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ON THE POINCARÉ POLYNOMIAL OF A COMPLEMENT OF HYPERPLANES

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1. Introduction

In this note we study the Poincaré polynomial

$$\mathrm{Poin}(Y;t) = \sum_{i=1}^{n} t^i \dim_{\mathbb{C}} H^i(Y,\mathbb{C}),$$

where $Y = \mathbb{C}^n - \cup_I A_I$ is the complement of a union of hyperplanes—an *arrangement* of hyperplanes. Two are our main goals. The first objective is to find natural geometric conditions in order for the Poincaré polynomial to *factorize*, that is

$$\mathrm{Poin}(Y;t) = \prod_j P_j(t), \tag{1}$$

where the $P_j(t)$ are polynomials with *positive integer coefficients*. The second goal is to identify the coefficients of the polynomials $P_j(t)$ — hence also the Betti numbers of Y — with natural characteristic numbers.

The Poincaré polynomial plays a prominent role in the study of the topology of complements of hyperplanes (see [7]). A motivation for our work is in fact a theorem of Terao [10: Main Theorem] stating that, for the complement of a central and essential free arrangement, the factorization (1) occurs into linear polynomials. Here an arrangement $\cup_I A_I \subset \mathbb{C}^n$ is said to be *central and essential* when the total intersection $\cap_I A_I = \{0\}$, the origin in $\mathbb{C}^n$. As for the notion of freeness, recall the general definition of the *sheaves of logarithmic p–forms* on a complex manifold X with poles along any hypersurface $D \subset X$. These are the $\mathcal{O}_X$–sheaves $\Omega^p_X(\log D)$ of meromorphic forms ω on X which satisfy the following local property on any $U \subset X$: If f is a local defining function for D on U, both $f\omega$ and $fd\omega$ are holomorphic throughout U. Accordingly, an arrangement $\cup_I A_I \subset \mathbb{C}^n$ is *free* if $\Omega^1_{\mathbb{C}^n}(\log \cup_I A_I)$ is a locally free $\mathcal{O}_{\mathbb{C}^n}$–module. As pointed out by Saito in [8], this property should be related to the vanishing

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of the higher homotopy groups of Y in a manner whose precise extent is yet to be understood.

In order to calculate the cohomology of Y we shall consider any smooth compactification X of Y, birationally isomorphic to $\mathbb{P}^n$ and in which Y is realized as the complement of a hypersurface D with normal crossings. For the sake of brevity, we shall henceforth refer to such $X \xrightarrow{\sigma} \mathbb{P}^n$ as to a *good compactification*; it can be constructed via a composite σ of blow–ups along multiple intersections of the hyperplanes in $\mathbb{P}^n$ obtained by completing at infinity the components A_I of the arrangement. Since D is normal crossing, the sheaves $\Omega_X^p(\log D)$ of logarithmic p–forms are automatically locally free, i.e., holomorphic vector bundles on X. Our interpretation of the factorization property is suggested by the elementary observation that the occurrence of (1) for a hyperplane complement Y might be the manifestation of a product structure—however disguised in the complexity of the arrangement—of some complex computing $H^*(Y, \mathbb{C})$. The obvious model of a factorizable polynomial $\mathrm{Poin}(C^*, t) = \sum t^i \dim H^i(C^*)$ is in fact that of the cohomology of a product complex $C^* = \otimes_j C_j^*$, for which a formula like (1) holds with $P_j(t) = \mathrm{Poin}(C_j^*, t)$. For an affine algebraic manifold like Y, (1) would be a trivial consequence of Künneth formula if, for example, Y were topologically a product of $\times_j Y_j$ of affine manifolds $Y_j \cong X_j - D_j$. Theorem 1 below shows that a natural weakening of this factorization scheme is in fact provided by the notion of decomposability of $\Omega_X^1(\log D)$ into holomorphic subbundles $\mathcal{E}_j$.

The following theorem applies directly to the complement of an *essential* arrangement, that is a union of affine hyperplanes whose lowest dimensional multiple intersections consist of isolated points. Note that this is in effect no loss of generality, for the complement Y of any non–empty arrangement in $\mathbb{C}^n$ is always isomorphic to the cartesian product $Y' \times \mathbb{C}^{n-m}$, where Y' is the complement of an essential arrangement in $\mathbb{C}^m$ (for some $m \leq n$).

Theorem 1.1. *Let X be a good compactification of the complement Y of an essential arrangement of hyperplanes in $\mathbb{C}^n$. Let D denote the normal crossing hypersurface $X - Y$. If $\Omega_X^1(\log D)$ is isomorphic to the direct sum $\oplus_j \mathcal{E}_j$ of holomorphic vector bundles $\mathcal{E}_j$, then*

(i) *For all j, the higher cohomology groups of the exterior powers of $\mathcal{E}_j$ vanish, i.e., $H^i(X, \wedge^p \mathcal{E}_j) = 0$ for $i > 0$ and all p;*

(ii) *The complex cohomology of Y is given by the global sections of the exterior powers of the $\mathcal{E}_j$'s,*

$$H^p(Y, \mathbb{C}) \cong \bigoplus_{\sum p_j = p} \bigotimes_j \Gamma(X, \wedge^{p_j} \mathcal{E}_j)$$

for all p.

(*Convention: $\wedge^0 \mathcal{E}_j \equiv \mathcal{O}_X$, the trivial line bundle*).

Notice that, as an immediate consequence of (i) together with Hirzebruch–Riemann–Roch theorem, one has $\dim \Gamma(\wedge^i \mathcal{E}_j) = \chi(\wedge^i \mathcal{E}_j) = \int_X \mathrm{todd}(X)\, \mathrm{ch}(\wedge^i \mathcal{E}_j)$

for all i and j. It is now a matter of straighforward algebra to verify that the following corollary holds true.

Corollary 1.2. *Under the same assumptions as in Theorem* 1.1*, the Poincaré polynomial of* Y *factorizes,*

$$\mathrm{Poin}(Y;t) = \prod_j P_j(t), \qquad P_j(t) = 1 + \beta_{j,1} t + \cdots + \beta_{j,\mathrm{rk}(\mathcal{E}_j)} t^{\mathrm{rk}(\mathcal{E}_j)},$$

where $\mathrm{rk}(\mathcal{E}_j)$ *denotes the rank of* $\mathcal{E}_j$ *and the various coefficients are given by the Riemann–Roch formula*

$$\beta_{j,i} = \chi\big(\wedge^i \mathcal{E}_j\big) = \int_X \mathrm{todd}(X)\,\mathrm{ch}(\wedge^i \mathcal{E}_j).$$

In order to illustrate the content of the theorem and of its corollary, let us consider the following two limiting situations.

Example 1.3. When $\Omega^1_X(\log D)$ does not necessarily decompose, Theorem 1.1 states that $H^i\big(X, \Omega^1_X(\log D)\big) = 0$ for $i > 0$ and that $H^*(Y, \mathbb{C}) \cong \Gamma\big(X, \Omega^*_X(\log D)\big)$, a result of Esnault, Schechtman and Viehweg [4]. Corollary 1.2 gives the Betti numbers of Y in terms of the Chern classes of $\Omega^1_X(\log D)$,

$$\mathrm{Poin}(Y;t) = \sum_{i=1}^{n} t^i \int_X \mathrm{todd}(X)\,\mathrm{ch}\big(\Omega^i_X(\log D)\big).$$

Example 1.4. If the splitting $\Omega^1_X(\log D) \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ is into *line* bundles $\mathcal{L}_j$, then one concludes that all higher cohomology groups of the $\mathcal{L}_j$ vanish, and

$$H^p(Y, \mathbb{C}) \cong \bigoplus_{1 \le j_1 < \cdots < j_p \le n} \Gamma(X, \mathcal{L}_{j_1}) \otimes \cdots \otimes \Gamma(X, \mathcal{L}_{j_p})$$

for all p. In this case the Poincaré polynomial of Y factorizes into polynomials of degree 1,

$$\mathrm{Poin}(Y;t) = \prod_{j=1}^{n} (1 + \beta_j\, t),$$

where the numbers $\beta_1, \ldots, \beta_n$ are

$$\beta_j = \chi(\mathcal{L}_j) = \int_X \mathrm{todd}(X)\,\mathrm{ch}\big(\mathcal{L}_j\big).$$

The factorization theorem of [9][10] thus follows from the observation that the splitting property of $\Omega^1_X(\log D)$ into line bundles is verified whenever Y is the complement of a central and essential free arrangement. We write $\mathcal{O}_{\mathbb{P}^n}(1)$ for the hyperplane line bundle on $\mathbb{P}^n$, $\mathcal{O}_{\mathbb{P}^n}(m) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$ for the m–th power. Let also the integers $\alpha_1, \ldots, \alpha_n \ge 1$ denote the "exponents" of the free arrangement (see [7: Definition 4.25] and Section 5 below). The sharper statement is as follows:

Theorem 1.5. *Let Y be the complement of a central and essential free arrangement. Then there is a good compactification $X \xrightarrow{\sigma} \mathbb{P}^n$ of Y so that $\Omega^1_X(\log D)$ splits into line bundles. Explicitly, let $E_1, \ldots, E_\nu$ be the exceptional components of D. Then there are non–negative integers $\mu_{a,i}$ (see Section 5) so that*

$$\Omega^1_X(\log D) \cong \bigoplus_{j=1}^{n} \underbrace{\left(\sigma^* \mathcal{O}_{\mathbb{P}^n}(\alpha_j - 1) \otimes \left(-\sum_{a=1}^{\nu} \mu_{a,j} E_a\right)\right)}_{\mathcal{L}_j}.$$

Corollary 1.6. *The cohomology of the complement Y of a central and essential free arrangement decomposes, $H^p(Y, \mathbb{C}) \cong \oplus_{1 \leq j_1 < \cdots < j_p \leq n} \Gamma(X, \mathcal{L}_{j_1}) \otimes \cdots \otimes \Gamma(X, \mathcal{L}_{j_p})$, with $\mathcal{L}_j$ as in Theorem 1.5. Thus, in particular,* $\mathrm{Poin}(Y; t) = \prod_{j=1}^{n} (1 + \beta_j\, t)$ *with $\beta_j = \chi\left(\sigma^* \mathcal{O}_{\mathbb{P}^n}(\alpha_j - 1) \otimes (-\sum_{a=1}^{\nu} \mu_{a,j} E_a)\right)$.*

Comparing with [10], one *deduces* that in effect $\beta_j = \alpha_j$ for all j. It should be possible, applying the analysis of Sections 4 and 5 below to a concrete resolution σ, to give a direct general proof of these equalities. In Section 6 we carry out such computation for a general central and essential arrangement in $\mathbb{C}^2$.

Section 2 is a brief summary of the relevant properties of logarithmic forms. Section 3 contains our proof of Theorem 1.1. In Section 4 we study the behaviour of logarithmic forms under blow–ups. The splitting theorem of Section 5 is due to H. Terao.

2. The complex of logarithmic forms

Let D be a normal crossing hypersurface in a smooth algebraic variety X ($\dim_{\mathbb{C}} X = n$) so that the complement $X - D$ is affine. On a sufficiently small open set $U \subset X$ of a point where precisely m ($m = 0, \ldots, n$) irreducible components of D intersect one can choose local coordinates $(z_1, \ldots, z_n)$ so that a local defining equation for $D \cap U$ is given by $f = z_1 \cdots z_m = 0$. It directly follows from the definition that

(i) $\Omega^1_X(\log D)$ is the sheaf of $\mathcal{O}_X$–modules generated on U by $\left\{\dfrac{dz_1}{z_1}, \ldots, \dfrac{dz_m}{z_m}, dz_{m+1}, \ldots, dz_n\right\}$;

(ii) For $p \geq 1$, $\Omega^p_X(\log D) = \bigwedge^p \Omega^1_X(\log D)$;

(iii) When provided with the holomorphic de Rham differential d, the $\Omega^p_X(\log D)$ form the differential complex of sheaves

$$0 \to \Omega^0_X(\log D) \xrightarrow{d} \Omega^1_X(\log D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X(\log D) \to 0,$$

where we put $\Omega^0_X(\log D) = \mathcal{O}_X$.

One should note that the sheaves $\Omega^p_X(\log D)$ are locally free and are generated by closed differential forms.

Upon introducing, as usual, complexes of Čech cochains $C^*\big(X, \Omega^p_X(\log D)\big)$ with coboundary δ, and in view of the anticommutativity of d with δ, one easily verifies that d induces a well–defined differential d_1 on Čech cohomology groups

$$0 \to H^q\big(X, \Omega^0_X(\log D)\big) \xrightarrow{d_1} H^q\big(X, \Omega^1_X(\log D)\big) \xrightarrow{d_1} \cdots \xrightarrow{d_1} H^q\big(X, \Omega^n_X(\log D)\big) \to 0$$

for every q. It is a well–known result of [3] that these complexes, which appear as the E_1–term in the Hodge–Deligne spectral sequence

$$E_1^{pq} = H^q\big(X, \Omega^p_X(\log D)\big) \quad \Longrightarrow \quad H^{p+q}(X - D, \mathbb{C}),$$

are all *trivial*, i.e., $d_1 \equiv 0$. Hence the spectral sequence degenerates at E_1 and the complex cohomology of Y is given by

$$H^i(X - D, \mathbb{C}) = \bigoplus_{p+q=i} H^q\big(X, \Omega^p_X(\log D)\big). \tag{2}$$

However—as observed by Esnault, Schechtman and Viehweg—the case of present interest affords the following remarkable simplification. (For related work, see also [2]).

[4: Lemma on page 558]**.** *Let X be a good compactification of the complement Y of an arrangement of hyperplanes in $\mathbb{P}^n$; $D = X - Y$. Then one has the cohomology vanishing*

$$H^q\big(X, \Omega^p_X(\log D)\big) = 0 \quad \textit{for } q > 0 \textit{ and all } p,$$

so that $H^i(Y, \mathbb{C}) = \Gamma\big(X, \Omega^i_X(\log D)\big)$ for $i = 0, \ldots, n$.

We briefly recall the simple argument, especially to the purpose of pointing out a feature (Remark 2.1) which will be required in the next section. Let Y be the complement in $\mathbb{C}^n$ of a union of affine hyperplanes A_I defined as the zero loci of some linear holomorphic functions f_I $(I = 1, \ldots, N)$. In view of a result of Brieskorn [1] and of Orlik and Solomon [6], the cohomology algebra $H^*(Y, \mathbb{C})$ is identified with a quotient of the exterior algebra E (over $\mathbb{C}$) generated by the deRham classes $\big\{1, \big[\frac{1}{2\pi i} d \log f_I\big]\big\}_{I=1,\ldots,N}$. (In fact, the same identification holds true also over $\mathbb{Z}$). One can easily verify that every p–fold wedge $d \log f_{I_1} \wedge \cdots \wedge d \log f_{I_p}$ is a section of Ω^p_Y on the open subset $Y \subset X$ extending to a differential form on X with logarithmic poles along D, i.e., to a section—*a fortiori* global—of $\Omega^p_X(\log D)$. It follows that the global sections $\Gamma\big(X, \Omega^p_X(\log D)\big)$ generate $H^p(Y, \mathbb{C})$; thus, in view of the decomposition (2), that necessarily $H^q\big(X, \Omega^p_X(\log D)\big) = 0$ for $q > 0$ and $\Gamma\big(X, \Omega^p_X(\log D)\big) = H^p(Y, \mathbb{C})$.

Remark 2.1. Clearly the same argument also shows that the natural map

$$\wedge^p H^1(Y, \mathbb{C}) = \wedge^p \Gamma\big(X, \Omega^1_X(\log D)\big) \xrightarrow{\pi} \Gamma\big(X, \Omega^p_X(\log D)\big) = H^p(Y, \mathbb{C})$$

is surjective for every $p \geq 1$. (The kernel of π is generated by the ideal of relations in $E = \wedge^* \Gamma\big(X, \Omega^1_X(\log D)\big)$).

3. Splitting and factorization

Throughout this section we shall be working under the hypotheses of Theorem 1.1. By the splitting assumption $\Omega^1_X(\log D) \cong \oplus_{j=1}^k \mathcal{E}_j$, where of course $\sum_{j=1}^k \mathrm{rk}(\mathcal{E}_j) = n$. One has

$$\begin{aligned}\Omega^p_X(\log D) = \wedge^p \Omega^1_X(\log D) &\cong \bigoplus_{1\le j_1\le\cdots\le j_p\le k} \mathcal{E}_{j_1}\wedge\cdots\wedge\mathcal{E}_{j_p} \\ &\cong \bigoplus_{\sum p_j = p} \bigotimes_{j=1}^k \wedge^{p_j}\mathcal{E}_j\end{aligned}$$

for all p. In view of Esnault–Schechtman–Viehweg vanishing, then all wedge products of vector bundles $\mathcal{E}_j$ must have vanishing higher cohomologies,

$$H^q\big(X, \otimes_{j=1}^k \wedge^{p_j} \mathcal{E}_j\big) = 0 \quad \text{for } q>0 \text{ and all } p_1,\ldots,p_k,$$

and

$$\begin{aligned}H^p(Y,\mathbb{C}) &\cong \bigoplus_{1\le j_1\le\cdots\le j_p\le k} \Gamma\big(X, \mathcal{E}_{j_1}\wedge\cdots\wedge\mathcal{E}_{j_p}\big) \\ &\cong \bigoplus_{\sum p_j=p} \Gamma\big(X, \otimes_{j=1}^k \wedge^{p_j} \mathcal{E}_j\big).\end{aligned}$$

Consider now the natural maps

$$\bigotimes_{j=1}^k \Gamma(X, \wedge^{p_j}\mathcal{E}_j) \xrightarrow{\psi} \Gamma\big(X, \otimes_{j=1}^k \wedge^{p_j} \mathcal{E}_j\big) \tag{3}$$

sending $\gamma = s_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} s_k \in \bigotimes_{j=1}^k \Gamma(X,\wedge^{p_j}\mathcal{E}_j) \mapsto \psi(\gamma)\in\Gamma\big(X, \otimes_{j=1}^k \wedge^{p_j}\mathcal{E}_j\big)$, the section such that $\psi(\gamma)_x = (s_1)_x \otimes_{\mathcal{O}_x}\cdots\otimes_{\mathcal{O}_x}(s_k)_x$ at every $x\in X$. In order to prove Theorem 1.1 one must show that, for all $p_1,\ldots,p_k$, the maps ψ are isomorphisms.

Surjectivity of (3) is a consequence of Remark 2.1. For, in view of the decomposition of the $\Gamma\big(X,\Omega^1_X(\log D)\big) \cong \bigoplus_{j=1}^k \Gamma(X,\mathcal{E}_j)$ of log 1–forms, one obtains $\wedge^p\Gamma\big(X,\Omega^1_X(\log D)\big) \cong \bigoplus_{\sum p_j=p}\bigotimes_{j=1}^k \wedge^{p_j}\Gamma(X,\mathcal{E}_j)$, whereas $\Gamma\big(X,\Omega^p_X(\log D)\big)\cong \bigoplus_{\sum p_j=p}\Gamma\big(X,\otimes_{j=1}^k\wedge^{p_j}\mathcal{E}_j\big)$. By Remark 2.1, then, the various maps

$$\bigotimes_{j=1}^k \wedge^{p_j}\Gamma(X,\mathcal{E}_j) \xrightarrow{\pi} \Gamma\big(X, \otimes_{j=1}^k \wedge^{p_j}\mathcal{E}_j\big)$$

are surjective. Note that π factors through ψ as shown in the following commutative diagram

$$\begin{array}{ccc} \bigotimes_{j=1}^{k} \wedge^{p_j}\Gamma(X,\mathcal{E}_j) & \xrightarrow{\ \pi\ } & \Gamma\big(X, \otimes_{j=1}^{k} \wedge^{p_j} \mathcal{E}_j\big) \\ {\scriptstyle \pi'}\downarrow & & \| \\ \bigotimes_{j=1}^{k} \Gamma(X,\wedge^{p_j}\mathcal{E}_j) & \xrightarrow{\ \psi\ } & \Gamma\big(X, \otimes_{j=1}^{k} \wedge^{p_j} \mathcal{E}_j\big). \end{array}$$

Since $\pi = \psi\pi'$ is surjective, so must be ψ.

In order to extablish the injectivity of ψ in (3) we shall need a couple of preparatory lemmas. Recall that a holomorphic vector bundle $\mathcal{E}_X$ on X is said to be generated by global sections if the sheaf map $\Gamma(X,\mathcal{E}_X)\otimes_{\mathbb{C}}\mathcal{O}_X \xrightarrow{\varphi} \mathcal{E}_X$, defined at every point $x \in X$ by $\varphi(s\otimes f)_x = f_x s_x$, is surjective. For an arrangement $\cup_I H_I = \{\prod_I f_I = 0\} \subset \mathbb{C}^n$ the pull–backs $\sigma^* d\log f_I$ to $X \xrightarrow{\sigma} \mathbb{P}^n$ furnish a basis of global sections of $\Omega^1_X(\log D)$. One easily sees that at every $x \in X$ there are n of them such that $\sigma^*(d\log f_{I_1})_x \wedge \cdots \wedge \sigma^*(d\log f_{I_n})_x \neq 0$ if and only if arrangement is essential:

Lemma 3.1. *Let X be a good compactification of the complement Y of an arrangement of hyperplanes in $\mathbb{C}^n$; $D = X - Y$ Then $\Omega^1_X(\log D)$ is generated by global sections if and only if the arrangement is essential.*

Under the assumptions of Theorem 1.1, then $\Omega^1_X(\log D) \cong \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ is generated by global sections. This implies that also the maps

$$\Gamma(X,\mathcal{E}_j)\otimes_{\mathbb{C}}\mathcal{O}_X \xrightarrow{\varphi} \mathcal{E}_j, \qquad \varphi(s\otimes f)_x = f_x s_x$$

are surjective for all j, i.e., each $\mathcal{E}_j$ is globally generated. Observe that, since the global sections in $\Gamma(X,\mathcal{E}_j)$ are d–closed by Deligne degeneration, *the bundles $\mathcal{E}_j$ are generated by closed 1–forms*. As a wedge of local frames consisting of closed 1-forms is also a closed form, the latter conclusion applies to any wedge product of the $\mathcal{E}_j$'s. This observation has the following consequence.

Lemma 3.2. *On any open set $U \cong \mathbb{C}^n$ in X, not intersecting D, there are local analytic coordinates $\big(y_j^{i_j}\big)_{j=1,\dots,k}^{i_j=1,\dots,\mathrm{rk}(\mathcal{E}_j)}$ such that any global section $s \in \Gamma(X,\mathcal{E}_j)$ takes the local form*

$$s|_U = \sum_{i=1}^{\mathrm{rk}(\mathcal{E}_j)} f^i\, dy_j^i,$$

where the $f^i \equiv f^i(y_j^1,\dots,y_j^{\mathrm{rk}(\mathcal{E}_j)})$ are functions of $y_j^1,\dots,y_j^{\mathrm{rk}(\mathcal{E}_j)}$ only. Analogously, let J be any subset of the index set $\{1,\dots,k\}$. Then, any global section $s \in \Gamma(X,\otimes_{j\in J}\wedge^{p_j}\mathcal{E}_j)$ takes the local form

$$s|_U = \sum_{\{i\}} f^{\{i\}} \bigwedge_{j\in J} dy_j^{i_1}\wedge\cdots\wedge dy_j^{i_{p_j}},$$

where the $f^{\{i\}}$ *depend on the variables* $\big(y_j^{i_j}\big)_{j\in J}^{i_j=1,\dots,\,\mathrm{rk}(\mathcal{E}_j)}$ *only.*

Proof. Let $\big\{\sigma_j^{i_j}\big\}_{j=1,\dots,\,k}^{i_j=1,\dots,\,\mathrm{rk}(\mathcal{E}_j)}$ be a local frame of $\Omega^1_X(\log D)$ on an open set $U\subset X$ so that, for each j, $(\sigma_j^1,\dots,\sigma_j^{\mathrm{rk}(\mathcal{E}_j)})$ is a local frame of $\mathcal{E}_j$. By the previous observation, the $\sigma_j^{i_j}$ can be chosen to be the restrictions to U of suitable global sections of $\Omega^1_X(\log D)$, in which case they are closed 1–forms. If $U\cap D=\emptyset$, then the $\sigma_j^{i_j}$ are holomorphic and, by the Poincaré lemma, $\sigma_j^{i_j}=dy_j^{i_j}$ for some holomorphic functions $y_j^{i_j}$. Since the Jacobian determinant $\wedge_{j,i_j}\sigma_j^{i_j}$ is nowhere zero on U, the n functions $\big(y_j^{i_j}\big)_{j=1,\dots,\,k}^{i_j=1,\dots,\,\mathrm{rk}(\mathcal{E}_j)}$ define local analytic coordinates on U. The restriction to U of any global section $s\in\Gamma(X,\mathcal{E}_j)$ is written as $s=\sum_{i=1}^{\mathrm{rk}(\mathcal{E}_j)} f^i\,dy_j^i$. But both s and the dy_j^i are d–closed, $0=ds=\sum_{i=1}^{\mathrm{rk}(\mathcal{E}_j)} df^i\wedge dy_j^i$, which implies—by Cartan lemma—that $df^i=\sum_i g^i\,dy_j^i\ \in\mathcal{E}_j$, i.e., $\dfrac{\partial f^i}{\partial y_{j'}^{i'}}=0$ whenever $j'\neq j$. In other words, the f^i must be functions of $y_j^1,\dots,y_j^{\mathrm{rk}(\mathcal{E}_j)}$ only. Finally, a quite analogous argument applied to the global sections of $\otimes_{j\in J}\wedge^{p_j}\mathcal{E}_j$ yields the second part of the statement.

Injectivity of (3) results from an inductive application of our next proposition.

Lemma 3.3. *For any sequence* $p_1,\dots,p_k$*, the natural map of global sections*

$$\phi\colon \Gamma(X,\wedge^{p_i}\mathcal{E}_i)\otimes_{\mathbb{C}}\Gamma\big(X,\otimes_{j\neq i}\wedge^{p_j}\mathcal{E}_j\big)\to\Gamma\big(X,\otimes_j\wedge^{p_j}\mathcal{E}_j\big)$$

induced from $\Gamma(X,\wedge^{p_i}\mathcal{E}_i)\otimes_{\mathbb{C}}\big(\otimes_{j\neq i}\wedge^{p_j}\mathcal{E}_j\big)\xrightarrow{\varphi}\otimes_j\wedge^{p_j}\mathcal{E}_j$*, with* $\varphi(s\otimes_{\mathbb{C}}s')_x=s_x\otimes_{\mathcal{O}_x}s'_x$*, is injective.*

Proof. In order to simplify notations, in the course of this proof we shall usually write $\mathcal{E}$ for $\wedge^{p_i}\mathcal{E}_i$ and $\mathcal{E}'$ for $\otimes_{j\neq i}\wedge^{p_j}\mathcal{E}_j$. Let s and s' be elements of $\Gamma(X,\mathcal{E})$ and $\Gamma(X,\mathcal{E}')$ respectively. In terms of local frames $\{\sigma_x^a\}$, $\{\sigma_x'^b\}$ of $\mathcal{E}$, $\mathcal{E}'$ at $x\in X$, they have the local expressions $s_x=\sum_a f_x^a\sigma_x^a$ and $s'_x=\sum_b f_x'^b\sigma_x'^b$ for some $f_x^a,f_x'^b\in\mathcal{O}_x$. As observed above, we can and will choose the local frames to be given by closed forms. The image of a general element $\gamma=\sum_t s_t\otimes_{\mathbb{C}}s'_t\in\Gamma(X,\mathcal{E})\otimes_{\mathbb{C}}\Gamma(X,\mathcal{E}')$ is the section $\phi(\gamma)\in\Gamma(X,\mathcal{E}\otimes\mathcal{E}')$ such that $\phi(\gamma)_x=\sum_t\sum_{a,b}f_{t,x}^a\sigma_x^a\otimes_{\mathcal{O}_x}f_{t,x}'^b\sigma_x'^b$. The kernel of ϕ is thus given by

$$(\ker\phi)_x=\left\{\sum_t\sum_{a,b}f_{t,x}^a\sigma_x^a\otimes_{\mathbb{C}}f_{t,x}'^b\sigma_x'^b\neq 0\mid\sum_t f_{t,x}^a f_{t,x}'^b=0\text{ for all }a,b\right\}.$$

By Lemma 3.2, around any point x disjoint from D there is an open neighborhood U with local analytic coordinates $\big(y_j^{i_j}\big)_{j=1,\dots,\,k}^{i_j=1,\dots,\,\mathrm{rk}(\mathcal{E}_j)}$ centered at x so that the functions $f_{t,x}^a$ depend on the $\mathrm{rk}(\mathcal{E}_i)$ coordinates y_i only and the functions $f_{t,x}'^b$ on the $\sum_{j\neq i}\mathrm{rk}(\mathcal{E}_j)$ coordinates y_j with $j\neq i$. Denoting by $\{z\}$ and $\{z'\}$ the two respective collections of coordinates, the $f^a_{t,x}$ are $f'^b_{t,x}$ are, respectively, identified with elements of the local rings of convergent power series $\mathbb{C}\{z\}$ and $\mathbb{C}\{z'\}$. But then the condition that, for all a, b, $\sum_t f^a_{t,x}(\{z\}) f'^b_{t,x}(\{z'\}) = 0$ as an element of $\mathcal{O}_x = \mathbb{C}\{y\}$, is equivalent to the vanishing of $\sum_t \sum_{a,b} f^a_{t,x}\sigma^a_x \otimes_{\mathbb{C}} f'^b_{t,x}\sigma'^b_x$. This argument being valid at all points x of an open set $U \subset X$, one concludes that a section in $\ker\phi$ would have to identically vanish on U, hence also on all of X. So ϕ is injective.

Theorem 1.1 has now been proved.

4. Blow–up of logarithmic forms

Let X be a smooth algebraic variety of complex dimension n. For *any* hypersurface $D = \cup_I D_I$ in X, the sheaves $\Omega^p_X(\log D)$ of logarithmic forms, as defined in the Introduction, are coherent and their direct sum is closed both under wedge multiplication and under exterior differentiation. (For the former two properties see [8: (1.3)]; the last fact is clear by definition). They are generally not locally free. The following result reduces the question as to when they are locally free to a single condition. Let us denote by $\mathcal{D}$ the *divisor* $\sum_I D_I$ given by the irreducible components of D each counted with multiplicity 1; we will implicitly identify $\mathcal{D}$ with the associated line bundle on X. Note that, by definition, $\Omega^n_X(\log D)$ is always the line bundle $\Omega^n_X \otimes \mathcal{D}$.

Saito criterion [8:Theorem 1.8]**.** $\Omega^1_X(\log D)$ *is locally free if and only if the top exterior power* $\wedge^n\Omega^1_X(\log D) = \Omega^n_X \otimes \mathcal{D}$*. In this case one has* $\Omega^p_X(\log D) = \wedge^p\Omega^1_X(\log D)$*, all locally free.*

In order to compute with logarithmic forms we shall need the following observation.

Remark 4.1: Local presentation. Consider a sufficiently small open $U \subset X$ in which the hypersurface $D \cap U = \{f = 0\}$ is reducible, i.e., $f = gh$, where $\{g = 0\}$ and $\{h = 0\}$ may well be further reducible. From the definition, a p–form ω on U is logarithmic with poles along $D \cap U$ if and only if $gh\omega$ and $d(gh) \wedge \omega$ are holomorphic. Putting $\omega' = g\omega$, one easily verifies that these are equivalent conditions to ω' being a logarithmic p–form with poles along $\{h = 0\}$ and such that both $h\,(d\log g \wedge \omega') \in \Omega^{p+1}_U$ and $dh \wedge (d\log g \wedge \omega') \in \Omega^{p+2}_U$. Let hence $\Omega^p_U\big(\log\{h = 0\}\big)$ denote the logarithmic p–forms on U with poles along $\{h = 0\}$.

Locally over U*,* $\Omega^p_X(\log D)(U)$ *is given by the forms* $\omega = \frac{1}{g}\xi$*, with* $\xi \in \Omega^p_U\big(\log\{h = 0\}\big)$ *such that* $\dfrac{dg}{g} \wedge \xi \in \Omega^{p+1}_U\big(\log\{h = 0\}\big)$*.*

Though this local presentation does not make it evident that $\Omega^p_X(\log D) \wedge \Omega^q_X(\log D) \subset \Omega^{p+q}_X(\log D)$, see [8].

In this section we consider the blow–up $\hat{X} \xrightarrow{\sigma} X$ of X along a submanifold $V \subset D$. Thus σ leaves the complement $X - D \subset X - V$ unchanged; the inverse image $\hat{D} = \sigma^{-1}(D) = \tilde{D} \cup E$ is the union of the proper transform $\tilde{D} = \overline{\sigma^{-1}(D-V)}$ of D and the exceptional hypersurface $E = \sigma^{-1}(V)$. Note that, as a rule, the pull–back of a form $\omega \in \Omega^p_X(\log D)$ will no longer be logarithmic, in the sense that $\sigma^*\omega \notin \Omega^p_{\hat{X}}(\log \hat{D})$. Proposition 4.2 and Lemma 4.4 below measure to which precise extent the pull–back of a logarithmic form fails to remain logarithmic. If $\mathrm{mult}_V D$ denotes the multipilcity of D along V and $\mathrm{codim}\, V$ is the codimension of V in X , let us introduce the integer

$$\mu = \mathrm{mult}_V D - \mathrm{codim}\, V.$$

One can explicitly compute the pull–back of the logarithmic forms of top degree:

Proposition 4.2. $\sigma^*\Omega^n_X(\log D) = \Omega^n_{\hat{X}}(\log \hat{D}) \otimes \mu E.$

Proof. Since $\Omega^n_X(\log D) = \Omega^n_X \otimes \mathcal{D}$, one has $\sigma^*\Omega^n_X(\log D) = \sigma^*\Omega^n_X \otimes \sigma^*\mathcal{D}$. A simple local computation (see, e.g., [5: p. 605, 608]) gives, on the one hand

$$\sigma^*\Omega^n_X = \Omega^n_{\hat{X}} \otimes (1 - \mathrm{codim}\, V)\, E.$$

On the other hand, for $\mathcal{D} = \sum_I D_I$, let $\hat{\mathcal{D}} = \sum_I \tilde{D}_I + E$ be the divisor on $\hat{X}$ given by the components of $\hat{D} = \tilde{D} \cup E = \Big(\cup_I \tilde{D}_I\Big) \cup E$. One has

$$\sigma^*\mathcal{D} = \sum_I \tilde{D}_I + (\mathrm{mult}_V D)\, E = \hat{\mathcal{D}} + (\mathrm{mult}_V D - 1)\, E,$$

and the result follows from the fact that $\Omega^n_{\hat{X}}(\log \hat{D}) = \Omega^n_{\hat{X}} \otimes \hat{\mathcal{D}}$.

It will suffice for the problem at hand to work—in the remainder of this section—under the

Assumption 4.3. *V is a multiple intersection of components of D, a number* $\mathrm{codim}\, V$ *of which are smooth along V and intersect there like hyperplanes in $\mathbb{C}^n$.*

Note that in this case $\mu = \mathrm{mult}_V D - \mathrm{codim}\, V \ \geq 0$, where $\mu = 0$ precisely when D has normal crossings along V. Put $l = \mathrm{codim}\, V$ and let $U \subset X$ be a sufficiently small neighborhood of a point on V. By Assumption 4.3 there are on U local coordinates $(z_1, \ldots, z_n)$ so that $V \cap U = \{z_1 = \cdots = z_l = 0\}$, a local defining function for D on U being given by

$$f = z_1 \cdots z_l\, g,$$

with g analytic. Let $\Omega^p_U\big(\log\{z_1 \cdots z_l = 0\}\big)$ denote the logarithmic p–forms on U with poles along the normal crossing hypersurface $\{z_1 \cdots z_l = 0\}$, i.e., the *free* module generated over $\mathcal{O}_X(U)$ by the p–fold wedge products of the elements $\frac{dz_1}{z_1}, \ldots, \frac{dz_l}{z_l}, dz_{l+1}, \ldots, dz_n$. In view of Remark 4.1, $\Omega^p_X(\log D)(U)$ is the subsheaf of $\frac{1}{g}\,\Omega^p_U\big(\log\{z_1\cdots z_l = 0\}\big)$ given by those forms $\omega = \frac{1}{g}\,\xi$ such that

$$\frac{dg}{g} \wedge \xi \in \Omega^{p+1}_U\big(\log\{z_1\cdots z_l = 0\}\big).$$

For $\omega \in \Omega^p_X(\log D)$, let $\frac{1}{g}\,\xi$ be its local form on U. We say that ω *vanishes to order k along V* if so does ξ; that is if ξ is a linear combination of p–fold wedge products of $\frac{dz_1}{z_1}, \ldots, \frac{dz_l}{z_l}, dz_{l+1}, \ldots, dz_n$ with coefficients in $(\mathfrak{m}^k_V \mathcal{O}_X)(U)$, the holomorphic functions on U which vanish to order at least k along V.

Lemma 4.4. *If $\omega \in \Omega^p_X(\log D)$ vanishes to order k along V, then its pull–back $\sigma^*\omega \in \Omega^p_{\hat{X}}(\log \hat{D}) \otimes (\mu - k)E$.*

Proof. Since σ is an isomorphism outside $\sigma^{-1}(V)$, it is sufficient to analyze $\sigma^*\omega$ on the preimage $\hat{U}$ of U. In local coordinates on an open set $\hat{U}_1 \subset \hat{U}$, the map $\hat{U} \xrightarrow{\sigma} U$ is given by $(w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_n)$ with $z_1 = w_1$, $z_i = w_1 w_i$ for $i = 2, \ldots, l$ and $z_i = w_i$ for $i = l+1, \ldots, n$. The pull–back of the local defining function f of D is $\sigma^* f = w_1^{\mathrm{mult}_V D - 1}\,\hat{f}$ on $\hat{U}_1$, where $\mathrm{mult}_V D = l + \mu$ and

$$\hat{f} = w_1 w_2 \cdots w_l\,\hat{g}, \quad \text{with } \hat{g} = w_1^{-\mu}\,\sigma^* g,$$

is a local defining function of $\hat{D}$ on $\hat{U}_1$. Here $\{w_1 = 0\}$ is a local defining equation for the exceptional divisor E on $\hat{U}_1$. Locally on U, $\omega = \frac{1}{g}\,\xi$ and its pull–back is

$$\sigma^*\omega = \frac{1}{\sigma^* g}\,\sigma^*\xi = \frac{1}{w_1^{\mu}\,\hat{g}}\,\sigma^*\xi.$$

Suppose now ξ vanishes to order k along V. Since $\sigma^* d\log z_1 = d\log w_1$, $\sigma^* d\log z_i = d\log w_1 + d\log w_i$ for $i = 1, \ldots, l$ and $\sigma^* dz_i = dw_i$ for $i = l+1, \ldots, n$, one immediately sees that $\hat{\xi} = w_1^{-k}\,\sigma^*\xi \in \Omega^p_{\hat{U}_1}\big(\log\{w_1 w_2 \cdots w_l = 0\}\big)$. But then $w_1^{\mu-k}\,\sigma^*\omega = \frac{\hat{\xi}}{\hat{g}}$ is in $\Omega^p_{\hat{X}}(\log \hat{D})(\hat{U}_1)$, for

$$\begin{aligned}
\frac{d\hat{g}}{\hat{g}} \wedge \hat{\xi} &= -\mu \frac{dw_1}{w_1} \wedge \hat{\xi} + \frac{d\sigma^* g}{\sigma^* g} \wedge \hat{\xi} \\
&= -\mu \frac{dw_1}{w_1} \wedge \hat{\xi} + w_1^{-k}\,\sigma^*\left(\frac{dg}{g} \wedge \xi\right) \;\in\; \Omega^{p+1}_{\hat{U}_1}\big(\log\{w_1 w_2 \cdots w_l = 0\}\big).
\end{aligned}$$

The lemma has been proven.

By Lemma 4.4, for any form $\omega \in \Omega^p_X(\log D)$ there is thus a minimal integer μ_ω with $0 \le \mu_\omega \le \mu$ so that $\sigma^*\omega \in \Omega^p_{\hat{X}}(\log \hat{D}) \otimes \mu_\omega E$. Assume now that $\Omega^1_X(\log D)$ is locally free on U. If $\omega_1, \ldots, \omega_n$ is a local frame on U, let $\mu_i = \mu_{\omega_i}$ for $i = 1, \ldots, n$. The wedge $\omega_1 \wedge \cdots \wedge \omega_n$ pulls back under σ to an element of $\wedge_{i=1}^n \left(\Omega^1_{\hat{X}}(\log \hat{D}) \otimes \mu_i E\right) = \left(\wedge^n \Omega^1_{\hat{X}}(\log \hat{D})\right) \otimes \left(\sum_i \mu_i E\right)$ generating $\sigma^*\Omega^n_X(\log D) = \Omega^n_{\hat{X}}(\log \hat{D}) \otimes \mu E$ over $\sigma^{-1}(U)$. (Note that since $\wedge^n \Omega^1_{\hat{X}}(\log \hat{D}) \subset \Omega^n_{\hat{X}}(\log \hat{D})$, one always has $\sum_{I=1}^n \mu_i \ge \mu$).

Proposition 4.5. *Suppose that $\Omega^1_X(\log D)$ splits into line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$. Then $\Omega^1_{\hat{X}}(\log \hat{D})$ also splits: there are non–negative integers $\mu_1, \dots, \mu_n$ so that*

$$\Omega^1_{\hat{X}}(\log \hat{D}) \cong (\sigma^* \mathcal{L}_1 \otimes (-\mu_1 E)) \oplus \cdots \oplus (\sigma^* \mathcal{L}_n \otimes (-\mu_n E)) .$$

Proof. Choose a local frame so that ω_i generates $\mathcal{L}_i$ over U. Multiplication of a section $h\omega_i$ ($h \in \mathcal{O}_{\hat{X}}$) of $\sigma^*\mathcal{L}_i$ by a section of $(-\mu_i E)$ defines a map $\sigma^*\mathcal{L}_i \otimes (-\mu_i E) \to \Omega^1_{\hat{X}}(\log \hat{D})$ for every i, and hence an injective homomorphism

$$t\colon\ (\sigma^* \mathcal{L}_1 \otimes (-\mu_1 E)) \oplus \cdots \oplus (\sigma^* \mathcal{L}_n \otimes (-\mu_n E)) \to \Omega^1_{\hat{X}}(\log \hat{D}),$$
$$(\ell_1 \otimes e_1, \dots, \ell_n \otimes e_n) \mapsto \sum_{i=1}^n e_i \ell_i .$$

Consider the map induced on n–fold wedges,

$$\det t\colon\ \otimes_{i=1}^n\ (\sigma^* \mathcal{L}_i \otimes (-\mu_i E)) = \underbrace{\sigma^* \Omega^n_X(\log D) \otimes \big(-\sum_{i=1}^n \mu_i E\big)}_{\Omega^n_{\hat{X}}(\log \hat{D}) \otimes (\mu - \sum_i \mu_i)\, E} \to \wedge^n \Omega^1_{\hat{X}}(\log \hat{D}).$$

Since $\sum_{i=1}^n \mu_i = \mu$, $\det t$ is an isomorphism; hence t is an isomorphism.

5. The case of free arrangements: Proof of Theorem 1.5

Let Y be the complement in $\mathbb{C}^n$ of a central and essential arrangement $A = \cup_{I=1}^N A_I$. For every $I = 1, \dots, N$, let H_I denote the hyperplane in $\mathbb{P}^n$ obtained by completing A_I at infinity. Thus $Y = \mathbb{P}^n - H$ is the complement of the projective arrangement $H = \cup_{I=0}^N H_I$ where we write H_0 for the hyperplane at infinity. In homogeneus coordinates $[Z_0 : \cdots : Z_n]$ a defining function for H has the form

$$F(Z) = Z_0 \prod_{I=1}^N F_I(Z_1, \dots, Z_n)$$

where the $F_I(Z_1, \dots, Z_n)$ are $N \geq n$ linear homogeneus polynomials. On each open set $U_{(i)} = \{Z_i \neq 0\}$ of the standard cover of $\mathbb{P}^n$ we shall use local coordinates $(z_{(i)0}, \dots, z_{(i)n})$ given by $z_{(i)j} = \frac{Z_j}{Z_i}$ and the local defining function for H

$$f_{(i)}(z_{(i)}) = \left. \frac{F(Z)}{Z_i^{N+1}} \right|_{\{Z_j = Z_i \cdot z_{(i)j}\}} .$$

Note in particular that $f_{(0)}$ is a homogeneus polynomial of degree N in $z_{(0)1}, \dots, z_{(0)n}$.

If the arrangement is free, a frame for the holomorphic vector bundle $\Omega^1_{\mathbb{P}^n}(\log H)$ over $U_{(0)}$ is given by n logarithmic forms $\omega_{(0)1}, \ldots, \omega_{(0)n}$ on $U_{(0)}$ such that

$$\omega_{(0)1} \wedge \cdots \wedge \omega_{(0)n} = \frac{1}{f_{(0)}}\, dz_{(0)1} \wedge \cdots \wedge dz_{(0)n}.$$

By [7: Corollary 4.77] each $\omega_{(0)i}$ can be chosen to have the form

$$\omega_{(0)i} = \frac{1}{f_{(0)}} \sum_{j=1}^{n} P_{i,j}(z_{(0)1}, \ldots, z_{(0)n})\, dz_{(0)j}, \tag{4}$$

where the $P_{i,j}$ are *homogeneus* polynomials of degree $\deg P_{i,1} = \cdots = \deg P_{i,n} = N - \alpha_i$. Here the integers $\alpha_1, \ldots, \alpha_n$ (the *exponents* of the free arrangement) must be strictly positive; they satisfy the condition $\sum_{i=1}^{n} \alpha_i = N$.

After these preliminaries, the first step toward proving Theorem 1.5 was provided to us by H. Terao in the form of an explicit splitting of the bundle of logarithmic forms *before* blowing–up. The proof given below is a version of [11].

Proposition 5.1 [Terao]**.** *If Y is the complement of a central and essential free arrangement in $\mathbb{C}^n$, let $H = \mathbb{P}^n - Y$ as above. Then*

$$\Omega_{\mathbb{P}^n}(\log H) \cong \bigoplus_{j=1}^{n} \mathcal{O}_{\mathbb{P}^n}(\alpha_j - 1).$$

Proof. We shall argue by explicitly computing the transition functions of $\Omega_{\mathbb{P}^n}(\log H)$ relative to the cover $\{U_{(i)}\}_{0\le i\le n}$. We first construct local frames on each of the $U_{(i)}$ in terms of the given local frame (4) on $U_{(0)}$. For each $i \neq 0$, let $\omega_{(i)1}, \ldots, \omega_{(i)n}$ be the forms on $U_{(i)}$ obtained by extending $\omega_{(0)1}, \ldots, \omega_{(0)n}$ from $U_{(0)} \cap U_{(i)}$ to all of $U_{(i)}$, i.e.,

$$\omega_{(i)j}\big|_{U_{(0)}\cap U_{(i)}} = \omega_{(0)j}.$$

From the relations among local coordinates $z_{(0)k} = \frac{Z_k}{Z_0} = \frac{Z_k}{Z_i}\frac{Z_i}{Z_0} = z_{(i)k}\, z_{(i)0}^{-1}$ on $U_{(0)} \cap U_{(i)}$, the relations among defining equations $f_{(0)} = z_{(i)0}^{-(N+1)} f_{(i)}$, and in view of the homogeneity of the polynomial coefficients in (4), one computes

$$\omega_{(i)j} = \frac{z_{(i)0}^{\alpha_j - 1}}{f_{(i)}} \left[-\left(\sum_{k=1}^{n} P_{j,k}(z_{(i)1}, \ldots, z_{(i)n})\, z_{(i)k} \right) dz_{(i)0} + \sum_{\substack{k=1 \\ k\neq i}}^{n} P_{j,k}(z_{(i)1}, \ldots, z_{(i)n})\, z_{(i)0}\, dz_{(i)k} \right].$$

Now:

Claim. *The forms defined as*

$$\omega'_{(i)1} = z_{(i)0}^{1-\alpha_1}\,\omega_{(i)1}, \quad \dots \quad ,\omega'_{(i)n} = z_{(i)0}^{1-\alpha_n}\,\omega_{(i)n}$$

give a local frame of $\Omega_{\mathbb{P}^n}(\log H)$ *over* $U_{(i)}$ *for each* i.

Proof of the claim. We must first verify that they are logarithmic, i.e., that both $f_{(i)}\,\omega'_{(i)j}$ and $df_{(i)} \wedge \omega'_{(i)j}$ are holomorphic throughout $U_{(i)}$. This is certainly true on $U_{(i)} \cap U_{(0)}$, since the $\omega'_{(i)j}$ are by definition local sections of $\Omega_{\mathbb{P}^n}(\log H)$ there. The fact that $f_{(i)}\,\omega'_{(i)j}$ is holomorphic on $U_{(i)}$ is evident. On the other hand, notice that

$$df_{(i)} \wedge \omega'_{(i)j} = \frac{df_{(i)}}{f_{(i)}} \wedge \left(-\sum_{k=1}^{n} P_{j,k}(z_{(i)1}, \dots, z_{(i)n})\, z_{(i)k} \right) dz_{(i)0} + \\ + \text{ non–singular “at infinity”, i.e., along } \{z_{(i)0} = 0\} \subset U_{(i)},$$

with $f_{(i)} = z_{(i)0} \prod_{I=1}^{N} F_I(z_{(i)1}, \dots, z_{(i)n})$. Thus $df_{(i)} \wedge \omega'_{(i)j}$ is holomorphic also at infinity. Moreover, we have

$$\omega_{(i)1} \wedge \cdots \wedge \omega_{(i)n}\big|_{U_{(i)}\cap U_{(0)}} = \omega_{(0)1} \wedge \cdots \wedge \omega_{(0)n}\big|_{U_{(i)}\cap U_{(0)}} = \\ \frac{1}{f_{(0)}}\, dz_{(0)1} \wedge \cdots \wedge dz_{(0)n}\Big|_{U_{(i)}\cap U_{(0)}} = \frac{z_{(i)0}^{N-n}}{f_{(i)}}\, dz_{(i)0} \wedge \cdots \wedge \widehat{dz_{(i)i}} \wedge \cdots \wedge dz_{(i)n}\Big|_{U_{(i)}\cap U_{(0)}}.$$

Since the sum $\sum_j \alpha_j = N$, then $\omega'_{(i)1} \wedge \cdots \wedge \omega'_{(i)n} = \frac{1}{f_{(i)}}\, dz_{(i)0} \wedge \cdots \wedge \widehat{dz_{(i)i}} \wedge \cdots \wedge dz_{(i)n}$ on $U_{(i)}$. Hence $\omega'_{(i)1}, \dots, \omega'_{(i)n}$ are linearly independent, which proves the claim.

Finally notice that for each $j = 1, \dots, n$, the collection $\{\omega'_{(0)j}, \dots, \omega'_{(n)j}\}$ is a section of the line bundle $\mathcal{O}_{\mathbb{P}^n}(\alpha_j - 1)$, as one sees from the transition functions

$$\frac{\omega'_{(i)j}}{\omega'_{(k)j}} = \frac{z_{(i)0}^{1-\alpha_j}}{z_{(k)0}^{1-\alpha_j}} \cdot \frac{\omega_{(i)j}}{\omega_{(k)j}} = \left(\frac{Z_i}{Z_k}\right)^{\alpha_j - 1}$$

on $U_{(i)} \cap U_{(k)}$. The proposition has been proven.

Next, consider a good compactification $X \xrightarrow{\sigma} \mathbb{P}^n$ of $X - D \xrightarrow{\sim} Y$, where $D = \sigma^{-1}(H)$. The birational map $\sigma = \sigma_\nu \cdots \sigma_1$ is the composite of a number of blow–ups,

$$X = X^\nu \xrightarrow{\sigma_\nu} X^{\nu-1} \xrightarrow{\sigma_{\nu-1}} \cdots \xrightarrow{\sigma_2} X^1 \xrightarrow{\sigma_1} X^0 = \mathbb{P}^n,$$

along submanifolds $V^a \subset X^a$ such that $V^a \subset D^a = \sigma_a^{-1} \cdots \sigma_1^{-1}(H)$ for every $a = 0, 1, \dots, \nu$. One computes the hypersurface $D = \sigma^{-1}(H) \subset X$ following step by step through the sequence of blow–ups. At every step, $D^a \subset X^a$ is the union of the proper transform $\overline{\sigma_a^{-1}(D^{a-1} - V^{a-1})}$ of D^{a-1} under σ_a and the exceptional divisor $\sigma_a^{-1}(V^{a-1})$. Let us denote by $\tilde{H}_I \subset X$ the proper transforms of the hyperplanes H_I under the total blow–up σ; also, denote by $E_a \subset X$ ($a = 1, \ldots, \nu$) the proper transform of the exceptional divisor $\sigma_a^{-1}(V^{a-1})$ under the subsequent blow–ups $\sigma_\nu \cdots \sigma_{a+1}$. Then

$$\begin{aligned} D = \sigma^{-1}(H) &= \left(\cup_{I=1}^{N+1} \tilde{H}_I\right) \cup (\cup_{a=1}^{\nu} E_a) \\ &= \tilde{H} \cup (\cup_{a=1}^{\nu} E_a)\,. \end{aligned}$$

Note that in order to produce a good compactification it is of course permitted but unnecessary to blow–up along submanifolds in which the components of D^a already have normal crossings. We can hence assume that each V^a is the pre–image under $\sigma_{a-1} \cdots \sigma_1$ of a multiple intersection of components of H so that $\operatorname{codim} V^a \geq 2$ and

$$\mu_a = \operatorname{mult}_{D^a} V^a - \operatorname{codim} V^a \geq 0$$

for all a. Using Proposition 4.5 at each step of the resolution gives immediately Theorem 1.5, where the integers $\mu_{a,i}$ are such that $\sum_i \mu_{a,i} = \mu_a$.

6. Example: arrangements of lines in $\mathbb{C}^2$

A general central and essential arrangement A of N lines in $\mathbb{C}^2$ is isomorphic to the locus defined by $f = z_1 z_2 \prod_{I=1}^{N-2} (z_1 + c_I z_2)$, where the c_I are pairwise distinct constants. Let hence $H = \left\{Z_0 Z_1 Z_2 \prod_{I=1}^{N-2} (Z_1 + c_I Z_2) = 0\right\} \subset \mathbb{P}^2$. In this case $\Omega^1_{\mathbb{P}^2}(\log H)$ is the vector bundle generated by $\omega_1 = \frac{dz_1}{z_1}$, $\omega_2 = \frac{1}{f}(-z_2 dz_1 + z_1 dz_2)$ over the open set $\{Z_0 \neq 0\}$. The exponents are thus $\alpha_1 = 1$ and $\alpha_2 = N-1$; by Theorem 5.1

$$\Omega_{\mathbb{P}^2}(\log H) \cong \mathcal{O}_{\mathbb{P}^2} \oplus \underbrace{\mathcal{O}_{\mathbb{P}^2}(N-2)}_{(N-2)\mathcal{H}}.$$

In this section we shall write $\mathcal{H}$ for the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$; also, we shall use the same symbol to denote either a divisor, or a line bundle, or its first Chern class. The additive notation for tensor products of line bundles will henceforth be assumed.

A good compactification of the complement $Y = \mathbb{C}^2 - A = \mathbb{P}^2 - H$ is given by the blow–up $X \xrightarrow{\sigma} \mathbb{P}^2$ at the origin $O = \{Z_1 = Z_2 = 0\}$ in $\{Z_0 \neq 0\}$. Notice that $\mu = \operatorname{mult}_O H - \operatorname{codim} O = N-2$. Let $E = \sigma^{-1}(O)$ and $D = \sigma^{-1}(H)$. We apply the analysis of Section 4 to this single blow–up: writing $g = \prod_{I=1}^{N-2} (z_1 + c_I z_2)$ and $\omega_1 = \frac{1}{g}\left(g\frac{dz_1}{z_1}\right)$, $\omega_2 = \frac{1}{g}\left(-\frac{dz_1}{z_1} + \frac{dz_2}{z_2}\right)$, one sees that $\mu_1 = 0$ and $\mu_2 = \mu = N-2$. Thus the decomposition of Theorem 1.5 has the form

$$\Omega^1_X(\log D) \cong \mathcal{O}_X \oplus \underbrace{((N-2)\sigma^*\mathcal{H} - (N-2)E)}_{(N-2)\tilde{\mathcal{H}}}.$$

Notice that, for any divisor $\mathcal{L}$ on $\mathbb{P}^2$, $\sigma^*\mathcal{L} = \tilde{\mathcal{L}} + (\mathrm{mult}_O\,\mathcal{L})E$ where $\tilde{\mathcal{L}}$ is the proper transform of $\mathcal{L}$. Since the multiplicity of the trivial line bundle is obviously zero and that of the hyperplane section is equal to 1, then $\sigma^*\mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_X$ and $\sigma^*\mathcal{H} = \tilde{\mathcal{H}} + E$.

Since in this case only one of the splitting line bundles is non–trivial, Theorem 1.1 is in effect a mere immediate consequence of Esnault–Schechtman–Viehweg vanishing:

$$H^1(Y,\mathbb{C}) \cong \Gamma(X,\mathcal{O}_X) \oplus \Gamma\big(X,(N-2)\tilde{\mathcal{H}}\big) = \mathbb{C} \oplus \Gamma\big(X,(N-2)\tilde{\mathcal{H}}\big),$$
$$H^2(Y,\mathbb{C}) \cong \Gamma(X,\mathcal{O}_X) \otimes \Gamma\big(X,(N-2)\tilde{\mathcal{H}}\big) = \Gamma\big(X,(N-2)\tilde{\mathcal{H}}\big).$$

We now show that the Euler characteristics of the splitting line bundles coincide with the exponents of the arrangement. The first equality $\chi(\mathcal{O}_X) = 1 = \alpha_1$ is clear. From the Riemann–Roch formula

$$\begin{aligned}\chi\big((N-2)\tilde{\mathcal{H}}\big) &= \int_X \mathrm{todd}(X)\,\mathrm{ch}\big((N-2)\tilde{\mathcal{H}}\big)\\ &= \int_X \left(\frac{1}{12}\big(c_1(X)^2 + c_2(X)\big) + \frac{(N-2)^2}{2}\,\tilde{\mathcal{H}}^2 + \frac{N-2}{2}\,c_1(X)\,\tilde{\mathcal{H}}\right)\\ &= \chi(\mathcal{O}_X) + \frac{(N-2)^2}{2}\,\tilde{\mathcal{H}}\cdot\tilde{\mathcal{H}} + \frac{N-2}{2}\,c_1(X)\cdot\tilde{\mathcal{H}}.\end{aligned}$$

But $\chi(\mathcal{O}_X) = 1$ and $\tilde{\mathcal{H}}\cdot\tilde{\mathcal{H}} = 0$. Moreover $c_1(X) = -c_1(\Omega^1_X) = -\Omega^2_X$, the dual to the canonical bundle, and

$$\Omega^2_X = \sigma^*\Omega_{\mathbb{P}^2} + (\mathrm{codim}\,O - 1)E = -3\sigma^*\mathcal{H} + E = -3\tilde{\mathcal{H}} - 2E.$$

Therefore $c_1(X)\cdot\tilde{\mathcal{H}} = 2E\cdot\tilde{\mathcal{H}} = 2$ and $\chi\big((N-2)\tilde{\mathcal{H}}\big) = \chi(\mathcal{O}_X) + \frac{N-2}{2}\,c_1(X)\cdot\tilde{\mathcal{H}} = N-1 = \alpha_2$, as claimed.

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