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A PRESENTATION OF THE MAPPING CLASS GROUPS

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Abstract. Given a compact orientable surface, the mapping class group is shown to have a presentation so that the generators are the set of all Dehn twists and the relations are supported in subsurfaces homeomorphic to the one-holed torus or the four-holed sphere. It turns out that all the relations were discovered by Dehn in 1938.

§1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be a compact oriented surface of genus g with r boundary components and $\mathcal{M}_{g,r} = \mathcal{M}(\Sigma)$ be the mapping class group $\text{Home}(\Sigma, \partial\Sigma)/\text{Iso}$ where homeomorphisms and isotopies leave points on $\partial\Sigma$ fixed. A presentation of the mapping class groups in terms of Dehn twists was obtained in the fundamental paper by Hatcher and Thurston [HT] where relations are supported in subsurfaces homeomorphic to $\Sigma_{2,3}$. Using the work of [HT], [Ha], and [Wa], Gervais in [Ge] obtains a presentation of $\mathcal{M}_{g,r}$ where relations are supported in subsurfaces homeomorphic to $\Sigma_{1,2}$. The goal of this note is to simplify Gervais' presentation and show that relations are actually supported in subsurfaces homeomorphic to $\Sigma_{1,1}$ or $\Sigma_{0,4}$.

We begin by introduce some notations. Let $\mathcal{S} = \mathcal{S}(\Sigma)$ be the set of isotopy classes of simple loops on the surface Σ. Given α, β in $\mathcal{S}$, define $I(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}$. We use $\alpha \cap \beta = \emptyset$ to denote $I(\alpha, \beta) = 0$; use $\alpha \perp \beta$ to denote $I(\alpha, \beta) = 1$; and use $\alpha \perp_0 \beta$ to denote $I(\alpha, \beta) = 2$ so that their algebraic intersection number is zero. If a, b are two arcs intersecting transversely at a point p, then the *resolution of $a \cup b$ at p from a to b* is defined as follows. Fix any orientation on a and use the orientation on the surface to determine an orientation on b. Then resolve the intersection according to the orientations (see figure 1). The resolution is evidently independent of the choice of the orientations on a. If $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, take $a \in \alpha$, $b \in \beta$ so that $|a \cap b| = I(\alpha, \beta)$. Then the curve obtained by resolving all intersection points in $a \cap b$ from a to b is again a simple loop denoted by ab. We define $\alpha\beta$ to be the isotopy class of ab. It follows from the definition that when $\alpha \perp \beta$ then $\alpha\beta \perp \alpha, \beta$ and when $\alpha \perp_0 \beta$ then $\alpha\beta \perp_0 \alpha, \beta$. Also the Dehn twist along α applied to β can be recovered from the operation as follows: if $\alpha \perp \beta$, then $D_\alpha(\beta) = \alpha\beta$; and if $\alpha \perp_0 \beta$, then $D_\alpha(\beta) = \alpha(\alpha\beta)$. Let $N(a)$ and $N(b)$ be two small regular neighborhoods of a

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and b. Then $N(a \cup b) = N(a) \cup N(b)$ is homeomorphic to $\Sigma_{1,1}$ when $\alpha \perp \beta$ and to $\Sigma_{0,4}$ when $\alpha \perp_0 \beta$. Let $\partial(\alpha, \beta)$ be the isotopy class of the curve system $\partial N(a \cup b)$.

In terms of these notations, the result of Gervais is as follows.

Theorem. (Gervais) *For a compact oriented surface Σ, the mapping class group $\mathcal{M}(\Sigma)$ has the following presentation*:

generators: $\{D_\alpha : \alpha \in \mathcal{S}(\Sigma)\}$.

relations: (I) $D_\alpha = 1$ *if α is the isotopy class of the null homotopic loop.*

(II) $D_\alpha D_\beta = D_\beta D_\alpha$ *if* $\alpha \cap \beta = \emptyset$.

(III) $D_{\alpha\beta} = D_\alpha D_\beta D_\alpha^{-1}$ *if* $\alpha \perp \beta$.

(III′) $D_{D_\alpha(\beta)} = D_\alpha D_\beta D_\alpha^{-1}$ *if* $\alpha \perp_0 \beta$.

(IV) $D_\alpha D_\beta D_{\alpha\beta} = D_{\partial(\alpha,\beta)}$ *if* $\alpha \perp_0 \beta$.

(V′) $(D_\alpha D_\beta D_\gamma)^4 = D_{\epsilon_1} D_{\epsilon_2}$ *where α, β, γ and ϵ_i are as shown in figure* 1.

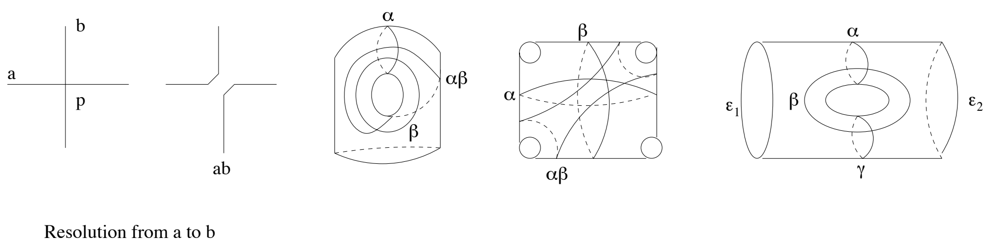

Figure 1 Right-hand orientation on the front face

If we set ϵ_1 in (V′) to be the trivial class, then (V′) becomes the relation

$$\text{(V)} \qquad (D_\alpha D_\beta D_\alpha)^4 = D_{\partial(\alpha,\beta)} \qquad \text{if } \alpha \perp \beta.$$

Our observation is that relations (I), (II), (III), (IV), and (V) form a complete set of relations.

Theorem. *For compact oriented surface Σ, the mapping class group $\mathcal{M}(\Sigma)$ has a presentation where the generators are the set of all Dehn twists and the relations are* (I), (II), (III), (IV), *and* (V).

Remarks. 1. The relations (I)-(V) are well known. Evidently relations (I), (II) hold. Relation (IV) is the lantern relation which was discovered by Dehn ([De], pp. 333) in 1938 and rediscovered independently by Johnson [Jo] in 1979. Relations (III) and (V) were also discovered by Dehn ([De], pp. 287, and the last sentence on pp. 310).

2. For surface $\Sigma \cong \Sigma_{1,1}$ or $\Sigma_{0,4}$, let $\mathcal{S}'(\Sigma)$ be the set of isotopy classes of essential non-boundary parallel simple loops on Σ. It is well known that there is a bijection (the slope map) $\pi : \mathcal{S}'(\Sigma) \to \hat{\mathbf{Q}}$ so that $\pi(\alpha) = p/q$, $\pi(\beta) = p'/q'$ satisfy $p'q - pq' = \pm 1$ if and only if $\alpha \perp \beta$ or $\alpha \perp_0 \beta$. Futhermore, $\alpha\beta = (p + \lambda p')/(q + \lambda q')$ where $\lambda = pq' - p'q$. Thus the three classes α, β and $\alpha\beta$ in relations (III), (IV), and (V) form an ideal triangle under the slope map in

the modular configuration $(\hat{\mathbf{Q}}, PSL(2,\mathbf{Z}))$. This shows that the mapping class group can be reconstructed explicitly (in terms of presentation) from the modular relation and the disjoint relation on $\mathcal{S}(\Sigma)$.

§2. Proof of the Theorem

As a convention, all surfaces drawn in the note have the right-hand orientation on the front face.

It suffices to show that relations (III′) and (V′) are the consequences of the relations (I)-(V).

To derive (III′) that $D_\gamma = D_\alpha D_\beta D_\alpha^{-1}$ where $\gamma = D_\alpha(\beta)$ with $\alpha \perp_0 \beta$, we note that $\alpha\beta \perp_0 \alpha$ and $\gamma = \alpha(\alpha\beta)$. Furthermore, $\partial(\alpha,\beta) = \partial(\alpha,\alpha\beta)$. Denote $\partial(\alpha,\beta)$ by δ. By relation (IV) for $\alpha \perp_0 \beta$ and $\alpha\beta \perp_0 \beta$, we obtain $D_\alpha D_\beta D_{\alpha\beta} = D_\delta$ and $D_\alpha D_{\alpha\beta} D_\gamma = D_\delta$. By relation (II), D_δ commutes with D_α, D_β, and $D_{\alpha\beta}$. By cancelling $D_{\alpha\beta}$ and D_δ from the two equations, we obtain $D_\gamma = D_\alpha D_\beta D_\alpha^{-1}$ which is the relation (III′).

To derive the relation (V′) from (I)-(V), we need two lemmas.

Lemma 1. *If $\alpha \perp \beta$, then $(\alpha\beta)\alpha = \alpha(\beta\alpha) = \beta$. In particular, as a consequnce of relation* (III), *we obtain Artin's relation*

(VI) $$D_\alpha D_\beta D_\alpha = D_\beta D_\alpha D_\beta \qquad \text{if } \alpha \perp \beta.$$

Proof. (Well known). To show $(\alpha\beta)\alpha = \beta$, consider figure 2.

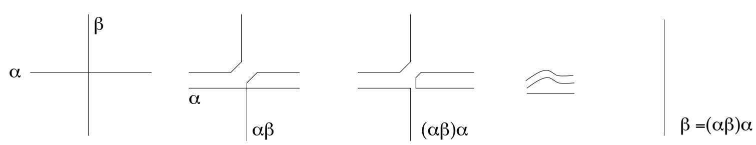

Figure 2

By (III) applied to $\alpha \perp \beta$, we obtain $D_{\alpha\beta} = D_\alpha D_\beta D_\alpha^{-1}$. Since $\alpha\beta \perp \alpha$ and $(\alpha\beta)\alpha = \beta$, by (III), we obtain $D_\beta = D_{\alpha\beta} D_\alpha D_{\alpha\beta}^{-1}$. Combining these two equations, we obtain (VI). □

Now to show (V′) that $(D_\alpha D_\beta D_\gamma)^4 = D_{\epsilon_1} D_{\epsilon_2}$ where α, β, γ and ϵ_i are as in figure 1, we need the following lemma (lemma 6.2, in [Lu]) whose proof is given in figure 3.

Lemma 2. *Let δ be $\partial(\alpha,\beta)$. Then $\beta\gamma \perp \delta\gamma$ and $(\beta\alpha)\alpha = (\beta\gamma)(\delta\gamma)$.*

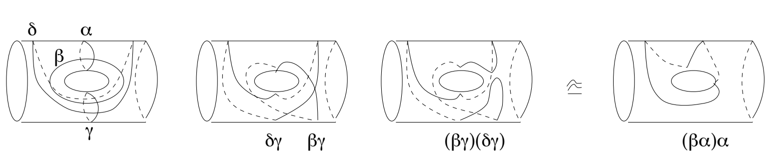

Figure 3

For simplicity, we use the letters A, B, C, D, E_i to denote the Dehn twists on $\alpha, \beta, \gamma, \delta, \epsilon_i$ respectively. For instance, (V′) becomes $(ABC)^4 = E_1E_2$.

By $\beta \perp \gamma$ and (III), we obtain

$$D_{\beta\gamma} = BCB^{-1}. \tag{1}$$

By $\alpha \perp \beta$, $\beta\alpha \perp \alpha$ and (III), we obtain $D_{\beta\alpha} = BAB^{-1}$ and $D_{(\beta\alpha)\alpha} = D_{\beta\alpha}D_\alpha D_{\beta\alpha}^{-1} = BAB^{-1}ABA^{-1}B^{-1}$. Using (VI) that $ABA = BAB$ and its equivalent forms $BAB^{-1} = A^{-1}BA$, $ABA^{-1} = B^{-1}AB$, we obtain

$$D_{(\beta\alpha)\alpha} = A^{-2}BA^2. \tag{2}$$

By $\alpha \perp \beta$ with $\partial(\alpha, \beta) = \delta$ and (V), we obtain

$$D = (ABA)^4. \tag{3}$$

By $\delta \perp_0 \gamma$ with $\partial(\delta, \gamma) = \epsilon_1 \cup \epsilon_2 \cup \alpha \cup \alpha$ and (IV), we obtain $DCD_{\delta\gamma} = E_1E_2A^2$. By (II) that E_i commutes with A, B, C, D, we obtain

$$D_{\delta\gamma} = E_1E_2C^{-1}D^{-1}A^2. \tag{4}$$

Finally, by lemma 2 that $\beta\gamma \perp \delta\gamma$, $(\beta\alpha)\alpha = (\beta\gamma)(\delta\gamma)$ and (III), we obtain

$$D_{(\beta\alpha)\alpha} = D_{\beta\gamma}D_{\delta\gamma}D_{\beta\gamma}^{-1}. \tag{5}$$

Substitute (1)-(4) into (5) and use relation (II), we obtain

$$A^{-2}BA^2BCB^{-1}A^{-2}(ABA)^4CBC^{-1}B^{-1} = E_1E_2. \tag{6}$$

We claim that under relations (I)-(VI), the left-hand side of the equation (6) is $(ABC)^4$. Here is the calculation. In each step of the derivation below, we apply one of the relations (II) that $AC = CA$, (VI) that $ABA = BAB$, $BCB = CBC$ or the cancellation law $XX^{-1} = 1$ to the letters underlined.

$$\begin{aligned}
&A^{-2}BA^2BCB^{-1}A^{-2}(ABA)^4CBC^{-1}B^{-1}\\
&\quad = A^{-2}BA^2BCB^{-1}A^{-2}ABAABAABA\underline{ABA}\quad\underline{CBC^{-1}}B^{-1}\\
&\quad = A^{-2}BA^2BCB^{-1}\underline{A^{-2}A}BAABAABABA\underline{BB^{-1}}C\underline{BB^{-1}}\\
&\quad = A^{-2}BA^2BC\underline{B^{-1}A^{-1}B}AABAABABAC\\
&\quad = A^{-2}BA^2BCAB^{-1}\underline{A^{-1}A}ABAABABAC\\
&\quad = A^{-1}\underline{A^{-1}BA}ABCA\underline{B^{-1}AB}AABABAC\\
&\quad = \underline{A^{-1}BA}B^{-1}AB\underline{CAA}B\underline{A^{-1}A}ABABAC\\
&\quad = BAB^{-1}B^{-1}\underline{ABA}ACBABABAC\\
&\quad = BAB^{-1}\underline{B^{-1}B}ABACBABABAC\\
&\quad = BA\underline{B^{-1}AB}ACBABABAC\\
&\quad = BAAB\underline{A^{-1}A}CBABABAC\\
&\quad = BAA\underline{BCB}ABABAC\\
&\quad = BA\underline{AC}B\underline{CA}BABAC\\
&\quad = BAC\underline{ABA}CBABAC\\
&\quad = BACBA\underline{BCB}ABAC\\
&\quad = BACBACB\underline{CA}BAC\\
&\quad = (BAC)^4\\
&\quad = (BCA)^4.
\end{aligned}$$

Thus $(BCA)^4 = E_1E_2$. But E_i commutes with B and C by (II). Thus $(ABC)^4 = E_1E_2$ after a conjugation by B. This finishes the proof.

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