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TORSION INVARIANTS OF Spin^c-STRUCTURES ON 3-MANIFOLDS

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Introduction

Recently there has been a surge of interest in the Seiberg-Witten invariants of 3-manifolds, see [3], [4], [7]. The Seiberg-Witten invariant of a closed oriented 3-manifold M is a function SW from the set of Spin^c-structures on M to $\mathbb{Z}$. This function is defined under the assumption $b_1(M) \geq 1$ where $b_1(M)$ is the first Betti number of M; in the case $b_1(M) = 1$ the function SW depends on the choice of a generator of $H^1(M;\mathbb{Z}) = \mathbb{Z}$. The definition of SW runs parallel to the definition of the SW-invariant of 4-manifolds: one counts the gauge equivalence classes of solutions to the Seiberg-Witten equations.

It was observed by Meng and Taubes [4] that the function $SW(M)$ is closely related to a Reidemeister-type torsion of M. The torsion in question was introduced by Milnor [5]; the refined version used by Meng and Taubes is due to the author [12]. Considered up to sign, this torsion is equivalent to the Alexander polynomial of the fundamental group of M, see [5], [8].

The aim of this paper is to discuss relationships between Spin^c-structures and torsions. We use the torsions introduced by the author in [9], [12], [13] to define a numerical invariant of Spin^c-structures on closed oriented 3-manifolds. Presumably, in the case $b_1 \geq 1$, this invariant is equivalent to the one arising in the Seiberg-Witten theory.

A related question of finding topological invariants of Spin-structures on 3-manifolds was studied in [11] in connection with a classification problem in the knot theory. It was observed in [11] that an orientation of a link in the 3-sphere S^3 induces a Spin-structure on the corresponding 2-sheeted branched covering of S^3. To distinguish Spin-structures on 3-manifolds one can use torsions, see [13]. As a specific application, note the homeomorphism classification of Spin-structures on 3-dimensional lens spaces: a lens space $L(p,q)$ with even p admits an orientation-preserving self-homeomorphism permuting the two Spin-structures on $L(p,q)$ if and only if $q^2 = p + 1 (\bmod 2p)$, see [13], Theorem C.3.1. This implies (the hard part of) the classification of oriented links with two bridges in S^3 first established by Schubert in a different way.

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The technique introduced in [13] applies in any dimension; it associates torsion invariants with so-called Euler structures on manifolds. Our main observation here is that in dimension 3 the Euler structures are equivalent to the Spin^c-structures. This allows us to use torsions to study Spin^c-structures on 3-manifolds.

Notation. Throughout the paper the homology and cohomology of manifolds and CW spaces are taken with integer coefficients unless explicitly indicated to the contrary.

Organization of the paper. In Sect. 1 we review the theory of smooth Euler structures on manifolds following [13] and establish the equivalence between Spin^c-structures and Euler structures on 3-manifolds. In Sect. 2 we recall the definition of the Reidemeister-Franz torsion of a CW space and review the refined torsions following [12], [13]. In Sect. 3 we review the torsion τ introduced in [9]. In Sect. 4 we show that the torsion τ of a 3-manifold is a finite linear combination of homology classes. In Sect. 5 we define a numerical invariant of Spin^c-structures on 3-manifolds.

1. Spin^c-structures and Euler structures

1.1. The group $\mathrm{Spin}_{\mathbb{C}}(3)$. Recall that $SO(3) = SU(2)/\{\pm 1\} = U(2)/U(1)$ where $U(1)$ lies in $U(2)$ as the diagonal subgroup. The projection $U(2) \to SO(3)$ is a principal circle bundle over $SO(3)$. Remember that the isomorphism classes of principal circle bundles over a CW space X are numerated by the elements of $[X, BU(1)] = [X, K(\mathbb{Z}, 2)] = H^2(X)$. The circle bundle $U(2) \to SO(3)$ is nontrivial and corresponds to the nonzero element of $H^2(SO(3)) = H^2(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$. Recall finally that $\mathrm{Spin}(3) = SU(2)$ and

$$\mathrm{Spin}_{\mathbb{C}}(3) = (U(1) \times \mathrm{Spin}(3))/\{\pm 1\} = (U(1) \times SU(2))/\{\pm 1\} = U(2).$$

1.2. Spin^c-structures on 3-manifolds. Let M be a closed oriented 3-manifold. Endow M with a Riemannian metric and consider the associated principal $SO(3)$-bundle of oriented orthonormal frames $f_M : Fr \to M$. A Spin^c-structure on M is a lift of f_M to a principal $U(2)$-bundle. More precisely, a Spin^c-structure on M is an isomorphism class of a pair (a principal $U(2)$-bundle $F \to M$, an isomorphism α of the principal $SO(3)$-bundle $F/U(1) \to M$ onto $f_M : Fr \to M$).

An equivalent definition: a Spin^c-structure on M is an element of $H^2(Fr)$ whose reduction to every fiber is the nonzero element of $H^2(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. To observe the equivalence of these definitions, it suffices to associate with any pair $(F \to M, \alpha)$ as above the element of $H^2(Fr)$ corresponding to the circle bundle $\alpha \circ \mathrm{proj} : F \to F/U(1) \approx Fr$. The set of Spin^c-structures on M is denoted by $\mathcal{S}(M)$.

The group $H_1(M) = H^2(M)$ acts on $H^2(Fr)$ via the pull-back homomorphism $f_M^* : H^2(M) \to H^2(Fr)$ and addition. This action preserves $\mathcal{S}(M) \subset H^2(Fr)$. The induced action of $H_1(M)$ on $\mathcal{S}(M)$ is free and transitive. This follows from the fact that M is parallelisable, so that $Fr = M \times SO(3)$ and by the

Künneth theorem, $H^2(Fr) = H^2(M) \oplus (\mathbb{Z}/2\mathbb{Z})$. The notion of a Spin^c-structure on M is essentially independent of the choice of a Riemannian metric on M.

1.3. Smooth Euler structures. (cf. [13]). Let M be a smooth closed connected oriented manifold of dimension $m \geq 2$ with $\chi(M) = 0$. By a vector field on M we mean a *nonsingular* tangent vector field on M. Vector fields u and v on M are called homologous if for some closed m-dimensional ball $D \subset M$ the restrictions of u and v to $M \backslash \mathrm{Int} D$ are homotopic in the class of (nonsingular) vector fields. The homology class of a vector field u on M is denoted by $[u]$ and called an Euler structure on M. The set of Euler structures on M is denoted by $\mathrm{vect}(M)$.

If u, v are two vector fields on M, then the first obstruction to their homotopy lies in $H^{m-1}(M) = H_1(M)$ and depends only on $[u], [v] \in \mathrm{vect}(M)$. This obstruction is denoted by $[u]/[v]$. It is easy to show that for any $h \in H_1(M)$, $e \in \mathrm{vect}(M)$ there is a unique Euler structure $he \in \mathrm{vect}(M)$ such that $he/e = h$. Thus, $H_1(M)$ acts freely and transitively on $\mathrm{vect}(M)$. This action and the group operation in $H_1(M)$ will be written multiplicatively.

For $e = [u] \in \mathrm{vect}(M)$, consider the opposite vector field $-u$ on M and set $e^{-1} = [-u] \in \mathrm{vect}(M)$. Clearly, $(e^{-1})^{-1} = e$. Set $c(e) = e/e^{-1} \in H_1(M)$. One can show that the class $c(e)$ is dual to the Euler class of the $(m-1)$-dimensional vector bundle $u^{\perp}$ formed by the tangent vectors orthogonal to u. Note that $(he)^{-1} = h^{\varepsilon} e^{-1}$ and $c(he) = h^{1-\varepsilon} c(e)$ for $h \in H_1(M)$ and $\varepsilon = (-1)^m$.

An equivalent definition of Euler structures on M can be given in terms of the spherical fiber bundle of unit tangent vectors $SM \to M$. An Euler structure on M is an element of $H^{m-1}(SM)$ whose reduction to every fiber $S_x M$, $x \in M$ is the generator of $H^{m-1}(S_x M) = H^{m-1}(S^{m-1}) = \mathbb{Z}$ determined by the orientation of M at x. The group $H_1(M) = H^{m-1}(M)$ acts on such elements freely and transitively via the pull-back homomorphism $H^{m-1}(M) \to H^{m-1}(SM)$ and addition. The equivalence of definitions is established as follows. Let u be a vector field of M. The mapping $x \mapsto u(x)/|u(x)| : M \to SM$ defines an m-cycle in SM. We orient SM so that the intersection number of this cycle with every oriented fiber $S_x M$ equals $+1$ (for any u). The element of $H^{m-1}(SM)$ represented by this cycle is an Euler structure on M in the sense of the second definition.

1.4. Lemma. *Let M be a closed oriented 3-manifold. There is a canonical $H_1(M)$-equivariant bijection $\mathrm{vect}(M) = \mathcal{S}(M)$.*

Proof. Consider the mapping $p : SO(3) \to S^2$ assigning to an orthonormal triple of vectors (e_1, e_2, e_3) in $\mathbb{R}^3$ the first vector $e_1 \in S^2$. This mapping is a circle fiber bundle whose fiber represents the nonzero element of $H_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. The pull-back homomorphism $p^* : H^2(S^2) \to H^2(SO(3))$ sends any generator g of $H^2(S^2) = \mathbb{Z}$ to the nonzero element of $H^2(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. Indeed, the Poincaré dual of g in $H_0(S^2) = \mathbb{Z}$ is represented by a point $x \in S^2$ so that the Poincaré dual of $p^*(g)$ is represented by the circle $p^{-1}(x)$.

Endow M with a Riemannian metric. Consider the principal $SO(3)$-bundle $f_M : Fr \to M$ and the spherical bundle $SM \to M$. Denote by p the bundle morphism $Fr \to SM$ assigning to an orthonormal frame (e_1, e_2, e_3) at a point of M the vector e_1. It follows from the results of the previous paragraph that the pull-back homomorphism $p^* : H^2(SM) \to H^2(Fr)$ sends $\mathrm{vect}(M) \subset H^2(SM)$ to $\mathcal{S}(M) \subset H^2(Fr)$. The resulting mapping $\mathrm{vect}(M) \to \mathcal{S}(M)$ is $H_1(M)$-equivariant and therefore bijective.

1.5. Remarks. 1. One can see directly that a vector field u on an oriented 3-manifold M gives rise to a Spin^c-structure on M. The tangent vector bundle TM splits as a direct sum $u^\perp \oplus \mathbb{R}u$. This reduces the structure group of TM to $U(1) = U(1) \oplus (1) \subset U(2)$.

2. For a Spin^c-structure s on a 3-manifold M, one can consider the first Chern class $c_1(s) \in H^2(M) = H_1(M)$ of the associated 2-dimensional complex vector bundle on M. This class equals (at least up to sign) to $c(e_s)$ where $e_s \in \mathrm{vect}(M)$ corresponds to s.

2. Torsion invariants of Euler structures

2.1. Torsions of chain complexes. (cf. [6]). Let $C = (C_m \to C_{m-1} \to \dots \to C_0)$ be a finite dimensional chain complex over a field F. We suppose that for each i we have fixed a basis c_i for C_i and a basis h_i for $H_i(C)$. (A 0-dimensional vector space has an empty basis.) For each i, let $\hat{h}_i$ be a sequence of vectors in $\mathrm{Ker}(\partial_{i-1} : C_i \to C_{i-1})$ which is a lift of h_i. Let b_i be a sequence of vectors in C_i whose image under ∂_{i-1} is a basis in $\mathrm{Im}\,\partial_{i-1}$. Set $b_0 = b_{m+1} = \emptyset$. The torsion of C is defined by

$$\tau(C) = \prod_{i=0}^{m} [\partial_i(b_{i+1})\hat{h}_i b_i/c_i]^{(-1)^{i+1}} \in F\backslash 0, \tag{2.1.a}$$

where $[\partial_i(b_{i+1})\hat{h}_i b_i/c_i]$ is the determinant of the matrix transforming c_i into the basis $\partial_i(b_{i+1}), \hat{h}_i, b_i$ of C_i. The torsion $\tau(C)$ depends only on $C, \{c_i, h_i\}_i$.

We need a version of $\tau(C)$ defined by $\hat{\tau}(C) = (-1)^{N(C)}\tau(C) \in F\backslash 0$ where

$$N(C) = \sum_{i=0}^{m} (\sum_{j=0}^{i} \dim C_j)\,(\sum_{j=0}^{i} \dim H_j(C))$$

(cf. [12]). Note that if C is acyclic, then $\hat{\tau}(C) = \tau(C)$.

2.2. The Reidemeister-Franz torsion. The torsion is defined for a triple (a finite connected CW space X, a field F, a group homomorphism $\varphi : H_1(X) \to F\backslash 0$). Consider the maximal abelian covering $\tilde{X}$ of X with its induced CW structure. The group $H = H_1(X)$ acts on $\tilde{X}$ via covering transformations permuting the cells in $\tilde{X}$ lying over any cell in X. A family of cells in $\tilde{X}$ is said to be *fundamental* if over each cell of X lies exactly one cell of this family. Choose a fundamental family of cells in $\tilde{X}$ and orient and order these cells in an arbitrary way. This yields a basis for the cellular chain complex $C_*(\tilde{X}) = C_*(\tilde{X};\mathbb{Z})$ over the group ring $\mathbb{Z}[H]$. Consider the induced basis for the chain complex

$$C_*^{\varphi}(X) = F \otimes_{\mathbb{Z}[H]} C_*(\tilde{X}).$$

If this based chain complex is acyclic, then we have its torsion $\tau(C_*^{\varphi}(X)) \in F\backslash 0$. A different choice of the fundamental family, cell orientations and the order would replace $\tau(C_*^{\varphi}(X))$ with a product $\pm\varphi(h)\,\tau(C_*^{\varphi}(X))$ where $h \in H$. The set of all such products is denoted by $\pm\tau^{\varphi}(X)$. Thus, $\pm\tau^{\varphi}(X) = \pm\varphi(H)\,\tau(C_*^{\varphi}(X))$ is an element of $F\backslash 0$ defined up to multiplication by -1 and elements of $\varphi(H)$. If the chain complex $C_*^{\varphi}(X)$ is not acyclic then we set $\tau^{\varphi}(X) = 0 \in F$.

2.3. The sign-refined torsions. (cf. [12]). Assume that the CW space X is homology oriented in the sense that an orientation of the vector space $H_*(X;\mathbb{R}) = \oplus_{i\geq 0} H_i(X;\mathbb{R})$ is given. We define a refined version of the Reidemeister-Franz torsion getting rid of the sign indeterminacy. Choose a fundamental family of cells $\tilde{e}$ in $\tilde{X}$ and orient and order these cells in an arbitrary way. As above, this yields a basis in the chain complex $C_*^{\varphi}(X)$ and allows to consider its torsion $\tau \in F$ (equal to 0 if the complex is not acyclic). Since the cells of $\tilde{e}$ bijectively correspond to the cells of X, the orientation and the order for the cells of $\tilde{e}$ induce an orientation and an order for the cells of X. This yields a basis of the cellular chain complex $C_*(X;\mathbb{R})$ over $\mathbb{R}$. Provide the homology of $C_*(X;\mathbb{R})$ with a basis determining the given homology orientation of X. Compute the torsion $\hat{\tau} \in \mathbb{R}\backslash\{0\}$ of the resulting based chain complex with based homology. Consider the sign $\mathrm{sign}(\hat{\tau}) = \pm 1$ of $\hat{\tau}$. It turns out that the product $\mathrm{sign}(\hat{\tau})\,\tau \in F$ is well defined up to multiplication by $\varphi(H)$. This yields a sign-refined torsion $\tau^{\varphi}(X) \in F/\varphi(H)$. Considered up to sign, this is the torsion discussed in Sect. 2.2. The opposite choice of the homology orientation leads to multiplication of $\tau^{\varphi}(X)$ by -1.

Note that any closed oriented manifold M of odd dimension m has a canonical homology orientation determined by any basis in $\oplus_{i<m/2} H_i(M;\mathbb{R})$ followed by the Poincaré dual basis in $\oplus_{i>m/2} H_i(M;\mathbb{R})$.

The sign-refined torsions were introduced in [12] in order to construct the multivariable Conway polynomial of oriented links in S^3. This polynomial is a sign-refined version of the multivariable Alexander polynomial of links.

2.4. Combinatorial Euler structures. (cf. [13]). Let X be a finite connected CW space with $\chi(X) = 0$. An *Euler chain* in X is a 1-dimensional singular chain ξ in X with

$$\partial\xi = \sum_a (-1)^{\dim a} \alpha_a,$$

where a runs over all (open) cells of X and α_a is a point in a. For Euler chains ξ, η in X, we define a homology class $\xi/\eta \in H_1(X)$ as follows. For each cell a, choose a path $x_a : [0,1] \to a$ from the point $\alpha_a = a \cap \partial\xi$ to the point $a \cap \partial\eta$. The class $\xi/\eta \in H_1(X)$ is represented by the 1-cycle $\xi - \eta + \sum_a (-1)^{\dim a} x_a$. The Euler chains ξ, η define the same *Euler stucture* on X if $\xi/\eta = 1$. The group $H_1(X)$ acts on the set of Euler structures $\mathrm{Eul}(X)$ on X: if $[h] \in H_1(X)$ is represented by a 1-cycle h and $[\xi] \in \mathrm{Eul}(X)$ is represented by an Euler chain ξ then $[h][\xi] \in \mathrm{Eul}(X)$ is represented by the Euler chain $\xi + h$. This action is free and transitive.

An Euler structure on X induces an Euler structure on any cell subdivision X' of X. Moreover, there is a canonical $H_1(X)$-equivariant bijection $\mathrm{Eul}(X) = \mathrm{Eul}(X')$. This allows us to define the set of combinatorial Euler structures $\mathrm{Eul}(M)$ on a smooth compact connected manifold M with $\chi(M) = 0$; it is obtained by identification of the sets $\{\mathrm{Eul}(X)\}_X$ where X runs over C^1-triangulations of M. In the case $\partial M = \emptyset$, there is a canonical $H_1(M)$-equivariant bijection $\mathrm{Eul}(M) = \mathrm{vect}(M)$. The idea is as follows. Fix a C^1-triangulation X of M. There is a natural singular vector field ν on M defined in terms of the barycentric coordinates of X, see [2]. The singularities of ν are the barycenters of the simplices of X. Any Euler structure on X can be presented by a spider-like Euler chain consisting of oriented arcs joining a point of X with the barycenters of the simplices. The vector field ν is nonsingular outside a ball neighborhood of such a spider. Since $\chi(M) = 0$, this nonsingular vector field on the complement of a ball extends to a nonsingular vector field on M. This yields a bijection $\mathrm{Eul}(M) = \mathrm{vect}(M)$.

In the case $\partial M \neq \emptyset$, we define smooth Euler structures on M as the homotopy classes of nonsingular tangent vector fields on M directed outwards on ∂M. As above, the group $H_1(M)$ acts on $\mathrm{vect}(M)$ freely and transitively and there is a canonical $H_1(M)$-equivariant bijection $\mathrm{Eul}(M) = \mathrm{vect}(M)$.

2.5. The torsion of Euler structures. (cf. [13]). Let X be a homology oriented finite connected CW space with $\chi(X) = 0$. Let F be a field and $\varphi : H = H_1(X) \to F\backslash 0$ be a group homomorphism. For every Euler structure $e \in \mathrm{Eul}(X)$ we define a refinement $\tau^\varphi(X, e) \in F$ of the torsion $\tau^\varphi(X) \in F/\varphi(H)$.

Any fundamental family of cells $\tilde{e}$ in the maximal abelian covering $\tilde{X}$ gives rise to an Euler structure on X: consider a spider in $\tilde{X}$ formed by arcs in $\tilde{X}$ connecting a certain point $x \in \tilde{X}$ to points in these cells; the arc joining x to a point of an odd-dimensional (resp. even-dimensional) cell should be oriented towards x (resp. out of x). Projecting this spider to X we obtain an Euler chain in X. Its class in $\mathrm{Eul}(X)$ depends only on $\tilde{e}$ and does not depend on the choice of x and the arcs. It is clear that any Euler structure e on X arises in this way from a fundamental family of cells $\tilde{e}$ in $\tilde{X}$. Now, to define $\tau^\varphi(X, e) \in F$ we proceed as in Sect. 2.3 using such $\tilde{e}$.

It follows from definitions that $\tau^\varphi(X, he) = \varphi(h)\, \tau^\varphi(X, e)$ for any $e \in \mathrm{Eul}(X), h \in H$. Clearly, $\tau^\varphi(X) = \{\tau^\varphi(X, e) \,|\, e \in \mathrm{Eul}(X)\}$.

The main point of these definitions is that $\tau^\varphi(X, e)$ is invariant under cell subdivisions of X. Combining the constructions of this section with those of Sect. 2.4, we obtain the torsions of smooth Euler structures on manifolds.

2.6. Relative torsions. The constructions of Sections 2.2-2.5 extend to any finite CW pair (X, Y) with connected X and $\chi(X, Y) = 0$. A homology orientation in (X, Y) is an orientation in $H_*(X, Y; \mathbb{R})$. Euler chains and Euler structures on (X, Y) are defined as in Sect. 2.4 where a runs over the cells of X not lying in Y. The group $H = H_1(X)$ acts freely and transitively on the set of Euler structures $\mathrm{Eul}(X, Y)$.

Let F be a field and $\varphi : H \to F\backslash 0$ be a group homomorphism. For a homology orientation of (X, Y) and $e \in \mathrm{Eul}(X, Y)$, we define a torsion $\tau^\varphi(X, Y, e) \in F$ as above using the chain complex

$$C_*^\varphi(X, Y) = F \otimes_{\mathbb{Z}[H]} C_*(\tilde{X})/C_*(p^{-1}(Y)),$$

where $p : \tilde{X} \to X$ is the maximal abelian covering of X.

We state a theorem of multiplicativity for torsions refining the classical multiplicativity due to Whitehead [14]. Observe that the sum of an Euler chain in (X, Y) and an Euler chain in Y is an Euler chain in X. This induces a pairing $(e, e') \mapsto ee'$ from $\mathrm{Eul}(X, Y) \times \mathrm{Eul}(Y)$ to $\mathrm{Eul}(X)$. Assume that X and Y are homology oriented and provide the pair (X, Y) with the induced homology orientation such that the torsion of the exact homology sequence of (X, Y) with coefficients in $\mathbb{R}$ with respect to the bases in homologies determining these homology orientations is positive. Assume that $\chi(X) = \chi(Y) = 0$ and denote by j the inclusion homomorphism $H_1(Y) \to H_1(X)$.

2.6.1. Theorem. *If $\tau^\varphi(X, Y) \neq 0$ or $\tau^{\varphi \circ j}(Y) \neq 0$, then*

$$\tau^\varphi(X, ee') = (-1)^\mu \, \tau^\varphi(X, Y, e) \, \tau^{\varphi \circ j}(Y, e')$$

for any $e \in Eul(X, Y), e' \in Eul(Y)$ and

$$\mu = \sum_{i=0}^{\dim X} [(\beta_i + 1)(\beta_i' + \beta_i'') + \beta_{i-1}' \beta_i''] \, (mod\ 2) \in \mathbb{Z}/2\mathbb{Z},$$

where

$$\beta_i = \sum_{r=0}^{i} b_r(X), \ \beta_i' = \sum_{r=0}^{i} b_r(Y), \ \beta_i'' = \sum_{r=0}^{i} b_r(X, Y).$$

For a proof, see ([12], Sect. 3.4.) Note that if $H_*(X, Y; \mathbb{R}) = 0$ then $\mu = 0$.

2.7. The duality. One of the fundamental properties of torsions is the duality due to Franz [1] and Milnor [5]. We state a refined version following [12], [13]. Let M be a smooth closed connected oriented manifold of odd dimension $m \geq 3$. Let F be a field with involution $f \mapsto \overline{f} : F \to F$. Let $\varphi : H_1(M) \to F\backslash 0$ be a group homomorphism such that $\overline{\varphi(h)} = \varphi(h^{-1})$ for any $h \in H_1(M)$. Then for every $e \in \mathrm{vect}(M) = \mathrm{Eul}(M)$,

$$\overline{\tau^\varphi(M, e)} = (-1)^z \, \tau^\varphi \, (M, e^{-1}) = (-1)^z \, \varphi(c(e)) \, \tau^\varphi(M, e),$$

where e^{-1} is the opposite Euler structure on M, $c(e) \in H_1(M)$ is the Euler class of e, and $z = 0$ for $m = 3 \,(\mathrm{mod}\, 4)$ and $z = \sum_{i<m/2} b_i(M)$ for $m = 1 \,(\mathrm{mod}\, 4)$.

3. The torsion τ

3.1. Preliminaries. Let H be a finitely generated abelian group. Denote by $Q(H)$ the classical ring of quotients of the rational group ring $\mathbb{Q}[H]$, i.e., the localization of $\mathbb{Q}[H]$ by the multiplicative system of all non-zerodivisors. We show here that $Q(H)$ splits as a finite direct sum of fields. (Such a splitting is unique: the fields in question may be characterized as the minimal ideals of $Q(H)$.)

Set $T = \operatorname{Tors} H$. Each character $\sigma : T \to S^1 \subset \mathbb{C}$ extends to a $\mathbb{Q}$-linear ring homomorphism $\tilde{\sigma} : \mathbb{Q}[T] \to \mathbb{C}$. Its image is a cyclotomic field, K_σ. Two characters σ_1 and σ_2 of T are said to be equivalent if $K_{\sigma_1} = K_{\sigma_2}$ and $\tilde{\sigma}_1$ is a composition of $\tilde{\sigma}_2$ and a Galois automorphism of K_{σ_1} over $\mathbb{Q}$. It is well known that for any complete family of representatives $\sigma_1, ..., \sigma_n$ of the equivalence classes, the homomorphism $(\tilde{\sigma}_1, ..., \tilde{\sigma}_n) : \mathbb{Q}[T] \to \oplus_{i=1}^n K_{\sigma_i}$ is an isomorphism. This implies that $Q(T) = \mathbb{Q}[T]$ and proves our claim in the case $\operatorname{rank} H = 0$.

In the general case consider the free abelian group $G = H/T$. Then

$$\mathbb{Q}[H] = \mathbb{Q}[T \oplus G] = (\mathbb{Q}[T])[G] = \oplus_{i=1}^n K_{\sigma_i}[G].$$

The group ring $K_{\sigma_i}[G]$ is an integral domain. An element of $\mathbb{Q}[H]$ is a non-zerodivisor if and only if its projections to all the summands $K_{\sigma_i}[G]$ are nonzero. Inverting all non-zerodivisors in $\mathbb{Q}[H]$ we obtain

$$Q(H) = \oplus_{i=1}^n F_i, \tag{3.1.a}$$

where F_i is the field of fractions of $K_{\sigma_i}[G]$. We can view F_i as the field of rational functions in $\operatorname{rank} H = \operatorname{rank} G$ variables with coefficients in K_{σ_i}. Note that $H \subset \mathbb{Q}[H] \subset Q(H)$.

3.2. Definition of τ. Let X be a homology oriented finite connected CW space (or a homology oriented smooth compact connected manifold) with $\chi(X) = 0$. Set $H = H_1(X)$. Denote by φ_i the composition of the inclusion $H \hookrightarrow Q(H)$ and the projection $Q(H) \to F_i$ on the i-th term in (3.1.a). By Sect. 2, for any $e \in \operatorname{Eul}(X)$, we have $\tau^{\varphi_i}(X, e) \in F_i$. Set

$$\tau(X, e) = \sum_{i=1}^n \tau^{\varphi_i}(X, e) \in \oplus_{i=1}^n F_i = Q(H).$$

This is a well defined element of $Q(H)$. Clearly $\tau(X, he) = h\,\tau(X, e)$, for $h \in H$.

Set $\tau(X) = \{\tau(X, e) \,|\, e \in \operatorname{Eul}(X)\}$. We view $\tau(X)$ as an element of $Q(H)/H$.

3.3. The Milnor torsion. Let us numerate the fields $\{F_i\}$ in (3.1.a) so that F_1 corresponds to the trivial character $T \to 1$ of $T = \operatorname{Tors} H$. Then F_1 is the field of fractions of the group ring $\mathbb{Q}[G]$ where $G = H/T$. The projection proj : $Q(H) \to F_1$ along $\oplus_{i \geq 2} F_i$ is induced by the projection $H \to G$. The inclusion $Q(G) = F_1 \hookrightarrow Q(H)$ is the composition of the ring homomorphism $Q(G) \to$

$Q(H)$ induced by any section of the projection $H \to G$ and multiplication by $|T|^{-1}\sum_{g\in T} g \in \mathbb{Q}[H]$.

The torsion $\pm\tau^{\rm proj}(X) = \pm{\rm proj}\,(\tau(X)) \in Q(G)/\pm G$ was introduced by Milnor [5] for compact 3-manifolds with boundary. He computed this torsion in terms of the Alexander polynomial. This was extended to closed 3-manifolds in [8], cf. [12] and Sect. 4.4 below.

If ${\rm rank}\, H = 0$, then $Q(H) = \mathbb{Q}[H]$ and ${\rm proj} = {\rm aug} : \mathbb{Q}[H] \to \mathbb{Q}$ is summation of coefficients. Clearly ${\rm proj}\,(\tau(X)) = \tau^{\rm proj}(X) = 0$ where the last equality follows from the fact that the chain complex $C_*^{\rm proj}(X) = C_*(X;\mathbb{Q})$ has nontrivial homology.

3.4. Duality for τ. The projection $Q(H) \to F_i$ in (3.1.a) is equivariant with respect to the ring involution $a \mapsto \overline{a} : Q(H) \to Q(H)$ induced by the inversion $h \mapsto h^{-1} : H \to H$ and the ring involution in F_i extending the complex conjugation in K_{σ_i} and the inversion $h \mapsto h^{-1} : G \to G$. The duality theorem of Sect. 2.7 applies to each summand in the definition of $\tau(M,e)$ where M is a smooth closed connected oriented manifold of odd dimension and $e \in {\rm vect}(M) = {\rm Eul}(M)$. We obtain

$$\overline{\tau(M,e)} = (-1)^z\,\tau(M,e^{-1}) = (-1)^z\,c(e)\,\tau(M,e). \qquad (3.4.a)$$

4. The torsion τ for 3-manifolds

Throughout Sections 4 and 5 the symbol M denotes a smooth compact connected oriented homology oriented 3-manifold whose boundary is either empty or consists of 2-tori. If $\partial M = \emptyset$ then we assume that the homology orientation of M is induced by the orientation of M as in Sect. 2.3. Set $H = H_1(M)$, $G = H/{\rm Tors}\, H$, and $\Sigma = \sum_{h\in {\rm Tors} H} h \in \mathbb{Z}[H]$.

4.1. Theorem. *If $b_1(M) \geq 2$, then $\tau(M,e) \in \mathbb{Z}[H]$ for any $e \in Eul(M)$.*

Note that if the inclusion $\tau(M,e) \in \mathbb{Z}[H]$ holds for one Euler structure then it holds for all the others and for the opposite homology orientation.

4.1.1. Lemma. *Let x be an element of $Q(H)$ such that $x(h-1) \in \mathbb{Z}[H]$ for any $h \in H$. If ${\rm rank}\, H \geq 2$, then $x \in \mathbb{Z}[H]$.*

Proof. The lemma is obvious if H is torsion-free, since in this case $Q(H)$ is the ring of rational functions on ${\rm rank}\, H \geq 2$ variables with integer coefficients. Set $x' = {\rm proj}(x) \in Q(G)$ where ${\rm proj} : Q(H) \to Q(G)$ is the projection. It is clear that $x'(g-1) \in \mathbb{Z}[G]$ for any $g \in G$. Hence $x' \in \mathbb{Z}[G]$. Under the inclusion $Q(G) \hookrightarrow Q(H)$ the ring $\mathbb{Z}[G]$ is mapped into $\mathbb{Q}[H]$ and x' is mapped into $y = xn^{-1}\Sigma$ where $n = |{\rm Tors}\, H|$ (cf. Sect. 3.3). Therefore $y \in \mathbb{Q}[H]$. Set $z = x - y$. For $h \in {\rm Tors}\, H$, we have $y(h-1) = 0$ and $z(h-1) = x(h-1) \in \mathbb{Z}[H]$. Summing up over $h \in {\rm Tors}\, H$, we obtain $z(\Sigma - n) \in \mathbb{Z}[H]$. On the other hand

$$z\Sigma = x(1 - n^{-1}\Sigma)\Sigma = x(\Sigma - n^{-1}\Sigma^2) = x \cdot 0 = 0.$$

Thus, $nz \in \mathbb{Z}[H]$ so that $z \in \mathbb{Q}[H]$. This implies $x = y + z \in \mathbb{Q}[H]$. For a suitable $h \in H$, all the coefficients of the formal sum $x \in \mathbb{Q}[H]$ appear as coefficients of $x(h-1)$. Therefore, $x \in \mathbb{Z}[H]$.

4.1.2. Proof of Theorem 4.1. Consider first the case $\partial M = \emptyset$. It is well known that M admits a cell decomposition X consisting of one 0-cell, one 3-cell, and equal number, say m, of 1-cells and 2-cells. Consider the maximal abelian covering $\tilde{X}$ of X. Choose a fundamental family of cells $\tilde{e}$ in $\tilde{X}$ and orient and order these cells in an arbitrary way. This yields a basis for the cellular chain complex $C_*(\tilde{X}) = (C_3 \to C_2 \to C_1 \to C_0)$ where $C_0 = \mathbb{Z}[H]$, $C_1 = (\mathbb{Z}[H])^m$, $C_2 = (\mathbb{Z}[H])^m$, and $C_3 = \mathbb{Z}[H]$ with $H = H_1(M) = H_1(X)$. We can choose $\tilde{e}$ so that:

(i) the boundary homomorphism $\partial_0 : C_1 \to C_0$ is given by an $(m \times 1)$-matrix $(h_1 - 1, h_2 - 1, ..., h_m - 1)$ where $h_1, ..., h_m$ are the generators of H represented by the oriented 1-cells of X;

(ii) the boundary homomorphism $\partial_2 : C_3 \to C_2$ is given by an $(1 \times m)$-matrix $(g_1 - 1, g_2 - 1, ..., g_m - 1)$ where $g_r \in H$ is represented by a loop in X which pierces once the r-th 2-cell of X and is contained in the (open) 3-cell of X otherwise.

Denote by $\Delta_{r,s}$ the determinant of the matrix obtained from the $(m \times m)$-matrix, A, of the boundary homomorphism $C_2 \to C_1$ by deleting the r-th row and s-th column. Let e be the Euler structure on X corresponding to $\tilde{e}$. Consider the cellular chain complex $C_*(X; \mathbb{R})$ with the basis determined by $\tilde{e}$ and the basis in homology determining the homology orientation of M. Let $\xi = \pm 1$ be the sign of the corresponding torsion $\hat{\tau} \in \mathbb{R} \backslash \{0\}$. We claim that for any r, s,

$$\tau(X, e)(g_r - 1)(h_s - 1) = (-1)^{m+r+s+1}\, \xi\, \Delta_{r,s} \in \mathbb{Z}[H]. \tag{4.1.a}$$

To prove this, consider splitting (3.1.a). Let φ_i denote the projection $Q(H) \to F_i$. It suffices to show that for every i,

$$\varphi_i(\tau(X, e))\, \varphi_i(g_r - 1)\, \varphi_i(h_s - 1) = (-1)^{m+r+s+1}\, \xi\, \varphi_i(\Delta_{r,s}). \tag{4.1.b}$$

We distinguish four cases.

(1). Let $\varphi_i(g_r - 1) = 0$. Since any loop in M may be deformed into a loop transversal to the 2-skeleton of X, the elements $g_1, ..., g_m$ generate H. The assumption $b_1(M) \geq 2$ implies that $\varphi_i(H) \neq 1$. Therefore $\varphi_i(g_u) \neq 1$ for a certain $u \neq r$. The equality $\partial_1 \partial_2 = 0$ yields a linear relation between the rows of A. Apply φ_i to all terms of this relation. The resulting relation is nontrivial because the u-th row appears with coefficient $\varphi_i(g_u - 1) \neq 0$. On the other hand the r-th row does not appear in this relation because $\varphi_i(g_r - 1) = 0$. Therefore $\varphi_i(\Delta_{r,s}) = 0$.

(2). Let $\varphi_i(h_s - 1) = 0$. Since $\varphi_i(H) \neq 1$, we have $\varphi_i(h_u) \neq 1$ for a certain $u \neq s$. The equality $\partial_0 \partial_1 = 0$ yields a linear relation between the columns of A. Applying φ_i we obtain a nontrivial relation because $\varphi_i(h_u - 1) \neq 0$. The s-th column does not appear in this relation because $\varphi_i(h_s - 1) = 0$. Hence $\varphi_i(\Delta_{r,s}) = 0$.

(3). Let $\varphi_i(g_r - 1) \neq 0$, $\varphi_i(h_s - 1) \neq 0$, and $\varphi_i(\Delta_{r,s}) = 0$. It is easy to see that $\operatorname{rank} A \leq m - 2$. Therefore $H_2(C^{\varphi_i}(X)) \neq 0$ and $\varphi_i(\tau(X, e)) = 0$.

(4). Let $\varphi_i(g_r-1)\neq 0$, $\varphi_i(h_s-1)\neq 0$, and $\varphi_i(\Delta_{r,s})\neq 0$. In this case the complex $C^{\varphi_i}(X)=F_i\otimes C_*(\tilde X)$ is acyclic. By definition, $\varphi_i(\tau(X,e))=\xi\,\tau(C^{\varphi_i}(X))$. We compute the latter torsion using (2.1.a) where: $m=3$; $c_0,...,c_3$ are the bases for the chain modules of $C^{\varphi_i}(X)$ defined by $\tilde e$; the bases in homology are empty; b_1 is the s-th vector of c_1, b_2 is obtained from c_2 by omitting the r-th vector, and $b_3=c_3$. We have $[\partial_0(b_1)b_0/c_0]=\varphi_i(h_s-1)$, $[\partial_1(b_2)b_1/c_1]=(-1)^{m-s}\varphi_i(\Delta_{r,s})$, $[\partial_2(b_3)b_2/c_3]=(-1)^{r-1}\varphi_i(g_r-1)$, and $[b_3/c_3]=1$. This implies

$$\tau(C^{\varphi_i}(X))=(-1)^{m+r+s+1}\varphi_i\,(\Delta_{r,s})\,(\varphi_i(g_r-1))^{-1}(\,\varphi_i(h_s-1))^{-1}.$$

This is equivalent to (4.1.b).

Since $g_1,...,g_m$ (resp. $h_1,...,h_m$) generate H, $\tau(X,e)(g-1)(h-1)\in\mathbb{Z}[H]$ for any $g,h\in H$. Applying Lemma 4.1.1 twice, we obtain $\tau(M,e)=\tau(X,e)\in\mathbb{Z}[H]$.

Let $\partial M\neq\emptyset$. We can collapse M onto a 2-dimensional CW complex $X\subset M$ with one 0-cell and m one-cells. By $\chi(X)=\chi(M)=0$, the number of 2-cells of X is equal to $m-1$. As above, we present the boundary homomorphisms $C_3(\tilde X)\to C_2(\tilde X)\to C_1(\tilde X)$ by an $((m-1)\times m)$-matrix, A, and $(m\times 1)$-matrix $(h_1-1,...,h_m-1)$ where $h_1,...,h_m\in H$ are generators represented by the 1-cells of X. The rest of the argument goes as in the case of closed M; instead of (4.1.a) we have $\tau(X,e)(h_s-1)=(-1)^{m+s}\,\xi\,\Delta_s$ where Δ_s is the determinant of the matrix obtained from A by deleting the s-th column. By Lemma 4.1.1, $\tau(X)\in\mathbb{Z}[H]/H$. The invariance of τ under simple homotopy equivalences implies $\tau(M)\in\mathbb{Z}[H]/H$.

4.2. The case $b_1(M)=1$. Fix an element $t\in H$ whose image modulo $\operatorname{Tors} H$ is a generator $[t]$ of the infinite cyclic group $G=H/\operatorname{Tors} H$. Observe that $1-t$ is invertible in $Q(H)$. Recall that $\Sigma=\sum_{h\in\operatorname{Tors} H}h\in\mathbb{Z}[H]$.

4.2.1. Theorem. *Let $b_1(M)=1$ and $\partial M=S^1\times S^1$. Assume that the homology orientation of M is given by the basis $[pt]\in H_0, t\in H_1$. For $e\in Eul(M)$, set $\tau_t(M,e)=\tau(M,e)-(1-t)^{-1}\Sigma$. Then $\tau_t(M,e)\in\mathbb{Z}[H]$.*

It is clear that $\tau_{th}(M,e)=\tau_t(M,e)$ for $h\in\operatorname{Tors} H$. If we replace t with t^{-1} then the homology orientation of M is inversed so that

$$\tau_{t^{-1}}(M,e)+(1-t^{-1})^{-1}\Sigma=-\tau_t(M,e)-(1-t)^{-1}\Sigma.$$

Thus, $\tau_t(M,e)+\tau_{t^{-1}}(M,e)=-\Sigma$.

Proof of Theorem. An argument used in Lemma 4.1.1 yields the following.

(4.2.2) Let x be an element of $Q(H)$ such that $x(h-1)\in\mathbb{Z}[H]$ for any $h\in H$. Then $x=x_t+r(1-t)^{-1}\Sigma$ with $x_t\in\mathbb{Z}[H]$ and $r\in\mathbb{Z}$.

Using this proposition and the argument of Sect. 4.1.2, we obtain $\tau(M,e)=x_t(M,e)+r(1-t)^{-1}\Sigma$ with $x_t(M,e)\in\mathbb{Z}[H]$ and $r\in\mathbb{Z}$. It remains to show that

$r = 1$. Note that the image of $\tau(M, e)$ under the projection proj : $Q(H) \to Q(G)$ is the torsion $\tau^{\mathrm{proj}}(M, e) \in Q(G)$. It suffices to prove that the sum of coefficients of $\tau^{\mathrm{proj}}(M, e)(1 - [t]) \in \mathbb{Z}[G]$ equals $|\mathrm{Tors}\, H|$. This sum does not change if $\tau^{\mathrm{proj}}(M, e)$ is multiplied by an element of G. Therefore we can forget about e and deal just with the sign-refined torsion $\tau^{\mathrm{proj}}(M)$.

As in Sect. 4.1.2 we collapse M onto a 2-dimensional CW complex $X \subset M$ with one 0-cell and $m \geq 1$ one-cells. We can assume that the closure of one of the 1-cells of X is a circle representing $t^{\pm 1} \in H = H_1(X) = H_1(M)$. Denote this circle by Y. We orient Y so that it represents t and provide it with homology orientation $[pt] \wedge [Y]$. We use Theorem 2.6.1 to compute $\tau^{\mathrm{proj}}(M) = \tau^{\mathrm{proj}}(X)$. A direct computation shows that $\mu = 0$ and $\tau^{\mathrm{proj} \circ j}(Y) = (1 - [t])^{-1}$. Hence $\tau^{\mathrm{proj}}(X)(1 - [t]) = \tau^{\mathrm{proj}}(X, Y)$. Observe that the cellular chain complex $C_*^{\mathrm{proj}}(X, Y)$ is nontrivial only in dimensions 1 and 2. The boundary homomorphism $C_2 \to C_1$ is given by a $((m-1) \times (m-1))$-matrix, A, over $\mathbb{Z}[G]$. The integral matrix A^0 obtained from A by replacing every term with the sum of its coefficients is the matrix of the boundary homomorphism in the cellular chain complex $C_*(X, Y)$. Hence $\det A^0 = \pm |H_1(X, Y)| = \pm |\mathrm{Tors}\, H|$. It follows from definitions that $\tau^{\mathrm{proj}}(X, Y) = \mathrm{sign}(\det A^0) \det A$. The sum of coefficients of $\tau^{\mathrm{proj}}(X, Y)$ is equal to $\mathrm{sign}(\det A^0) \det A^0 = |\mathrm{Tors}\, H|$. Therefore $r = 1$.

4.2.3. Theorem. *Let $b_1(M) = 1$ and $\partial M = \emptyset$. Let $e \in \mathrm{vect}(M) = \mathrm{Eul}(M)$ and $K = K_t(e)$ be an integer such that $c(e) \in t^K\, \mathrm{Tors}\, H$. Set*

$$\tau_t(M, e) = \tau(M, e) - \frac{K-2}{2}(1-t)^{-1}\Sigma - (1-t)^{-2}\Sigma.$$

Then $\tau_t(M, e) \in \mathbb{Z}[H]$.

The number K is even: this follows from the identity $c(he) = h^2 c(e)$ and the parallelisability of M which implies the existence of an Euler structure e on M with $c(e) = 1$. Note that $K_{t^{-1}}(e) = -K_t(e)$ and $\tau_{th}(M, e) = \tau_t(M, e)$ for $h \in \mathrm{Tors}\, H$. An easy computation shows that $\tau_{t^{-1}}(M, e) = \tau_t(M, e) + (K_t(e)/2)\, \Sigma$.

Proof of Theorem. We begin with an analogue of (4.2.2).

(4.2.4) Let x be an element of $Q(H)$ such that $x(g-1)(h-1) \in \mathbb{Z}[H]$ for any $g, h \in H$. Then $x = x_t + r(1-t)^{-1}\Sigma + s(1-t)^{-2}\Sigma$ with $x_t \in \mathbb{Z}[H]$ and $r, s \in \mathbb{Z}$.

Using this proposition and the argument of Sect. 4.1.2, we obtain

$$\tau(M, e) = x_t(M, e) + r(1-t)^{-1}\Sigma + s(1-t)^{-2}\Sigma \tag{4.2.a}$$

with $x_t(M, e) \in \mathbb{Z}[H]$ and $r, s \in \mathbb{Z}$. Let us prove that $s = 1$. The number s does not change if $\tau(M, e)$ is multiplied by an element of H. Therefore we can forget about e and deal just with $\tau(M) = \{\tau(M, e) \,|\, e \in \mathrm{Eul}(M)\} \in Q(H)/H$.

Choose an embedded circle $\ell \subset M$ representing t and denote by E its exterior, i.e., the complement in M of its open regular neighborhood. Note that

E is a compact connected orientable 3-manifold with $H_1(E) = H_1(M) = H$ and $H_2(E) = H_3(E) = 0$. We provide E with homology orientation $[pt] \wedge t$. Since $M \backslash \mathrm{Int} E$ is a solid torus, the pair (M, E) has a relative cell decomposition consisting of one 2-cell, α^2, and one 3-cell, α^3. The orientation of M induces an orientation of α^3; we orient α^2 so that $\alpha^2 \cdot \ell = +1$. It is easy to compute that the homology orientation of the pair (M, E) induced by those in M and E is given by $[\alpha^2] \wedge [\alpha^3]$.

The image of $\tau(M)$ under the projection $\mathrm{proj} : Q(H) \to Q(G)$ is the torsion $\tau^{\mathrm{proj}}(M) \in Q(G)/G$. To compute the latter torsion we apply Theorem 2.6.1 to the pair (M, E). It is easy to check that $\mu = 0$. For an appropriate lift of the oriented cells α^2, α^3 to the maximal abelian covering of M, the boundary homomorphism $C_3 \to C_2$ of the cellular chain complex $C_*^{\mathrm{proj}}(M, E)$ is given by the (1×1)-matrix $[t] - 1$. By a direct computation, $\tau^{\mathrm{proj}}(M, E) = (1 - [t])^{-1}$. Theorem 2.6.1 implies that

$$\tau^{\mathrm{proj}}(E) = (1 - [t])\, \tau^{\mathrm{proj}}(M) = (1 - [t])\, \mathrm{proj}(\tau(M)) = z + s(1 - [t])^{-1}\Sigma,$$

where $z \in \mathbb{Z}[G]$. Now, Theorem 4.2.1 applied to E yields $s = 1$.

By Sect. 3.4, $\overline{\tau(M, e)} = c(e)\, \tau(M, e)$. Substituting (4.2.a), using the equalities $c(e)\Sigma = t^K \Sigma, \overline{\Sigma} = \Sigma$ and computing modulo $\mathbb{Z}[H]$ one obtains $r = (K - 2)/2$.

4.3. The case $b_1(M) = 0$. Under our assumptions on M, the condition $b_1(M) = 0$ implies that $\partial M = \emptyset$. Since $H = H_1(M)$ is finite, $\tau(M, e) \in Q(H) = \mathbb{Q}[H]$ for any $e \in \mathrm{Eul}(M)$. By Sect. 3.3, $\mathrm{aug}(\tau(M, e)) = 0$.

Recall the linking form $L : H \times H \to \mathbb{Q}/\mathbb{Z}$. To compute $L(g, h)$ for $g, h \in H$ one represents g, h by disjoint 1-cycles, say x, y. Take a nonzero integer n such that ny is the boundary of a 2-chain, α. One counts the intersection number $x \cdot \alpha \in \mathbb{Z}$ and sets $L(g, h) = n^{-1}(x \cdot \alpha) \bmod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. This is well defined because $H_2(M) = 0$ so that the intersection number of x with any 2-cycle is 0. Note that L is a non-degenerate symmetric bilinear form.

4.3.1. Theorem. *For any $e \in \mathrm{Eul}(M)$ and $g, h \in H$,*

$$\tau(M, e)(g - 1)(h - 1) = -L(g, h)\, \Sigma \, (\mathit{mod}\, \mathbb{Z}[H]). \tag{4.3.a}$$

Theorem 4.3.1 implies that in general $\tau(M, e)$ does not lie in $\mathbb{Z}[H]$. This theorem can be reformulated in terms of the coefficients of $\tau(M, e) = \sum_{h \in H} q(h) h$ where $q(h) \in \mathbb{Q}$. Namely, for any $g, h \in H$,

$$q(gh) - q(g) - q(h) + q(1) = -L(g, h)\, (\mathrm{mod}\, \mathbb{Z}).$$

Proof of Theorem. We shall use the notation of Sect. 4.1.2. Set $n = |H|$. The argument of Sect. 4.1.2 gives (4.1.b) for the projection $\varphi_i : \mathbb{Q}[H] \to F_i$ induced by any *nontrivial* character of H. Clearly, $\mathrm{aug}(\tau(M, e)(g_r - 1)(h_s - 1)) = 0$. Hence

$$\tau(M, e)(g_r - 1)(h_s - 1) = (-1)^{m+r+s+1} \xi\, (\Delta_{r,s} - \mathrm{aug}(\Delta_{r,s}) n^{-1}\Sigma). \tag{4.3.b}$$

We compute ξ. By definition, ξ is the sign of the torsion $\hat{\tau}$ of the cellular chain complex $C = C_*(X;\mathbb{R})$ with respect to the basis determined by the oriented and ordered cells of the fundamental family of cells $\tilde{e}$. It is clear that $C_0 = \mathbb{R}$, $C_1 = \mathbb{R}^m$, $C_2 = \mathbb{R}^m$, and $C_3 = \mathbb{R}$. The boundary homomorphisms $C_3 \to C_2$ and $C_1 \to C_0$ are zero. The boundary homomorphism $C_2 \to C_1$ is an isomorphism given by an integer $(m\times m)$-matrix, A^0. It is clear that $\det A^0 = \pm n$. As a basis in $H_*(C)$ we take $([pt],[M])$. We also assume that the orientation of the only 3-cell of X is induced by the orientation of M. It is easy to compute from definitions that $N(C) = m+1 \pmod 2$ and $\tau(C) = \det(A^0)$. Thus, $\hat{\tau}(C) = (-1)^{m+1}\det(A^0)$ and $\xi = (-1)^{m+1}\mathrm{sign}(\det A^0)$.

Let $(\tilde{\alpha}_1,...,\tilde{\alpha}_m)$ be the oriented and ordered 2-cells of $\tilde{e}$. Let $(\tilde{\beta}_1,...,\tilde{\beta}_m)$ be the oriented and ordered 1-cells of $\tilde{e}$. Fix $s = 1,...,m$ and consider the cellular 2-chain $\tilde{\alpha} = \sum_i (-1)^{i+s}\Delta_{i,s}\tilde{\alpha}_i \in C_2(\tilde{X};\mathbb{Z})$. It is clear that

$$\partial(\tilde{\alpha}) = \sum_{i,j}(-1)^{i+s}\Delta_{i,s}a_{i,j}\tilde{\beta}_j = \sum_j \delta_s^j \det(A)\,\tilde{\beta}_j = \det(A)\,\tilde{\beta}_s,$$

where $A = (a_{i,j})$ is the $(m\times m)$-matrix of the boundary homomorphism $\partial : C_3(\tilde{X};\mathbb{Z}) \to C_2(\tilde{X};\mathbb{Z})$ and δ is the Kronecker delta. Projecting $\tilde{\alpha}$ into X we obtain a cellular 2-chain $\alpha = \sum_i (-1)^{i+s}\mathrm{aug}(\Delta_{i,s})\,\alpha_i$ where α_i is the i-th oriented 2-cell in X. Clearly, $\partial\alpha = \mathrm{aug}(\det A)y_s$ where y_s is the oriented circle in X formed by the s-th 1-cell and the only 0-cell and representing $h_s \in H$. Observe that summation of coefficients transforms A into A^0 so that $\mathrm{aug}(\det A) = \det A^0 = (-1)^{m+1}\xi n$. Therefore $\partial\alpha = (-1)^{m+1}\xi n y_s$.

We present g_r by a loop x_r in M piercing once the r-th 2-cell of X and contained in the open 3-cell otherwise. By definition,

$$L(g_r,h_s) = ((-1)^{m+1}\xi n)^{-1}(x_r\cdot\alpha) = \sum_i (-1)^{m+1+i+s}\xi n^{-1}\,\mathrm{aug}(\Delta_{i,s})(x_r\cdot\alpha_i)$$

$$= \sum_i (-1)^{m+1+i+s}\xi n^{-1}\,\mathrm{aug}(\Delta_{i,s})\,\delta_i^r = (-1)^{m+r+s+1}\xi n^{-1}\,\mathrm{aug}(\Delta_{r,s}).$$

Comparing with (4.3.b), we obtain (4.3.a) with $g = g_r, h = h_s$. Since $g_1,...,g_m$ (resp. $h_1,...,h_m$) generate H and L is bilinear, this yields the claim of the theorem.

4.4. Remarks. 1. The torsion τ is closely related to the first elementary ideal E of the fundamental group of M. This is the ideal in $\mathbb{Z}[H]$ generated by the determinants $\{\Delta_{r,s}\}$ (resp. $\{\Delta_s\}$) appearing in Sect. 4.1.2 for $\partial M = \emptyset$ (resp. for $\partial M \neq \emptyset$). Denote by I the ideal of $\mathbb{Z}[H]$ generated by $\{h-1,\, h \in H\}$. For $b_1(M) \geq 1$, the proof of Theorem 4.1 yields $E = \tau(M)I^2$ if $\partial M = \emptyset$ and $E = \tau(M)I$ if $\partial M \neq \emptyset$. If $b_1(M) = 0$, then E is the pre-image of $\tau(M)I^2$ under the homomorphism $\mathbb{Z}[H] \to \mathbb{Q}[H]$ sending $h \in H$ to $h - |H|^{-1}\Sigma$. (The key point is the inclusion $\Sigma \in E$ which follows from the equality $\Sigma = \pm\det A$ essentially proven in Sect. 4.1.2.) For more on this, see [9], [10].

2. The Alexander polynomial $\Delta(M)$ of $\pi_1(M)$ is defined for $b_1(M) \geq 1$ as the greatest common divisor of the elements of $\text{proj}(E(\pi)) \subset \mathbb{Z}[H/\text{Tors}\, H]$. This implies

$$\Delta(M) = \begin{cases} \pm\text{proj}\,(\tau(M)), \text{ if } b_1(M) \geq 2, \\ \pm\text{proj}\,(\tau(M)(t-1)^2), \text{ if } b_1(M) = 1 \text{ and } \partial M = \emptyset, \\ \pm\text{proj}\,(\tau(M)(t-1)), \text{ if } b_1(M) = 1 \text{ and } \partial M \neq \emptyset. \end{cases}$$

3. The results of this section extend to nonorientable 3-manifolds. In particular, if M is a compact connected homology oriented nonorientable 3-manifold with $b_1(M) \geq 2$ then $\tau(M) \in \mathbb{Z}[H_1(M)]/H_1(M)$.

5. The torsion function T

We adhere to the notation introduced at the beginning of Sect. 4.

5.1. The case $b_1(M) \geq 2$. Let $e_0 \in \text{Eul}(M) = \text{vect}(M)$. By Theorem 4.1, $\tau(M, e_0) = \sum q^{e_0}(h)h$ where h runs over a finite subset of $H = H_1(M)$ and $q^{e_0}(h) \in \mathbb{Z}$. Composing the function $h \mapsto q^{e_0}(h) : H \to \mathbb{Z}$ with the bijection $e \mapsto e_0/e : \text{Eul}(M) \to H$ we obtain a function $T : \text{Eul}(M) \to \mathbb{Z}$ with finite support. By definition, for $e \in \text{Eul}(M)$,

$$T(e) = q^{e_0}(e_0/e) \in \mathbb{Z}. \tag{5.1.a}$$

In fact, T does not depend on the choice of e_0. Indeed, for $e_1 \in \text{Eul}(M)$,

$$\sum_{h \in H} q^{e_1}(h)h = \tau(M, e_1) = (e_1/e_0)\,\tau(M, e_0) = (e_1/e_0) \sum_{h \in H} q^{e_0}(h)h.$$

Therefore $q^{e_0}(h) = q^{e_1}((e_1/e_0)h)$. Substituting $h = e_0/e$, we obtain $q^{e_0}(e_0/e) = q^{e_1}(e_1/e)$. Setting $e_0 = e$ in (5.1.a), we obtain $T(e) = q^e(1)$.

Assume that $\partial M = \emptyset$. By (3.4.a), $q^e(h^{-1}) = q^{e^{-1}}(h)$ for any $e \in \text{Eul}(M), h \in H$. Setting $h = 1$ we obtain $T(e^{-1}) = T(e)$. By the results of Sections 1 and 2, $\mathcal{S}(M) = \text{vect}(M) = \text{Eul}(M)$ so that T is an integer-valued function on $\mathcal{S}(M)$. The results of Menge and Taubes [4] imply an intimate connection between the Seiberg-Witten invariant $SW : \mathcal{S}(M) \to \mathbb{Z}$ and T. We conjecture that $SW = T$.

5.2. The case $b_1(M) = 1$. Fix an element $t \in H$ as in Sect. 4.2. Let us call an element $h \in H$ negative if $h \in t^k \text{Tors}\, H$ with $k < 0$. Denote by Λ the Novikov ring of H consisting of integral series $\sum_{h \in H} q(h)h$ such that $q(h) = 0$ for all but a finite number of negative h. Multiplication in Λ is induced by the group operation in H. It is clear that $1 - t$ is invertible in Λ. Using Theorems 4.2.1 and 4.2.3 we can view the torsion $\tau(M, e)$ with $e \in \text{Eul}(M) = \text{vect}(M)$ as an element $\sum_{h \in H} q^e(h)h$ of Λ. We define a function $T_t : \text{Eul}(M) \to \mathbb{Z}$ as in Sect. 5.1 or by $T_t(e) = q^e(1)$.

It is easy to compute T_t in terms of $\tau_t(M,e) \in \mathbb{Z}[H]$, see Theorems 4.2.1 and 4.2.3. Let $\tau_t(M,e) = \sum_{h\in H} q_t^e(h)h$ with $q_t^e(h) \in \mathbb{Z}$. If $\partial M \neq \emptyset$, then $T_t(e) = q_t^e(1) + 1$. If $\partial M = \emptyset$, then $T_t(e) = q_t^e(1) + K_t(e)/2$.

In distinction to the case $b_1 \geq 2$, the function T_t has an infinite support. It is easy to compute that $T_t = T_{th}$ for $h \in \operatorname{Tors} H$, $T_{t^{-1}} = 1 - T_t$ if $\partial M \neq \emptyset$ and $T_{t^{-1}} = T_t - K_t/2$ if $\partial M = \emptyset$.

If $\partial M = \emptyset$, then $\mathcal{S}(M) = \operatorname{vect}(M) = \operatorname{Eul}(M)$ so that T_t is an integer-valued function on $\mathcal{S}(M)$. Note that both SW and T_t depend on the choice of a generator of $H/\operatorname{Tors} H$.

5.3. The case $b_1(M) = 0$. The constructions of Sect. 5.1 apply word for word with the only difference that here the function T takes values in $\mathbb{Q}$. By Sect. 3.3 and 4.3, $\sum_e T(e) = 0$ and $T(ghe) - T(ge) - T(he) + T(e) = -L(g,h) \pmod{\mathbb{Z}}$ for any $g,h \in H, e \in \operatorname{Eul}(M)$. It would be interesting to extend the SW-invariant to the case $b_1(M) = 0$ and to compare it with T.

5.4. Examples. The function T may happen to be identically zero. For instance, if M is a connected sum of two closed connected oriented 3-manifolds with $b_1 \geq 1$ then $\tau(M) = 0$ and $T = 0$. In some cases the function T can be used to distinguish Spin^c-structures on 3-manifolds up to homeomorphism. Consider for instance a closed connected oriented 3-manifold M with $H_1(M) = \mathbb{Z}/2\mathbb{Z} = (a \,|\, a^2 = 1)$. There are two Spin^c-structures on M, say e and $e' = ae$. We have $\tau(M,e) = k - ka$ and $\tau(M,e') = a(k - ka) = ka - k$ with $k \in \mathbb{Q}$. Theorem 4.3.1 implies that $k \neq 0$. Then $T(e) = k \neq T(e') = -k$. This implies that there is no orientation preserving homeomorphism $M \to M$ transforming e into e'.

5.5. Manifolds with boundary re-examined. Let $\partial M \neq \emptyset$. Following [4], denote by $\mathcal{S}$ the set of Spin^c-structures on M whose first Chern class $c_1 \in H^2(M)$ restricts to zero on every component of ∂M. (This is no constraint when ∂M is connected.) Denote by $\underline{\mathcal{S}}$ the set of pairs $(s,x) \in \mathcal{S} \times H^2(M,\partial M)/\text{Tors}$ such that the cohomology class $c_1(s) (\text{mod Tors}) \in H^2(M)/\text{Tors}$ equals the image of x under the natural homomorphism $H^2(M,\partial M)/\text{Tors} \to H^2(M)/\text{Tors}$. The Seiberg-Witten invariant of M is a function $SW : \underline{\mathcal{S}} \to \mathbb{Z}$. The following lemma suggests a relationships between SW and $T : \operatorname{vect}(M) \to \mathbb{Z}$.

5.5.1. Lemma. *There is a canonical embedding vect$(M) \hookrightarrow \underline{\mathcal{S}}$.*

Proof. By assumption, each component of ∂M is homeomorphic to $S^1 \times S^1$. Therefore it bears a nonsingular tangent vector field whose trajectories are the circles $[pt] \times S^1$. The homotopy class of this vector field is independent of the homeomorphism of the component onto $S^1 \times S^1$. (It is preserved under the Dehn twists along $[pt] \times S^1$ and $S^1 \times [pt]$; cf. also [13], Sect. 9.3.) Denote by v_0 the resulting nonsingular tangent vector field on ∂M.

Let u be a nonsingular tangent vector fields on M directed outwards on ∂M. The constructions of Sections 1.3, 1.4 yield a Spin^c-structure on M. The obstruction to the extension of v_0 to a nonsingular vector field on M transversal to u is an element of $H^2(M, \partial M)$ and we project it into $H^2(M, \partial M)/\text{Tors}$. This gives an embedding $\text{vect}(M) \hookrightarrow \underline{\mathcal{S}}$.

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