Mathematical Research Letters **4**, 725–733 (1997)

EXAMPLES PERTAINING TO GEVREY HYPOELLIPTICITY

Michael Christ

1. Introduction

The purpose of this note is to introduce certain examples which shed light on a conjecture concerning hypoellipticity in Gevrey classes for partial differential operators with multiple characteristics.

For $s \geq 1$ and any open set U, let $G^s(U)$ denote the class of all C^∞ functions f defined in U, such that for each compact subset $K \subset U$ there exists $C < \infty$ such that for all $x \in K$ and all multi-indices α,

$$|\partial^\alpha f(x)| \leq C^{1+|\alpha|}|\alpha|^{s|\alpha|}.$$

A linear partial differential operator L is said be G^s hypoelliptic in U if for any open subset $U' \subset U$ and any $u \in \mathcal{D}'(U')$ such that $Lu \in G^s(U')$, necessarily $u \in G^s(U')$. An operator L is said to be microlocally G^s hypoelliptic in a conic open set $\Gamma \subset T^*U$ if for any distribution u, there is an inclusion of G^s wave front sets: $WF_{G^s}(u) \cap \Gamma \subset WF_{G^s}(Lu) \cap \Gamma$.

The conjecture in question proposes a sufficient condition for the microlocal G^s hypoellipticity of operators $L = \sum_{1 \leq j \leq k} X_j^2$, where the X_j are real vector fields with real analytic coefficients in some open subset V of $\mathbb{R}^d$, under the hypothesis that $\{X_j\}$ satisfies the bracket hypothesis of Hörmander [9]. Its formulation requires several definitions.

Denote by σ_j the principal symbol of X_j, and by T^*V the cotangent bundle of V with the zero section deleted. Let $M \subset T^*V$ be a smooth submanifold. For the purposes of this paper, a submanifold $M' \subset M$ of positive dimension will be said[1] to be a bicharacteristic submanifold of M if the tangent space of M' is orthogonal to the tangent space of M with respect to the canonical symplectic form on T^*V, at every point of M'.

Define $\mathcal{I}_1$ to be the ideal, in the ring of germs of real analytic functions on T^*V, generated by all the symbols σ_j. Inductively define $\mathcal{I}_{j+1}$ to be the ideal generated by $\mathcal{I}_j$ together with all Poisson brackets $\{f, \sigma_i\}$ such that $f \in \mathcal{I}_j$ and $1 \leq i \leq k$. Define $\Sigma_j \subset T^*V$ to be the zero variety of $\mathcal{I}_j$. Then $\mathcal{I}_j \subset \mathcal{I}_{j+1}$ and $\Sigma_j \supset \Sigma_{j+1}$ for all $j \geq 1$. The bracket hypothesis at a point $x \in V$ implies that $\Sigma_m \cap T^*_x V = \emptyset$ for some finite m. Under that hypothesis, define $m(x)$ to be the

Received June 6, 1997.

Research supported by NSF grant DMS 96-23007.

[1] A notion of bicharacteristic leaf is defined slightly differently by Treves [12]; it is not defined in [2]. Likewise the Poisson stratification introduced in [12] differs from that of [2]. These fine distinctions are not relevant to the simple examples treated in this article.

smallest integer m such that $\Sigma_m \cap T_x^*V = \emptyset$. A more refined invariant $m(x,\xi)$, defined at each point of T^*V, is the smallest integer such that $(x,\xi) \notin \Sigma_m$. Assuming for simplicity that each Σ_j is a smooth manifold, define a second invariant, $\ell(x,\xi)$, to be the smallest index $j < m(x,\xi)$ such that for every conic neighborhood Γ of (x,ξ), $\Sigma_j \cap \Gamma$ contains a bicharacteristic submanifold, provided such an index exists. Define $\ell(x,\xi) = m(x,\xi)$ if no such $j < m(x,\xi)$ exists.

Conjecture 1. (Bove and Tartakoff [2]) *Let L be a sum of squares of C^ω real vector fields, satisfying the bracket hypothesis at x. Suppose that there exists a neighborhood V_0 of x such that each $\Sigma_j \cap T^*V_0$ is a smooth manifold. Then L is microlocally G^s hypoelliptic in a small conic neighborhood of (x,ξ) for every $s \geq m(x,\xi)/\ell(x,\xi)$.*

Modulo certain fine distinctions, this generalizes a conjecture of Treves [12] concerning the analytic case $s = 1$.

In [6] we showed that the operators $\partial_{x_1}^2 + x_1^{2p}\partial_{x_2}^2 + x_1^{2q}\partial_{x_3}^2$ are G^s hypoelliptic if and only if $s \geq \max(p/q, q/p)$, thereby demonstrating that the optimal exponent for Gevrey hypoellipticity is not always 1 or $m(x)$, but rather that a range of intermediate behavior arises. A refinement in terms of certain anisotropic generalizations of the Gevrey classes was then formulated and proved, by a different method, by Bove and Tartakoff [2]. Their conjecture is consistent with these examples.

In the present note basic examples of a different character will be analyzed.[2] Their import is twofold: First, G^s hypoellipticity may sometimes hold for a *larger* range of exponents than predicted by Conjecture 1. Second, the mechanism underlying the simpler examples of [6] is not the only factor influencing Gevrey hypoellipticity.

Consider

$$L_{m,p} = \partial_x^2 + (x^{m-1}\partial_t)^2 + (t^p\partial_t)^2 \tag{1.1}$$

in $\mathbb{R}^2$. Assume that $m \geq 2$ and $p \geq 1$ are integers. Then any such L is elliptic everywhere except where $(x,t) = 0$; with coordinates (x,t,ξ,τ) for $T^*\mathbb{R}^2$, its characteristic variety is the line $\{x = t = \xi = 0\}$.[3]

Theorem 2. *$L_{m,p}$ is G^s hypoelliptic for all s satisfying*

$$s^{-1} \leq 1 - p^{-1}(1 - m^{-1}). \tag{1.2}$$

[2] More general results were announced in [4], based on the argument used below to derive Theorems 2 and 3. That argument works when a certain polynomial Θ arising in the theory of [4] is nonnegative on $\mathbb{R}^2$ and certain higher order terms are dominated by it, but a more elaborate argument for the general case contained a gap; it yields a strictly weaker conclusion than the desired Gevrey class hypoellipticity. The correctness of the most general statements in [4] is doubtful.

[3] Since the characteristic variety of $L_{m,p}$ consists of a discrete set of rays, G^s hypoellipticity is equivalent to microlocal G^s hypoellipticity for $L_{m,p}$.

Modulo insignificant lower order terms, the operators (1.1) are generalizations of a fundamental example of Métivier [11]; their Poisson strata are discussed by Treves [12], Example 3.6. These operators fail to be analytic hypoelliptic, as follows from the method of [3] and [5].

In these examples $\Sigma_j = \{x = t = \xi = 0\}$ for all $1 \le j < m$, and $\Sigma_m = \emptyset$. Thus Conjecture 1 predicts G^s hypoellipticity if and only if $s^{-1} \le m^{-1}$. But $\ell(0,0,0,\tau) = 1$ for all $\tau \ne 0$, and when $p \ge 2$, the reciprocal of the optimal exponent for G^s hypoellipticity is $1 - p^{-1}(1 - m^{-1}) > 1 - (1 - m^{-1}) = m^{-1}$.

The following variant of Theorem 2 can be proved by the same technique, and was also obtained by Bernardi, Bove and Tartakoff [1] and Matsuzawa [10]. Consider

$$\mathcal{L}_{m,k,p} = \partial_x^2 + \big([x^{m-1} + x^{m-1-k}t^p]\partial_t\big)^2. \tag{1.3}$$

Define $\tilde{p} = p(m-1)/k$.

Theorem 3. *Suppose that $m-1, k, p$ are all even. Then $\mathcal{L}_{m,k,p}$ is G^s hypoelliptic for all $s^{-1} \le 1 - \tilde{p}^{-1}(1 - m^{-1})$.*

By an elaboration of the method of [3] and [5] we have shown the indicated range of s to be optimal in Theorems 2 and 3, but the proofs are more involved than those of the positive results and will not be indicated here.

One interpretation of Theorem 2 is that not only the symplectic geometry of the varieties Σ_j, but also the ideals $\mathcal{I}_j$ themselves, influence Gevrey class hypoellipticity for $s < 1$. We believe this also to be the case for $s = 1$. The following examples may be of interest: let $L = X^2 + Y^2$ in $\mathbb{R}^3$ with coordinates (x, y, t) where $X = \partial_x$ and $Y = \partial_y + a(x, y)\partial_t$, $a \in C^\omega$ is real valued, and $\partial a/\partial x = x^{2p} + x^2y^2 + y^{2p}$ for some $p \ge 2$. Hypoellipticity of these operators depends only on $\partial a/\partial x$, rather than on a itself. Conjecture 1 predicts analytic hypoellipticity for all $p \ge 2$. Indeed, $m = 6$ for all p; the varieties Σ_j are independent of p for all $j \ge 2$, and they equal the symplectic manifold $\{(x, y, t; \xi, \eta, \tau) : x = \xi = y = \eta = 0\}$ for $2 \le j < 6$, and are empty for $j = 6$. L is known to be analytic hypoelliptic for $p = 2$ [8], but existing methods of proof do not appear to be applicable for $p > 2$. The ideals $\mathcal{I}_j$ have a somewhat different character when $p > 2$ than when $p = 2$.

After this paper was circulated we received preprints of Bernardi, Bove and Tartakoff [1] and of Matsuzawa [10] containing Theorems 2 and 3, with different methods of proof. The latter paper contains more general results as well.

2. Proofs

The method of proof of Theorem 2 is the same as that used in [5] and [6] to prove results in the positive direction.[4] Fix m, p. For any linear partial

[4]This method does apply in somewhat greater generality, but our aim here is the analysis of the simplest relevant examples.

differential operator L, denote by L^* its adjoint. Write $y = (x, t)$, $\eta = (\xi, \tau)$. The coordinate t will sometimes be complex, whereas x, ξ, τ will remain real.

For any compactly supported distribution u in $\mathbb{R}^2$, consider the FBI transform

$$\mathcal{F}u(y, \eta) = \int u(y')\, \alpha(y - y') e^{i(y-y')\cdot\eta - \frac{1}{2}\langle\eta\rangle(y-y')^2}\, dy', \tag{2.1}$$

where $(y - y')^2$ is defined to be $(x - x')^2 + (t - t')^2$, $\langle\eta\rangle = (1 + \eta^2)^{1/2}$, $\alpha(x, t) = (1 + \frac{i}{2}x\xi\langle\eta\rangle^{-1})(1 + \frac{i}{2}t\tau\langle\eta\rangle^{-1})$, and the integral is interpreted in the sense of distributions if $u \notin L^1$. Then $u \in G^s$ in a neighborhood of some point y_0, if and only if there exist a neighborhood V of y_0 and $\delta > 0$ such that

$$\mathcal{F}u(y, \eta) = O(\exp(-\delta\langle\eta\rangle^{1/s})) \tag{2.2}$$

for all $(y, \eta) \in V \times \mathbb{R}^2$.

In proving G^s hypoellipticity near y_0, we may assume u to be supported in a small neighborhood of y_0, and $\mathcal{F}(L_{m,p}u)(y, \eta)$ to satisfy (2.2) in $V \times \mathbb{R}^d$ for some smaller neighborhood V. Operators which are microlocally elliptic are microlocally G^s hypoelliptic, so since the characteristic variety of $L_{m,p}$ is the line $x = t = \xi = 0$, it suffices to prove (2.2) for y near 0 and where $\eta = (\xi, \tau)$ with $|\tau| \geq |\xi|$ and $|\eta|$ large. Thus $|\tau| \sim |\eta|$.

Define

$$\gamma(m, p) = 1 - p^{-1}(1 - m^{-1}). \tag{2.3}$$

Then $0 < \gamma(m, p) < 1$, and we aim to prove G^s hypoellipticity for all $s \geq \gamma(m, p)^{-1}$.

The main step is the following lemma. Let $B_\delta = \{y \in \mathbb{C}^2 : |y| < \delta\}$. Let $\tilde{y} = (\tilde{x}, \tilde{t}) \in \mathbb{R}^2$ be any point near 0, and set

$$E(x, t) = \exp\big(i(\tilde{t} - t)\tau - \frac{1}{2}\langle\eta\rangle^\gamma(\tilde{t} - t)^2\big). \tag{2.4}$$

Lemma 2.1. *Let $L = L_{m,p}$ and $\gamma = \gamma(m, p)$. Then for any sufficiently small constants $0 < c_1 < c_2 < c_3$ there exists $\delta > 0$ such that for each $\tilde{y} \in B_{c_1} \cap \mathbb{R}^2$ and each $\eta = (\xi, \tau) \in \mathbb{R}^2$ satisfying $|\xi| \leq |\tau|$, there exists $g \in C^\infty(B_{c_3} \cap \mathbb{R}^2)$ satisfying the following three conditions.*

$$\begin{aligned} &L^*(gE)(y) = \\ &\alpha(\tilde{y} - y)e^{i(\tilde{x}-x)\xi - \frac{1}{2}\langle\eta\rangle(\tilde{x}-x)^2}E(y) + O(e^{-\delta\langle\eta\rangle^\gamma}) \quad \text{for } y \in B_{c_3} \cap \mathbb{R}^2, \end{aligned} \tag{2.5}$$

g extends to a holomorphic function of t in $B_{c_3} \cap \{|\operatorname{Im}(t)| < \langle\eta\rangle^{\gamma-1}\}$ and

$$g(y) = O(1) \text{ in the } L^2 \text{ norm for } y \in B_{c_3} \cap \{|\operatorname{Im}(t)| < \langle\eta\rangle^{\gamma-1}\}, \tag{2.6}$$

and

$$\begin{aligned} &g(x, t) = \\ &O(e^{-\delta\langle\eta\rangle^\gamma}) \text{ in the } L^2 \text{ norm for } (x, t) \in B_{c_3} \cap \mathbb{R}^2 \text{ where } |x| > c_2. \end{aligned} \tag{2.7}$$

A symbol "$O(\cdot)$" connotes a bound uniform in $\eta, \tilde{y}, y$. Before discussing the proof, we indicate how the lemma leads to Theorem 2.

Lemma 2.2. *Let L be any linear partial differential operator satisfying the conclusion of Lemma 2.1 for some $\gamma \in (0,1]$. Then for any $s \geq \gamma^{-1}$ and for any sufficiently small neighborhood U of 0 and any relatively compact $U' \Subset U$, for any $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $Lu \in G^s(U)$, there exists $\varepsilon > 0$ such that $\mathcal{F}u(y,\eta) = O\big(\exp(-\varepsilon|\eta|^{1/s})\big)$ as $|\eta| \to \infty$, uniformly for $y \in U'$, provided that $\eta = (\xi, \tau)$ where $|\tau| \geq |\xi|$.*

Sketch of proof. The easy proof is essentially identical to the argument immediately following the statement of Lemma 3.1 of [6], so we merely recall its outline. Suppose that $Lu \in G^s(U')$, where $s = \gamma(m,p)^{-1}$.

Begin by rewriting the integral defining $\mathcal{F}u$ by substituting

$$\alpha(y-y')\exp\big(i(y-y')\cdot\eta - \frac{1}{2}\langle\eta\rangle(y-y')^2\big) = L^*(gE) + O(\exp(-\delta\langle\eta\rangle^\gamma))$$

The second term leads to an error of the desired order of magnitude. Integrating by parts leads to a main term $\int gE \cdot Lu$; boundary terms are negligible because $\exp(-\frac{1}{2}\langle\eta\rangle(y-y')^2)$ is $O(\exp(-c\langle\eta\rangle))$ away from the diagonal.

Next, because $Lu \in G^s$, conclusion (5) of Theorem 2.3 of [6] asserts that it is possible to decompose Lu as $v + R$ where v is holomorphic with respect to t and is $O(1)$ in the region $|\operatorname{Im}(t)| < \langle\eta\rangle^{\gamma-1}$, and R is $O(\exp(-\varepsilon\langle\eta\rangle^\gamma))$ in the real domain. R again leads to an acceptable error. Finally the contribution of v is treated by shifting the contour of integration with respect to t into the complex domain so as to pick up a factor of $\exp(-c\langle\eta\rangle^\gamma)$ from the factor $\exp(i(\tilde{t}-t)\tau)$ in E.

Any linear differential operator with analytic coefficients is microlocally G^s hypoelliptic for all $s \geq 1$ in any conic open set where its principal symbol does not vanish. Therefore for any operator L that is elliptic where $|\xi| \geq |\tau|$, under the hypotheses of the preceding lemma, one has also a decay estimate $O\big(\exp(-\varepsilon|\eta|^{1/s})\big)$ wherever L is elliptic. In particular, the operators of Theorem 2 are elliptic where $|\xi| \geq |\tau|$.

To prove Theorem 2, we couple these decay estimates with the FBI transform characterization (2.2) of G^s, to conclude that any L that satisfies the conclusion of Lemma 2.1, and is elliptic where $|\xi| \geq |\tau|$, is G^s hypoelliptic in a neighborhood of the origin for all $s \geq \gamma^{-1}$. In particular, $L_{m,p}$ is G^s hypoelliptic for all $s \geq \gamma(m,p)^{-1}$. Thus Theorem 2 is proved, modulo the proof of Lemma 2.1. □

We now discuss the proof of Lemma 2.1. Fix $(\tilde{x}, \tilde{t})$ and $\eta = (\xi, \tau)$ where $|\tau| \geq |\xi|$. One has

$$\begin{aligned} &E^{-1}L^*E = \\ &\partial_x^2 + \big(x^{m-1}[\partial_t - i\tau + \langle\eta\rangle(\tilde{t}-t)]\big)^2 + \big([\partial_t - i\tau + \langle\eta\rangle(\tilde{t}-t)]t^p\big)^2. \end{aligned} \tag{2.8}$$

Write

$$E^{-1}L^*E = \mathcal{A} + \mathcal{R} \qquad \text{where} \qquad \mathcal{A} = \partial_x^2 - \tau^2 x^{2(m-1)} - \tau^2 t^{2p}. \tag{2.9}$$

$\mathcal{A}$ acts on functions of (x,t); we also write $A_t = \partial_x^2 - \tau^2 x^{2(m-1)} - \tau^2 t^{2p}$ to denote the same operator, acting on functions of x alone and depending on a parameter t.

The construction of the approximate solution g sought in Lemma 2.1 transpires in various Sobolev type spaces. Henceforth let $\gamma = \gamma(m,p)$. Define

$$w_\tau(x,t) = \Big(\tau^{2/m} + \tau^2 x^{2(m-1)} + \tau^2 |t|^{2p}\Big)^{1/2},$$

for $(x,t) \in \mathbb{R} \times \mathbb{C}$. Fix a nonnegative auxiliary function $v \in C^\infty(\mathbb{R})$ such that $v \equiv 0$ in a neighborhood of $\{|x| \le c_1\}$, and $v \equiv 1$ in a neighborhood of $\{|x| \ge c_2\}$. For any open set $\Omega \subset \mathbb{C}^1$, for $k \in \{0,1,2\}$, define $\mathcal{H}^k_\tau(\mathbb{R}\times\Omega)$ to consist of all measurable functions $f(x,t)$ defined on $\mathbb{R}\times\Omega$ that are holomorphic in t for almost every x, and for which the following norms are finite:

$$\|f\|^2_{\mathcal{H}^0_\tau(\mathbb{R}\times\Omega)} = \iint_{\mathbb{R}\times\Omega} |f(x,t)|^2\, w_\tau(x,t)^{-2} e^{\rho|\tau| v(x)}\, dx\, dt\, d\bar t$$

$$\|f\|^2_{\mathcal{H}^1_\tau(\mathbb{R}\times\Omega)} = \iint_{\mathbb{R}\times\Omega} \Big(|\partial_x f(x,t)|^2\, w_\tau(x,t)^{-2} + |f(x,t)|^2\Big) e^{\rho|\tau| v(x)}\, dx\, dt\, d\bar t$$

$$\|f\|^2_{\mathcal{H}^2_\tau(\mathbb{R}\times\Omega)} = \iint_{\mathbb{R}\times\Omega} \Big(|\partial_x^2 f(x,t)|^2\, w_\tau(x,t)^{-2} + |\partial_x f(x,t)|^2 + |f(x,t)|^2 w_\tau(x,t)^2\Big) e^{\rho|\tau| v(x)}\, dx\, dt\, d\bar t\ .$$

These spaces and norms depend on the parameter ρ, which may for the present be any real number but will ultimately be chosen to be small but strictly positive. There are corresponding spaces of functions defined on $\mathbb{R}$, depending on a parameter $t \in \mathbb{C}$:

$$\|f\|^2_{\mathcal{H}^0_{\tau,t}(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|^2\, w_\tau(x,t)^{-2} e^{\rho|\tau| v(x)}\, dx$$

$$\|f\|^2_{\mathcal{H}^1_{\tau,t}(\mathbb{R})} = \int_{\mathbb{R}} \Big(|\partial_x f(x)|^2\, w_\tau(x,t)^{-2} + |f(x)|^2\Big) e^{\rho|\tau| v(x)}\, dx$$

$$\|f\|^2_{\mathcal{H}^2_{\tau,t}(\mathbb{R})} = \int_{\mathbb{R}} \Big(|\partial_x^2 f(x)|^2\, w_\tau(x,t)^{-2} + |\partial_x f(x)|^2 + |f(x)|^2 w_\tau(x,t)^2\Big) e^{\rho|\tau| v(x)}\, dx$$

The definitions ensure that $\mathcal{A}$ maps $\mathcal{H}^2_\tau(\mathbb{R}\times\Omega)$ boundedly to $\mathcal{H}^0_\tau(\mathbb{R}\times\Omega)$, uniformly in Ω, τ, under the standing hypotheses that $|\tau| \ge |\xi|$ and $|\tau| \ge 1$. Likewise A_t maps $\mathcal{H}^2_{\tau,t}(\mathbb{R})$ boundedly to $\mathcal{H}^0_{\tau,t}(\mathbb{R})$, uniformly in $\tau \in \mathbb{R}, t \in \mathbb{C}$.

Lemma 2.3. *There exists $c_0 > 0$ such that for all sufficiently small $|\rho|$ and all $\tau \neq 0$, $A_t : \mathcal{H}^2_{\tau,t}(\mathbb{R}) \mapsto \mathcal{H}^0_{\tau,t}(\mathbb{R})$ is invertible, uniformly in $t \in \mathbb{C}, \tau \in \mathbb{R}$ provided that*

$$|\operatorname{Im}(t)| \le c_0 |\tau|^{\gamma(m,p)-1}. \tag{2.10}$$

Proof. The proof is based on the inequality

$$-\operatorname{Re}\langle A_t f,\, f\rangle \ge c \int_{\mathbb{R}} \left(|\partial_x f|^2 + w(x,t)^2 |f|^2\right) dx \quad \text{for all } f \in C^2_0(\mathbb{R}), \tag{2.11}$$

where $\langle f,\, g\rangle = \int_{\mathbb{R}} f\overline{g}\, dx$. To prove this write

$$-\operatorname{Re}\langle A_t f,\, f\rangle = \|\partial_x f\|^2 + \int_{\mathbb{R}} x^{2(m-1)} \tau^2 |f|^2\, dx + \int \tau^2 \operatorname{Re}(t^{2p}) |f|^2\, dx.$$

One has

$$\|\partial_x f\|^2 + \int x^{2(m-1)} \tau^2 |f|^2\, dx \ge c\tau^{2/m} \int |f|^2\, dx,$$

as follows from the case $\tau = 1$ by scaling. Moreover

$$\operatorname{Re}(t^{2p}) \ge c(\operatorname{Re}(t))^{2p} - C(\operatorname{Im}(t))^{2p}$$

for some $c, C \in \mathbb{R}^+$. The hypothesis (2.10) restricting the imaginary part of t implies

$$\tau^2 (\operatorname{Im}(t))^{2p} \le c_0^{2p} \tau^{2+2p(\gamma-1)}.$$

The exponent is $2 + 2p(\gamma - 1) = 2 - 2p(p^{-1}(1 - m^{-1})) = 2m^{-1}$. Combining all these ingredients yields (2.11), provided that c_0 is chosen to be sufficiently small.

The conclusion of the lemma follows easily from (2.11) as in [3], Lemma 3.1 and [6], Lemma 3.3, because

$$e^{\rho|\tau|v/2} A_t e^{-\rho|\tau|v/2} - A_t = O(|\rho|)$$

as an operator from $\mathcal{H}^2_{\tau,t}(\mathbb{R})$ to $\mathcal{H}^0_{\tau,t}(\mathbb{R})$; this holds because $v \equiv 0$ in a neighborhood of the origin while the term $|\tau|x^{m-1}$ in the definition of w is strictly positive on the support of v. For further details see the proof of Lemma 3.3 of [6].

Corollary 2.4. *If c_0 is chosen to be sufficiently small then for any open set $\Omega \subset \mathbb{C}^1$ contained in the region where $|\operatorname{Im}(t)| < c_0|\tau|^{\gamma(m,p)-1}$, the operator $\mathcal{A} : \mathcal{H}^2_\tau(\mathbb{R} \times \Omega) \mapsto \mathcal{H}^0_\tau(\mathbb{R} \times \Omega)$ is invertible, uniformly in τ, Ω.*

Let

$$\Omega_1 = \{t \in \mathbb{C} : |\operatorname{Re}(t)| < 2 \text{ and } |\operatorname{Im}(t)| < \frac{c_0}{2}|\tau|^{\gamma-1}\},$$
$$\Omega_\infty = \{t \in \mathbb{C} : |\operatorname{Re}(t)| < 1 \text{ and } |\operatorname{Im}(t)| < \frac{c_0}{4}|\tau|^{\gamma-1}\},$$

Let $\Lambda \in \mathbb{R}^+$ be a large constant to be chosen below. Given a large τ, choose an integer N so that $|N - \Lambda^{-1}|\tau|^\gamma| < 1$. For $2 \le j \le 2N$ construct open sets $\Omega_j \subset \mathbb{C}$, depending on τ, with $\Omega_\infty = \Omega_{2N} \Subset \Omega_{2N-1} \Subset \cdots \Subset \Omega_1$ satisfying

$$\operatorname{distance}(\Omega_{j+1}, \partial\Omega_j) \ge c\Lambda|\tau|^{-1}.$$

Here c is a small constant, independent of τ, Λ, j.

Lemma 2.5. *$\mathcal{R} : \mathcal{H}^2_\tau(\mathbb{R}\times\Omega_j) \mapsto \mathcal{H}^0_\tau(\mathbb{R}\times\Omega_{j+2})$ is bounded, with norm $O(\Lambda^{-1} + c_1 + c_0)$, uniformly in τ.*

Proof. By Cauchy's inequality relating the derivative of a holomorphic function to its L^1 norm over a ball, ∂_t maps each space $\mathcal{H}^k_\tau(\mathbb{R}\times\Omega_j)$ boundedly to $\mathcal{H}^k_\tau(\mathbb{R}\times\Omega_{j+1})$, with norm $O(\operatorname{distance}(\Omega_{j+1}, \partial\Omega_j)^{-1}) = O(\Lambda^{-1}|\tau|)$. The norms are defined so that the multiplication operators τt^p and τx^{m-1} map $\mathcal{H}^k_\tau(\mathbb{R}\times\Omega_j)$ to $\mathcal{H}^{k-1}_\tau(\mathbb{R}\times\Omega_j)$ with uniformly bounded norms. Furthermore, the extra factors of $\tilde{t}-t$ in the definition (2.8),(2.9) for $\mathcal{R}$ contribute an additional factor to these bounds which is $O(c_1 + c_0)$. Combining these estimates yields the lemma. For further details see the proofs of Lemma 3.4 of [6], and of the first display at the top of page 319 of [3].

Set $\psi(y) = \alpha(\tilde{y} - y)e^{i(\tilde{x}-x)\xi - \frac{1}{2}\langle\eta\rangle(\tilde{x}-x)^2}$. To attempt to solve $(\mathcal{A}+\mathcal{R})g \approx \psi$ we define

$$g = \sum_{j=0}^{N}(-1)^j\big(\mathcal{A}^{-1}h\mathcal{R})^j\mathcal{A}^{-1}\psi, \tag{2.12}$$

where $h \in C^\infty_0(\mathbb{R})$ is $\equiv 1$ where $|x| \le c_3$. Thus

$$(\mathcal{A} + h\mathcal{R})g = \psi \pm \mathcal{E},$$

where

$$\mathcal{E} = (h\mathcal{R}\mathcal{A}^{-1})^{N+1}\psi.$$

Note that $\psi \in \mathcal{H}^0_\tau(\mathbb{R}\times\Omega_1)$ with norm $O(1)$, provided $\rho > 0$ is chosen to be sufficiently small. If Λ is chosen to be sufficiently large and c_0, c_3 to be sufficiently small then applying Lemmas 2.3 and 2.5 in turn N times yields

$$\mathcal{E} = O(\exp(-\varepsilon N)) = O(\exp(-\varepsilon'|\tau|^\gamma))$$

for some $\varepsilon, \varepsilon' > 0$, in the $\mathcal{H}^0_\tau(\mathbb{R}\times\Omega_\infty)$ norm.

Because $v(x) > 0$ for $|x| \ge c_2$ and $\rho > 0$, the weight $e^{\rho|\tau|v(x)}$ in the definitions of the $\mathcal{H}^k_\tau$ norms ensures that $g = O(\exp(-\varepsilon|\tau|^\gamma))$ in the $L^2(dx\,dt)$ norm for such x. In the region $|x| < c_3$ of interest, the auxiliary function h is $\equiv 1$, hence $(\mathcal{A}+\mathcal{R})g \equiv \psi + \mathcal{E}$. This approximate solution g thus has all properties required of it in Lemma 2.1. □

The main change needed to obtain Theorem 3 is to modify the weight $w(x,t)$ used in the definitions of the $\mathcal{H}^k_\tau$ and $\mathcal{H}^k_{\tau,t}$ norms to

$$\left(\tau^{2/m} + \tau^{2/(m-1-k)}|t|^{2p} + \tau^2[x^{2(m-1)} + x^{2(m-1-k)}|t|^{2p}]\right)^{1/2}.$$

Remark. The limiting effect preventing this analysis from establishing G^s hypoellipticity for a larger range of exponents s is the failure of A_t to be invertible for t outside of a complex region which shrinks to the real axis as $|\tau| \to \infty$; the rate of shrinkage dictates the optimal Gevrey class G^s. This phenomenon is the essence of [5] and [3].

For the operators studied in [6] and [2], the limitation on s comes about in a different way. Application of the FBI transform $\mathcal{F}$ as above leads to unacceptable error terms, so variants $\mathcal{F}_\gamma$ adapted to specific Gevrey classes were employed instead in [6] in order to obtain smaller error terms.

References

1. E. Bernardi, A. Bove, and D. Tartakoff, *On a conjecture of F. Treves: Analytic hypoellipticity and Poisson strata*, preprint.
2. A. Bove and D. Tartakoff, *Optimal non-isotropic Gevrey exponents for sums of squares of vector fields*, preprint.
3. M. Christ, *The Szegö projection need not preserve global analyticity*, Ann. of Math. **143** (1996), 301–330.
4. ———, *On local and global analytic and Gevrey hypoellipticity*, Journees "Equations aux derivees partielles", Saint-Jean-de-Monts, Mai-Juin 1995.
5. ———, *Analytic hypoellipticity in dimension two*, MSRI preprint 1996-009, submitted.
6. ———, *Intermediate optimal Gevrey exponents occur*, Comm. Partial Diff. Eq. **22** (1997), 359–379.
7. ———, *Hypoellipticity in the infinitely degenerate regime*, preprint.
8. A. Grigis and J. Sjöstrand, *Front d'onde analytique et sommes de carrés de champs de vecteurs*, Duke Math. J. **52** (1985), 35–51.
9. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.
10. T. Matsuzawa, *Gevrey hypoellipticity for Grushin operators*, Pub. RIMS Kyoto University, to appear.
11. G. Métivier, *Non-hypoellipticité analytique pour* $D_x^2 + (x^2+y^2)D_y^2$, C. R. Acad. Sci. Paris **292** (1981), 401–404.
12. F. Treves, *Symplectic geometry and analytic hypo-ellipticity*, preprint.

Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: mchrist@math.berkeley.edu