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THE ABSOLUTELY CONTINUOUS SPECTRUM OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH DECAYING POTENTIALS

Michael Christ, Alexander Kiselev, and Christian Remling

In this announcement, we are interested in the spectral theory of one-dimensional Schrödinger operators

$$H = -\frac{d^2}{dx^2} + V(x), \tag{1}$$

acting on the Hilbert space $L_2(0,\infty)$. One also needs a boundary condition at $x = 0$ in order to obtain self-adjoint operators. The operator (1) describes the motion of a quantum mechanical particle, and the spectral properties of H are intimately connected to the physics of this system (see, e.g., [19]).

We will view V as a perturbation of the free Hamiltonian $H_0 = -d^2/dx^2$. It is natural to expect that suitable smallness assumptions on V guarantee stability of the absolutely continuous part of H_0. This problem is one of the basic questions in quantum mechanics, and it has been studied extensively. It has been known for a long time that the spectrum of H is purely absolutely continuous on $(0,\infty)$ if $V \in L_1$. More information is available for potentials satisfying certain additional assumptions. We mention the classical result of Weidmann [27] on potentials of bounded variation, a series of works on oscillating potentials of the type $\frac{\sin x^\alpha}{x^\beta}$ [2, 3, 9, 15, 28], and works on potentials satisfying additional conditions on the derivatives, see, e.g., [1, 4, 10]. However, in all these results, the potential is required to have further properties, in addition to decaying sufficiently rapidly. In fact, a result going back to von Neumann and Wigner says that potentials $V(x) = O(1/x)$ can already have positive eigenvalues [26], and one can construct potentials with decay arbitrarily close to $O(1/x)$ and dense point spectrum in $(0,\infty)$ [18, 24]. In other words, decay conditions that are essentially weaker than $V \in L_1$ do not imply *purely* absolutely continuous spectrum on $(0,\infty)$.

However, as was first noticed by one of us [11, 12], it is still true that the absolutely continuous spectrum is preserved if $V(x) = O(x^{-\alpha})$ ($\alpha > 2/3$) (see [12]), although, according to the above remarks, embedded singular spectrum

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can occur. Subsequently, Molchanov presented an alternate proof of the same result [17].

Our first result is a sharp version of this theorem. Recall that S is called an essential support of the measure μ if $\mu(\mathbb{R} \setminus S) = 0$ and $\mu(T) > 0$ for every subset $T \subset S$ of positive Lebesgue measure.

Theorem 1. [5, 21] *Suppose $|V(x)| \le C(1+x)^{-\alpha}$ with $\alpha > 1/2$. Then $\Sigma_{ac} = (0,\infty)$ is an essential support of the absolutely continuous part of the spectral measure. Moreover, for almost every $E > 0$, one can find solutions to the Schrödinger equation $Hy = Ey$ with WKB asymptotic behavior*:

$$y_{\pm}(x,E) = \exp \pm \left(i\sqrt{E}x - \frac{i}{2\sqrt{E}} \int_0^x V(t)\,dt \right) (1+o(1)) \quad (x \to \infty).$$

This result is optimal, because the work on decaying random potentials [6, 7, 13, 14, 22] has shown that there are potentials $|V(x)| \le C(1+x)^{-1/2}$, such that the corresponding Schrödinger operator has purely singular spectrum.

We have independently found two different proofs of Theorem 1. These proofs will be given in two separate publications [5, 21]. The approach of [5] is based on ideas developed in [11, 12] and, in particular, on new norm estimates for certain multilinear transformations which may be of independent interest. The method of [21] uses ideas from both proofs of the 2/3 result [12, 17].

Actually, our methods also yield a number of extensions and generalizations of Theorem 1. For example, we do not really need a pointwise bound on V; we prove the result under considerably more general assumptions which allow, among other things, local singularities of V. However, we do need a certain amount of regularity in the decay of V; we are as yet unable to treat general L_p potentials (see also the open questions below). We refer the reader to [5, 21] for details.

We also obtain a general criterion for the stability of the absolutely continuous spectrum of perturbed Schrödinger operators. We now consider the following situation: Given a Schrödinger operator $H_0 = -d^2/dx^2 + U$ with absolutely continuous spectrum on some set S, we ask under what conditions this spectrum is stable under perturbations by V. Again, we give the result in the simplest form.

Hypothesis 2. *Assume that for all $E \in S$, the Schrödinger equation*

$$-y'' + Uy = Ey \tag{2}$$

has only bounded solutions. Assume further that one can choose a solution $\theta(\cdot,E)$ $(E \in S)$ of (2), such that the operator $K : L_2((0,\infty),dx) \to L_2(S,dE)$, defined for bounded functions f of compact support by

$$(Kf)(E) = \int_0^\infty \theta(x,E)^2 \exp\left(\frac{i}{\operatorname{Im}\theta\overline{\theta}'} \int_0^x V(t)|\theta(t,E)|^2\,dt \right) f(x)\,dx, \tag{3}$$

is norm bounded.

Note that Hypothesis 2 in particular implies that $\Sigma_{ac}(H_0) \supset S$ [23, 25]. The quantity $\operatorname{Im} \theta\overline{\theta}'$ is independent of x, since it is a multiple of the Wronskian of the two solutions $\theta, \overline{\theta}$. Moreover, it is non-zero precisely if θ and $\overline{\theta}$ are linearly independent. We have the following result, first obtained by methods of [5] (it can also be shown by methods of [21]):

Theorem 3. *Suppose that Hypothesis* 2 *holds. If* $|V(x)| \leq C(1+x)^{-\alpha}$ *with* $\alpha > 1/2$, *then* $\Sigma_{ac}(H_0 + V) \supset S$. *Moreover, for almost every* $E \in S$, *one can find solutions* $y, \overline{y}$ *to the Schrödinger equation* $(H_0 + V)y = Ey$ *with WKB type asymptotic behavior*:

$$y(x, E) = \theta(x, E) \exp\left(\frac{i}{2 \operatorname{Im} \theta\overline{\theta}'} \int_0^x V(t)|\theta(t, E)|^2 \, dt \right) (1 + o(1)) \quad (x \to \infty).$$

Hypothesis 2 can be verified for $U = 0$ and for periodic U (see [12]). Given this, it is clear that, in particular, Theorem 1 follows from Theorem 3.

We also have a result on decay conditions which imply *purely* absolutely continuous spectrum on $(0, \infty)$. This result improves the elementary remark on L_1 potentials (on the power scale). This problem was brought to our attention by S. Molchanov, who has independently obtained related (but weaker) results using different methods [17].

Theorem 4. [21] *If* $C := \limsup_{x \to \infty} x\,|V(x)| < \infty$, *then* H_α *is purely absolutely continuous on* $((2C/\pi)^2, \infty)$. *In particular, if* $V(x) = o(1/x)$, *then* H_α *is purely absolutely continuous on* $(0, \infty)$.

The point of this Theorem is the absence of singular *continuous* spectrum. That $E = (2C/\pi)^2$ is a (sharp) bound on possible embedded eigenvalues appears already in [8, Section 3.2]. See also [13, Theorem 4.1] for further information on embedded eigenvalues.

We would like to conclude this paper with some open questions. We think that these questions are interesting, but they also look rather difficult at present.

1. Does Theorem 1 still hold under the assumption $V \in L_2$ (or $V \in L_p$ for some $p < 2$)? Currently, we can show that if $x^\epsilon V \in L_p$ for some $\epsilon > 0$ and $p \leq 2$, Theorem 1 holds. Still extending this result to L_p seems hard.

2. Are there potentials $V(x) = O(x^{-\alpha}), \alpha > 1/2$ with embedded singular *continuous* spectrum? We expect that the answer is yes. In this case, it would be interesting to construct such potentials.

We would also like to point out that in recently constructed examples with embedded singular continuous spectrum [16, 20], the essential support of the absolutely continuous part Σ_{ac} does not have full measure in the absolutely continuous spectrum σ_{ac} (which is the essential closure of the set Σ_{ac}).

3. Formulate general conditions on U which imply boundedness of the integral operator from (3).

These problems will be the subject of continuing research.

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Department of Mathematics, University of California, Berkeley CA 94720, USA
E-mail address: mchrist@math.berkeley.edu

Mathematical Sciences Research Institute, Berkeley CA 94720, USA
E-mail address: kiselev@msri.org, akiselev@cco.caltech.edu

Universität Osnabrück, Fachbereich Mathematik/Informatik, 49069 Osnabrück, GERMANY
E-mail address: cremling@mathematik.uni-osnabrueck.de