_{(x,\bar{w}_0)}^2 \to \mathbb{C}^{n'}.$$

By the choice of w_0 one has $x \in N_{(x,\bar{w}_0)}^2$. Then (3.) is implied by (93). This finishes the proof of Theorem 1.2.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, 72076 TÜBINGEN, GERMANY
E-mail address: dmitri.zaitsev@uni-tuebingen.de