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THE DOUBLE LOGARITHM AND MANIN'S COMPLEX FOR MODULAR CURVES

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To Sasha Beilinson for his 40-th birthday

1. Introduction

Multiple polylogarithms are defined via power series expansion:

$$(1)\qquad Li_{n_1,...,n_m}(x_1,...,x_m) \ = \sum_{0<k_1<k_2<...<k_m} \frac{x_1^{k_1}x_2^{k_2}...x_m^{k_m}}{k_1^{n_1}k_2^{n_2}...k_m^{n_m}}.$$

Here $w := n_1 + ... + n_m$ is called the weight and m the depth of the multiple polylogarithm. These power series obviously generalize both Euler's classical polylogarithms $Li_n(x)$ (m=1), and Euler's sums [E], often called multiple ζ-numbers ($x_1 = ... = x_m = 1$) and studied in [Dr], [Z1-3], [G3-4], ...:

$$(2)\qquad \zeta(n_1,...,n_m) := \sum_{0<k_1<k_2<...<k_m} \frac{1}{k_1^{n_1}k_2^{n_2}...k_l^{n_m}} \qquad n_m > 1.$$

Multiple polylogarithms are periods of mixed Tate motives (s.12 of [G2], [G3]).

Let $\Gamma_1(N;m) \subset GL_m(\mathbb{Z})$ be the subgroup of the matrices whose bottom row is congruent to $(0,...,0,1)$ modulo N and V_m the standard m-dimensional representation of $GL_m(\mathbb{Z})$. In this note we begin to study a mysterious connection between the multiple polylogarithms of depth m at *N-th roots of unity* and cohomology of $\Gamma_1(N;m)$ with coefficients in $S^{w-m}(V_m)$ where $m > 1$. We will work out in details the simplest case: $m = 2, n_1 = n_2 = 1$, p is prime. In s.4 we touch the general situation and will return to it in [G1], see also [G4].

So we investigate the double logarithm function

$$(3)\qquad Li_{1,1}(x,y) \ = \sum_{0<k_1<k_2} \frac{x^{k_1}y^{k_2}}{k_1k_2}.$$

The series are convergent for $|x|,|y| \leq 1$, $(x,y) \neq (1,1)$. They admit analytic continuation to a multivalued analytic function.

We show that the double logarithm is a period of a mixed Tate motive $\tilde{L}i_{1,1}(x,y)$ equipped with an additional data: 2-framing. Let p be a prime. If

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x and y are p-th roots of unity the corresponding mixed Tate motive is defined over the scheme $S_p := \mathrm{Spec}\,\mathbb{Z}[x]/(x^p - 1)$.

The 2-framed mixed motives form a $\mathbb{Q}$-vector space. Let $\mathcal{L}(p)_2$ be the space generated by all 2-framed mixed Tate motives over the scheme S_p, and $\mathcal{C}(p)_2 \otimes \mathbb{Q}$ the subspace genertated by the motives $\tilde{L}i_{1,1}(x, y), x^p = y^p = 1$.

Let C_p be the group of cyclotomic units in $\mathbb{Z}[\zeta_p]$. We will construct a complex

$$\delta : \mathcal{C}(p)_2 \otimes \mathbb{Q} \to \Lambda^2 C_p \otimes \mathbb{Q}, \tag{4}$$

which encodes the essential information about the double logarithm at p-th roots of unity. One can show that $\mathrm{Ker}\,\delta = K_3(\mathbb{Z}[\zeta_p]) \otimes \mathbb{Q}$.

Consider the classical triangulation of the hyperbolic plane by the images of the geodesic triangle with vertices at $0, 1, \infty$ under the action of $SL_2(\mathbb{Z})$. Projecting it down we get canonical triangulation of a modular curve X. By definition Manin's complex $C_\bullet(X)$ is the chain complex of this triangulation.

We will construct a natural correspondence between the numbers $Li_{1,1}(\zeta_p^a, \zeta_p^b)$ and the triangles on the modular curve $X_1(p)$. More precisely, let $C_\bullet(X_1(p))_+$ be the coinvariants of the complex conjugation acting on the complex points of $X_1(p)$. We will construct a canonical isomorphism between (truncated) Manin's complex $\tau_{[2,1]} C_\bullet(X_1(p))_+$ and the quotient of complex (4) along $\mathrm{Ker}\,\delta$. This implies that $\mathcal{L}(p)_2/\mathcal{C}(p)_2 \otimes \mathbb{Q}$ is canonically isomorphic to $H_1(X_1(p), \mathbb{Q})_+$.

One may compare this with the classical theory of cyclotomic units, where $\{$ units $\}/\{$ cyclotomic units $\}$ is a finite group of order h_p^+ (the plus part of the class number of of the cyclotomic field). Notice that $L_1(x) = -log(1 - x)$, so restricting x to the p-th roots of unity we get $-\log(1 - \zeta_p^\alpha)$. Thus we consider $\mathcal{L}(p)_2$ as the "weight 2" analog of the group of units $\mathbb{Z}[\zeta_p]^*$, and $\mathcal{C}(p)_2$ as an analog of the subgroup of the cyclotomic units.

We suggest that the "higher cyclotomy theory" should study the multiple polylogarithm motives at roots of unity.

2. The double logarithm

1. Properties of the double logarithm function. (See [G1]-[G4] for general properties of multiple polylogarithms). Consider the following iterated integral:

$$I_{1,1}(a_1, a_2) := \int_0^1 \frac{dt}{t - a_1} \circ \frac{dt}{t - a_2} := \int\int_{0<t_1<t_2<1} \frac{dt_1}{t_1 - a_1} \wedge \frac{dt_2}{t_2 - a_2}.$$

Key Lemma. *The double logarithm can be written as an iterated integral*:

$$Li_{1,1}(x, y) = \int_{0<t_1<t_2<1} \frac{dt_1}{t_1 - (xy)^{-1}} \frac{dt_2}{t_2 - y^{-1}} = I_{1,1}\Big((xy)^{-1}, y^{-1}\Big).$$

Proof. Using $\frac{dt}{t-a} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{a^k}$ we get

$$I_{1,1}(a_1, a_2) = \int_0^1 \Big(\int_0^{t_2} (\sum_{k_1=1}^{\infty} \frac{t_1^{k_1-1}}{a_1^{k_1-1}}) dt_1 \Big) \cdot (\sum_{k_2=1}^{\infty} \frac{t_2^{k_2-1}}{a_2^{k_2-1}}) dt_2 =$$

$$\sum_{k_1,k_2=1}^{\infty} \frac{1}{k_1} \int_0^1 \frac{t_2^{k_1+k_2-1}}{a_1^{k_1} a_2^{k_2}} dt_2 \quad = \quad \sum_{k_1,k_2=1}^{\infty} \frac{(a_2/a_1)^{k_1} \cdot (1/a_2)^{k_1+k_2}}{k_1(k_1+k_2)}.$$

This lemma provides an analytic continuation of the double logarithm.

Symmetry relations. The double logarithm enjoys the following properties:

$$\log(1-x)\log(1-y) = Li_{1,1}(x,y) + Li_{1,1}(y,x) + Li_2(xy) \tag{5}$$

$$\log(1-x^{-1})\log(1-y^{-1}) = I_{1,1}(x,y) + I_{1,1}(y,x)$$

(Notice that $I_{1,1}(x,y) + I_{1,1}(y,x) = Li_{1,1}(\frac{y}{x}, \frac{1}{y}) + Li_{1,1}(\frac{x}{y}, \frac{1}{x})$).

Indeed, multiplying the power series for $\log(1-x)$ and $\log(1-y)$ we get

$$(\sum_{0<k_1<k_2} + \sum_{0<k_1=k_2} + \sum_{0<k_2<k_1}) \frac{x^{k_1} y^{k_2}}{k_1 k_2}$$

which is just the right hand side of the first identity.

The second follows from the product formula for iterated integrals:

$$\int_0^1 \frac{dt}{t-x} \cdot \int_0^1 \frac{dt}{t-y} \quad = \quad \int_0^1 \frac{dt}{t-x} \circ \frac{dt}{t-y} + \int_0^1 \frac{dt}{t-y} \circ \frac{dt}{t-x}.$$

The distribution relations. For any $n|N$ and $|x|, |y| \leq 1$ one has

$$Li_{1,1}(x_1, x_2) = \sum_{y_i^n = x_i} Li_{1,1}(y_1, y_2), \qquad Li_2(x) = n \cdot \sum_{y^n = x} Li_2(y). \tag{6}$$

This is easy to show using the power series expansion.

The differential equation. One has

$$dLi_{1,1}(x,y) = \log(1-xy) d\log\frac{(1-y)x}{(1-x)} + \log(1-y) d\log(1-x).$$

Indeed,

$$dLi_{1,1}(x,y) = \sum_{0<k_1<k_2} (\frac{x^{k_1}}{k_1} y^{k_2-1} dy + x^{k_1-1} \frac{y^{k_2}}{k_2} dx) =$$

$$\sum_{0<k_1} \frac{x^{k_1}}{k_1} \frac{y^{k_1}}{1-y} dy + \sum_{0<k_2} \frac{(x^{k_2-1}-1)}{x-1} \frac{y^{k_2}}{k_2} dx =$$

$$\log(1-xy) d\log(1-y) - \log(1-xy) \frac{dx}{x(x-1)} + \log(1-y) d\log(1-x).$$

Using $dx/x(x-1) = (1/(x-1) - 1/x)dx$ we get the formula.

Relation with the dilogaritm. It is easy to check by differentiation that

$$Li_{1,1}(x,y) = Li_2(\frac{xy-y}{1-y}) - Li_2(\frac{y}{y-1}) - Li_2(xy).$$

Substituting this to (5) we get the famous five term relation for the dilogarithm:

$$Li_2(\frac{xy-y}{1-y}) + Li_2(\frac{xy-x}{1-x}) - Li_2(\frac{y}{y-1}) - Li_2(\frac{x}{x-1}) - Li_2(xy) = \log(1-x)\cdot\log(1-y).$$

2. A variation of mixed Hodge structures related to the double logarithm. Let

$$A_{1,1}(x,y) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \log(1-xy) & 2\pi i & 0 & 0 \\ \log(1-y) & 0 & 2\pi i & 0 \\ Li_{1,1}(x,y) & 2\pi i\cdot\log\frac{(1-y)x}{(1-x)} & 2\pi i\cdot\log x & (2\pi i)^2 \end{pmatrix}.$$

The matrix $A_{1,1}(x,y)$ defines a variation of mixed Hodge structures over $\mathbb{C}^2\backslash\{x=0, x=1, y=1, xy=1\}$ as follows.

Let C_i be the $i-$th column of the matrix $A_{1,1}(x,y)$, $i=0,...,3$. For given (x,y) let $H_{1,1}(x,y) :=< C_0,...,C_3 >_{\mathbb{Q}}$ be the $\mathbb{Q}$-linear combinations of columnes of the matrix $A_{1,1}(x,y)$. The monodromy properties of the function $Li_{1,1}(x,y)$ and logarithms imply that $\mathbb{Q}$-vector spaces $H_{1,1}(x,y)$ form a local system, called $H_{1,1}$. It has a weight filtration defined as follows:

$$W_{-2k}H_{1,1}(x,y) :=< C_k,...,C_3 >, \quad W_{-2k+1} = W_{-2k}.$$

Let $e_0,...,e_3$ be the standard basis in $\mathbb{C}^4$ (the space of columnes). Then $H_{1,1}(x,y)\otimes_{\mathbb{Q}}\mathbb{C} = \mathbb{C}^4$. We define the Hodge filtration setting $F^{-k}H_{1,1}(x,y) = < e_0,...,e_k >_{\mathbb{C}}$.

Lemma 2.1. *$H_{1,1}$ is a variation of mixed Hodge structures.*

Proof. The Griffith transversality condition is just equivalent to the differential equation for the double logarithm function.

3. A 2-framed mixed Tate motive $\tilde{I}_{1,1}(a_1,a_2)$. Changing variables $u_i := t_i - a_i$ we get

$$I_{1,1}(a_1,a_2) := \int\int_{\Delta(a_1,a_2)} \frac{du_1}{u_1}\wedge\frac{du_2}{u_2},$$

where for appropriate real (a_1,a_2) one has

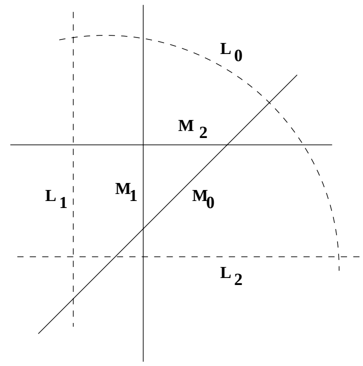
$$\Delta(a_1,a_2) := \{(u_1,u_2)\subset\mathbb{R}^2| \quad 0\le u_1+a_1\le u_2+a_2\le 1\}.$$

This suggests the following interpretation of the double logarithms as a period of a mixed Tate motive (More general motives were studied in [BMSV]).

Let $(u_0:u_1:u_2)$ be homogeneous coordinates in $\mathbb{P}^2$, $L_i = \{u_i = 0\}$ the coordinate lines and $L := L_0\cup L_1\cup L_2$ the coordinate triangle.

Set $M := M_0 \cup M_1 \cup M_2$ where

$$M_1 = \{u_1 + a_1 = 0\}, \quad M_2 = \{u_2 + a_2 = 0\}, \quad M_0 = \{u_1 + a_1 = u_2 + a_2\}$$



Let Δ_M be an oriented 2-chain in $\mathbb{CP}^2\backslash L$, which is the homeomorphic image of the triangle $(t_1, t_2)|t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1$, with sides on lines M_i. Its vertices coinside with the ones of M. Then Δ_M represents a generator of the group $H_2(\mathbb{CP}^2, M; \mathbb{Z})$. The formula $I_{1,1}(a_1, a_2) = \int_{\Delta_M} \omega_L$ provides an analytic continuation of the double logarithm.

Consider the mixed Hodge structure

$$H_{L,M} := H^2(\mathbb{CP}^2\backslash L, M).$$

The only non zero Hodge numbers are $h^{0,0}, h^{1,1}, h^{2,2}$, and in general $h^{0,0} = 1, h^{1,1} = 2, h^{2,2} = 1$.

The natural map $H_{L,M} \to H^2(\mathbb{CP}^2\backslash L)$ provides an isomorphism $gr_4^W H_{L,M} \longrightarrow H^2(\mathbb{CP}^2\backslash L) = \mathbb{Q}(-2)$, and similarly we get an isomorphism $\mathbb{Q}(0) = H^2(\mathbb{CP}^2, M) \longrightarrow W_0 H_{L,M}$. Thus we get distinguished elements

$$\omega_L \in \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(-2), gr_4^W H_{L,M}), \qquad [\Delta_M] \in (\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(0), W_0 H_{L,M}))^*$$

and $\int_{\Delta_M} \omega_L$ is a period of the mixed Hodge structure $H_{L,M}$.

4. The Tannakian formalism for mixed Tate categories: a review. (see also [BMS], [BGSV]). Let $\mathcal{M}$ be an abelian tensor $\mathbb{Q}$-category with an invertible object $\mathbb{Q}(1)$. Set $\mathbb{Q}(n) := \mathbb{Q}(1)^n$. We will say that $\mathcal{M}$ is a mixed Tate category if the objects $\mathbb{Q}(n)$ are mutually nonisomorphic, any simple object is isomorphic to one of them and $Ext^1_{\mathcal{M}}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ if $n \leq 0$.

Any object M of a mixed Tate category has a canonical weight filtration $W_\bullet M$ such that $gr^W_{2k} M = \oplus \mathbb{Q}(-k)$, $gr^W_{2k+1} M = 0$. The functor

$$\omega : \mathcal{M} \longrightarrow Vect_\bullet, \quad M \longmapsto \oplus_k \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(-k), gr^W_{2k} M)$$

to the category of graded $\mathbb{Q}$-vector spaces is a fiber functor. Let

$$L(\mathcal{M})_\bullet := Der(\omega) := \{F \in End\omega | F_{X\otimes Y} = F_X \otimes id_Y + id_Y \otimes F_Y\}$$

be the space of its derivations. It is a graded pro-Lie algebra over $\mathbb{Q}$.

Let $\tilde{\omega}$ be the fiber functor to the category of finite dimensional $\mathbb{Q}$-vector spaces obtained from ω by forgetting the grading. Then $Aut^{\otimes}\tilde{\omega}$ is a pro-algebraic group scheme over $\mathbb{Q}$. It is a semidirect product of $\mathbb{G}_m$ and a pro-unipotent group scheme G. The pro-Lie algebra $L(\mathcal{M})$ is the Lie algebra of the group scheme G. The action of $\mathbb{G}_m$ provides a grading on $L(\mathcal{M})$.

According to the Tannakian formalism the category $\mathcal{M}$ is canonically equivalent to the category of finite dinensional $\mathbb{Q}$-modules over the group scheme $Aut^{\otimes}\tilde{\omega}$. This category is naturally equivalent to the category of graded finite dimensional modules over the group scheme G. Since G is pro-unipotent, the last category is equivalent to the category graded finite dinensional modules over the graded pro-Lie algebra $L(\mathcal{M})_{\bullet}$.

Let $\mathcal{U}(\mathcal{M})_{\bullet} := End(\omega)$ be the space of all endomorphisms of the fiber functor ω. It is a Hopf algebra which is isomorphic to the universal enveloping algebra of the Lie algebra $L(\mathcal{M})_{\bullet}$.

The dual Hopf algebra $(\mathcal{U}(\mathcal{M})_{\bullet})^*$ can be identified with the Hopf algebra $\mathbb{Q}[G]$ of regular functions on G. The action of $\mathbb{G}_m$ on G provides a grading on it. Below we give a more concrete way to think about this Hopf algebra.

Let $n \geq 0$. Say that M is an n-framed object of $\mathcal{M}$ if it is supplied with a nonzero morphisms $v_n : \mathbb{Q}(-n) \longrightarrow gr^W_{2n}M$ and $f_0 : gr^W_0 M \longrightarrow \mathbb{Q}(0)$.

Consider the finest equivalence relation on the set of all n-framed objects for which $M_1 \sim M_2$ if there is a map $M_1 \rightarrow M_2$ respecting the frames. For example any n-framed object is equivalent to a one M with $W_{-2}M = 0$, $W_{2n}M = M$. Let $\mathcal{A}_n$ be the set of equivalence classes. It is an abelian group:

$$[M, v_n, f_0] + [M', v'_n, f'_0] = [M \oplus M', (v_n, v'_n), f_0 + f'_0]$$

$-[M, v_n, f_0] := [M, -v_n, f_0] = [M, v_n, -f_0]$. The neutral element is $\mathbb{Q}(0) \oplus \mathbb{Q}(-n)$ with the obvious frame. The composition $f_0 \circ v_0 : \mathbb{Q}(0) \rightarrow \mathbb{Q}(0)$ provides an isomorphism $\mathcal{A}_0 = \mathbb{Q}$.

The tensor product induces the commutative multiplication $\mu : \mathcal{A}_k \otimes \mathcal{A}_\ell \rightarrow \mathcal{A}_{k+\ell}$. Let us define the comultiplication

$$\Delta = \bigoplus_{0 \leq k \leq n} \Delta_{k,n-k} : \mathcal{A}_n \rightarrow \bigoplus_{0 \leq k \leq n} \mathcal{A}_k \otimes \mathcal{A}_{n-k}.$$

Choose a basis $p_1, ..., p_m$ of $\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(-k), gr^W_{2k}M)$ and the dual basis $p'_1, ..., p'_m$ of $(\mathrm{Hom}_{\mathcal{M}}(gr^W_{2k}M, \mathbb{Q}(-k))$. Then

$$\Delta_{k,n-k}[M, v_n, f_0] := \sum_{i=1}^{m} [M, v_n, p'_i](n-k) \otimes [M, p_i, f_0].$$

In particular $\Delta_{0,n} = id \otimes 1$ and $\Delta_{n,0} = 1 \otimes id$. Then $\mathcal{A}(\mathcal{M})_{\bullet} := \oplus \mathcal{A}(\mathcal{M})_n$ is a graded Hopf algebra with the commutative multiplication μ and the comultiplication Δ.

Theorem 2.2. *The Hopf algebra* $\mathcal{A}_{\bullet} := \oplus_{k=0}^{\infty} \mathcal{A}_k$ *is canonically isomorphic to the dual of the Hopf algebra* $\mathcal{U}(\mathcal{M})_{\bullet}$.

A canonical morphism $i : \mathcal{A}_\bullet \to \mathcal{U}(\mathcal{M})^*_\bullet$ is constructed as follows. Let $F \in End(\omega)_n$ and $[M, v_n, f_0] \in \mathcal{A}_n$. Then $< i([M, v_n, f_0]), F >:=< f_0, F(v_n) >$.

Set $\Delta'(X) := \Delta(X) - (X \otimes 1 + 1 \otimes X)$. Δ' provides the quotient $\mathcal{L}(\mathcal{M})_\bullet := \mathcal{A}(\mathcal{M})_\bullet/(\mathcal{A}(\mathcal{M})_{>0})^2$ with the structure of a Lie coalgebra with cobracket δ.

An example. A $\mathbb{Q}$-Hodge-Tate structure is a mixed $\mathbb{Q}$-Hodge structure with $h^{p,q} = 0$ if $p \neq q$. Let $\mathcal{HT}_\mathbb{Q}$ be the category of $\mathbb{Q}$-Hodge-Tate structures. Set $\mathcal{H}_\bullet := \mathcal{A}_\bullet(\mathcal{HT}_\mathbb{Q})$. Then

$$\mathcal{H}_1 = Ext^1_{\mathbb{Q}-MHS}(\mathbb{Q}(0), \mathbb{Q}(1)) = \frac{\mathbb{C}}{2\pi i \mathbb{Q}} = \mathbb{C}^*_\mathbb{Q}.$$

Under the isomorphism the extension provided by the mixed Hodge structure given by the columns of the matrix $\begin{pmatrix} 1 & 0 \\ \log(z) & 2\pi i \end{pmatrix}$ corresponds to $z \in \mathbb{C}^*$.

One has

$$\Delta' : \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_1 = \mathbb{C}^* \otimes_\mathbb{Q} \mathbb{C}^*, \quad \delta : \mathcal{H}_2/(\mathcal{H}_1)^2 \longrightarrow \Lambda^2_\mathbb{Q}\mathbb{C}^*.$$

Below a tilda over Li always means that we are dealing with the framed Hodge-Tate structure related to a multiple polylogarithm. See for the general construction s. 12 of [G2] or [G3].

Proposition 2.3.

$$\Delta' \tilde{L}i_{1,1}(x, y) = (1 - xy) \otimes \frac{(1 - y)x}{(1 - x)} + (1 - y) \otimes (1 - x) \in \mathbb{C}^* \otimes_\mathbb{Q} \mathbb{C}^*. \tag{7}$$

Proof. Follows from the description of the mixed Hodge structure given in §2.2.

3. The double logarithm at N-th roots of unity and Manin's complex for $X_1(N)$

1. Symmetry relations. In this section we study the 2-framed mixed Hodge structures related to the double logarithm at roots of unity.

Theorem 3.1. *Let a, b be N-th roots of unity. Then modulo N-torsion one has the following relations between the motivic double logarithms*:

The symmetry relations:

a) *In the depth one*:

$$\tilde{L}i_1(a^{-1}) = \tilde{L}i_1(a), \qquad \tilde{L}i_2(a^{-1}) + \tilde{L}i_2(a) = 0. \tag{8}$$

b) *In the depth two*:

$$\tilde{L}i_{1,1}(a, b) + \tilde{L}i_{1,1}(b, a) = \tilde{L}i_1(a)\tilde{L}i_1(b) - \tilde{L}i_2(ab), \tag{9}$$

$$\tilde{L}i_{1,1}(a, b) + \tilde{L}i_{1,1}(a^{-1}, ab) = \tilde{L}i_1(ab)\tilde{L}i_1(b). \tag{10}$$

The distribution relations: *For any $n|N$ one has*

$$\tilde{L}i_{1,1}(a_1,a_2) = \sum_{b_i^n=a_i} \tilde{L}i_{1,1}(b_1,b_2), \qquad \tilde{L}i_2(a) = n \cdot \sum_{b^n=a} \tilde{L}i_2(b). \tag{11}$$

Let Δ_{12} be the group of order 12 given by generators σ_1, σ_2 subject to the relations $\sigma_1^2 = \sigma_2^2 = 1, (\sigma_1\sigma_2)^6 = 1$. This group acts on the space of the depth two, weight two polylogarithms as follows:

$$\sigma_1 : \tilde{L}i_{1,1}(a,b) \longmapsto -\tilde{L}i_{1,1}(a^{-1}, ab) + \tilde{L}i_1(ab)\tilde{L}i_1(b), \tag{12}$$

$$\sigma_2 : \tilde{L}i_{1,1}(a,b) \longmapsto -\tilde{L}i_{1,1}(b,a) + \tilde{L}i_1(a)\tilde{L}i_1(b) - \tilde{L}i_2(ab). \tag{13}$$

The action on the depth one polylogarithms is trivial. The relations (9) are just the symmetry under the action of Δ_{12}. (Compare this with the Bass theorem on cyclotomic units.)

Remark. Consider slightly modified functions

$$L'_{1,1}(x,y) := Li_{1,1}(x,y) - \frac{1}{2}\log(1-xy)\cdot\log\frac{(1-y)x}{1-x} - \frac{1}{2}\log(1-x)\cdot\log(1-y)$$

$$L'_2(x) := Li_2(x) - \frac{1}{2}\log(1-x)\cdot\log x.$$

Then we rid off products of logarithms from the symmetry relations (9)

$$\tilde{L}'_{1,1}(x,y) + \tilde{L}'_{1,1}(y,x) + \tilde{L}'_2(xy) = 0, \quad \tilde{L}'_{1,1}(x,y) + \tilde{L}'_{1,1}(x^{-1},xy) = 0.$$

Definition 3.2. The subgroup $\mathcal{C}(N)_2 \subset \mathcal{H}_2/(\mathcal{H}_1)^2$ is generated by the 2-framed Hodge-Tate structures $\tilde{L}i_{1,1}(a,b)$, $a^N = b^N = 1$.

Thanks to (9), (10) we know that $\tilde{L}i_2(a) \in \mathcal{C}(N)_2$. Let $\mathcal{C}(N)_2^{(1)} \subset \mathcal{C}(N)_2$ be the subgroup generated by the motivic dilogarithms at N-th roots of unity, i.e. by $\tilde{L}i_2(a)$, $a^N = 1$. Set $\bar{C}(N)_2 := \mathcal{C}(N)_2/C(N)_2^{(1)}$

Definition 3.3. Let a,b,c be N-th roots of unity such that $abc = 1$. Denote by $\{a,b,c\}$ be the projection of $\tilde{L}i_{1,1}(a,b)$ to $\bar{\mathcal{C}}(N)_2$.

The symmetry under the action of the group Δ_{12} now looks neater:

$$\{a,b,c\} = -\{b,a,c\}, \quad \{a,b,c\} = -\{a,c,b\}, \quad \{a,b,c\} = \{a^{-1},b^{-1},c^{-1}\}. \tag{14}$$

In particular $\Delta_{12} = S_3 \times \mathbb{Z}/2\mathbb{Z}$.

2. The double logarithm complex. Let C_N^* be the subgroup of $\mathbb{C}^*$ generated by the elements $1-\zeta_N^\alpha$ where ζ_N is a primitive root of unity.

Lemma 3.4. a) *Modulo N-torsion one has*

$$\mathcal{C}(N)_2^{(1)} \subset \operatorname{Ker}\delta, \quad \delta(\mathcal{C}(N)_2) \subset \Lambda^2 C_N^*.\mathbb{Z}[\frac{1}{N}].$$

b) $\mathcal{C}(N)_2^{(1)} = K_3^{ind}(\mathbb{Z}[\zeta_N])$ *modulo torsion.*
c) *If $N=p$ is a prime then modulo p-torsion $\delta(\mathcal{C}(p)_2) \subset \Lambda^2 C_p$.*

Proof. a) Since ζ_N is an N-torsion element in $\mathbb{C}^*$ we get

$$\delta\tilde{L}_2(\zeta_N^\alpha) = \tilde{L}_1(\zeta_N^\alpha) \wedge \zeta_N^\alpha = 0 \quad \text{modulo } N\text{-torsion.}$$

Using formula (7) for $\Delta'\tilde{L}i_{1,1}(x,y)$ when x,y are N-th roots of unity we get modulo N-torsion

$$\begin{aligned}\delta\{a,b,c\} &= -(1-a)\wedge(1-b) - (1-b)\wedge(1-c) - (1-c)\wedge(1-a)\\ &= \frac{1-a}{1-c} \wedge \frac{1-b}{1-c}.\end{aligned} \tag{15}$$

b) This is a reformulation of the well known result about the cyclotomic elements in $K_3(\mathbb{Z}[\zeta_N])$.

c) Follows from (15) and the fact that $\frac{1-\zeta_p^\alpha}{1-\zeta_p^\beta}$ is a unit when p is a prime.

So we get a complex defined modulo N-torsion

$$\delta : \mathcal{C}(N)_2 \longrightarrow \Lambda^2 C_N^* \mathbb{Q}. \tag{16}$$

Below we will compare it with Manin's complex for the modular curve $X_1(N)$.

3. Modular symbols and Manin's complex. *Modular symbols.* Let x,y be two points on $P^1(\mathbb{Q})$ viewed as the boundary points of the hyperbolic plane $\mathcal{H}$. Let $\gamma_{x,y}$ be the geodesic connecting x and y. Let $\Gamma \subset PSL_2(\mathbb{Z})$ be a subgroup of finite index. Set $Y_\Gamma := \Gamma\backslash\mathcal{H}$. Let $\mathcal{H}^* := \mathcal{H}^* \cup \mathbb{P}^1(\mathbb{Q})$. Let $X_\Gamma = \Gamma\backslash\mathcal{H}^*$ be the compactification of Y_Γ and $P_\Gamma := X_\Gamma\backslash Y_\Gamma = \Gamma\backslash P^1(\mathbb{Q})$ the set of cusps. The projection of the geodesic $\gamma_{x,y}$ onto X_Γ defines an element $\{x,y\} \in H_1(X_\Gamma, P_\Gamma; \mathbb{Z})$, called a modular symbol. Let $X_1(N)_\mathbb{C} = \Gamma_1(N)\backslash\mathcal{H}^*$ be the modular curve of level N. $X_1(N)_\mathbb{C}$ is the set of complex points of an algebraic curve $X_1(N)$ which can be defined over $\mathbb{Q}$. So the complex conjugation acts on $X_1(N)(\mathbb{C})$. The projection $\Gamma_1(N)\backslash\mathcal{H}^* \longrightarrow X_1(N)(\mathbb{C})$ transforms the involution $z \to -\bar{z}$ of the hyperbolic plane to the complex conjugation on $X_1(N)(\mathbb{C})$.

Manin's complex. Let γ be the vertical geodesic from 0 to $i\infty$ on the hyperbolic plane. For any $g \in PSL_2(\mathbb{Z})$ we get a geodesic $g\gamma$. Its projection onto $X_1(N)$ depends only on the coset $\Gamma_1(N)\cdot g$. Set

$$E_N := \frac{\{<\alpha,\beta> \in (\mathbb{Z}/N\mathbb{Z})^2 \quad | \quad \text{g.c.d. } (\alpha,\beta,N)=1\}}{<\alpha,\beta> \quad \sim \quad <-\alpha,-\beta>}.$$

The group $PSL_2(\mathbb{Z})$ acts from the right on the rows $< \alpha, \beta >$ where $\alpha, \beta \in (\mathbb{Z}/N\mathbb{Z})^2/\pm 1$. The set E_N is the orbit of $< 0, 1 >$, so $E_N = \Gamma_1(N)\backslash PSL_2(\mathbb{Z})$. Thus for any $< \alpha, \beta > \in E_N$ we get a geodesic $\xi(\alpha, \beta)$ on $X_1(N)$.

Set

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Let $< \gamma, \tau\gamma, \tau^2\gamma >$ be the geodesic triangle with the vertices $i\infty, 0, 1$.

Consider the following cell decomposition of $X_1(N)$:

0-cells: cusps on $X_1(N)$.

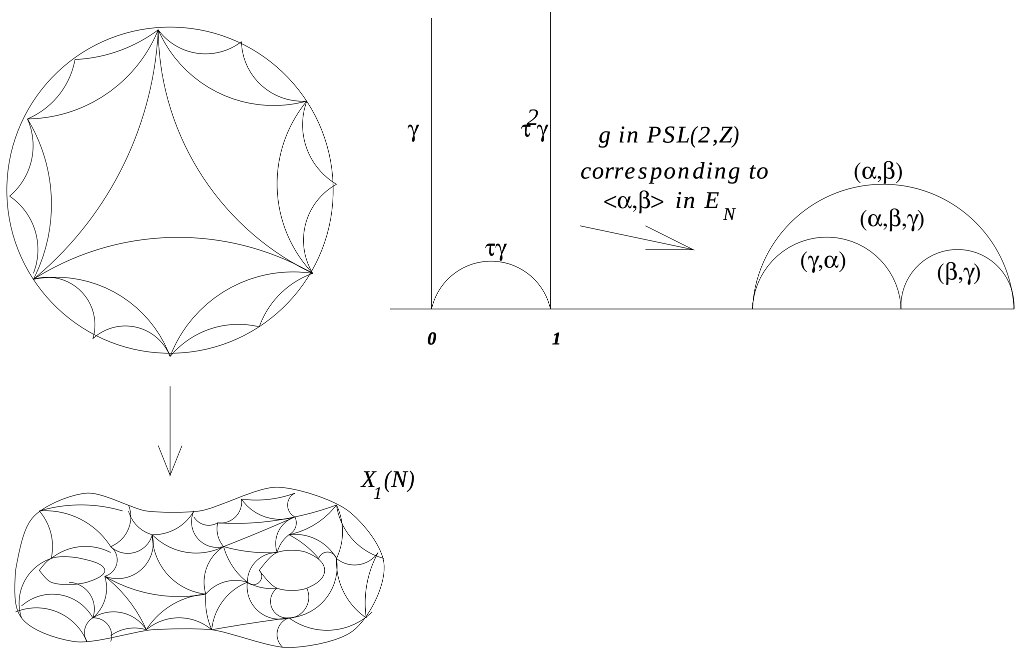
1-cells: geodesics $g \cdot \gamma$ on $X_1(N)$, $g \in PSL_2(\mathbb{Z})$.

2-cells: the projections onto $X_1(N)$ of the triangles $g\cdot < \gamma, \tau\gamma, \tau^2\gamma >$, where $g \in PSL_2(\mathbb{Z})$.

Set $C_i(X_1(N)) := \mathbb{Z}[i - cells]$, $\quad C_i^{\mathbb{Q}}(X_1(N)) := C_i(X_1(N)) \otimes \mathbb{Q}$.

We get the chain complex of this cell decomposition of $X_1(N)$:

$$C_2(X_1(N)) \xrightarrow{\partial} C_1(X_1(N)) \xrightarrow{\partial} C_0(X_1(N)).$$



Let $C_i(X_1(N))_+$ be the space of coinvariants of the complex conjugation.

4. Main construction. Let $\alpha, \beta, \gamma \in \mathbb{Z}/N\mathbb{Z}$ and $\alpha + \beta + \gamma = 0$. Choose $g \in PGL_2(\mathbb{Z})$ such that $\Gamma_1(N) \cdot g = < \alpha, \beta >$. Denote by $\xi(\alpha, \beta, \gamma)$ the oriented triangle $g\cdot < \gamma, \tau\gamma, \tau^2\gamma >$. Its sides are the geodesics $\xi(\alpha, \beta), \xi(\beta, \gamma), \xi(\gamma, \alpha)$. Let (α, β, γ) be the generator of the group $C_2(X_1(N))_+$ corresponding to this triangle. Denote by (α, β) the element of the group $C_1(X_1(N))_+$ corresponding to the geodesic $\xi(\alpha, \beta)$. Then

$$\partial(\alpha, \beta, \gamma) = (\alpha, \beta) + (\beta, \gamma) + (\gamma, \alpha). \tag{17}$$

Theorem 3.5. a) *Let $N > 1$ and $\zeta_N := e^{2\pi i/N}$. Then modulo N-torsion there is a morphism of complexes*

$$\begin{array}{ccc} C_2(X_1(N))_+ & \xrightarrow{\partial} & C_1(X_1(N))_+ \\ g_2 \downarrow & & g_1 \downarrow \\ \mathcal{C}(N)_2 & \xrightarrow{\delta} & \Lambda^2 C_N^* \end{array}$$

given on the generators by

$$g_1 : (\alpha, \beta, \gamma) \longmapsto -\{\zeta_N^\alpha, \zeta_N^\beta, \zeta_N^\gamma\}, \qquad \alpha + \beta + \gamma = 0$$

$$g_2 : (\alpha, \beta) \longmapsto (1 - \zeta_N^\alpha) \wedge (1 - \zeta_N^\beta), \qquad (0, \beta) \longmapsto 0.$$

b) *It is surjective if $N = p^a$ where p is a prime.*

Proof. The map g_1 is obviously surjective for any N.

Proposition 3.6. *g_2 is a well defined homomorphism.*

Proof of the proposition (3.6). The group $S_3 \times \mathbb{Z}/2\mathbb{Z}$ acts on the generators (α, β, γ) of the group $C_2(X_1(N))_+$. Namely, $\sigma \in S_3$ acts by a permutation and the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ by $(\alpha, \beta, \gamma) \longmapsto (-\alpha, -\beta, -\gamma)$.

Lemma 3.7. *One has*

$$(\alpha, \beta, \gamma) = (-\alpha, -\beta, -\gamma), \quad (\alpha, \beta, \gamma) = (\gamma, \alpha, \beta), \quad (\alpha, \beta, \gamma) = -(\beta, \alpha, \gamma).$$

Proof. The first equality follows from the definition of E_N. The second is true since the triangles $< g \cdot \gamma, g\tau \cdot \gamma, g\tau^2 \cdot \gamma >$ and $< g\tau \cdot \gamma, g\tau^2 \cdot \gamma, g \cdot \gamma >$ coinside. The third is valid thanks to formula (19) in the following lemma

Lemma 3.8. a) *The complex conjugation acts on the geodesic $\xi(\alpha, \beta)$ by*

$$\xi(\alpha, \beta) \longmapsto \xi(-\alpha, \beta,) \tag{18}$$

and on the generators of the group $C_2(X_1(N))$ by

$$(\alpha, \beta, \gamma) \longmapsto -(\beta, \alpha, \gamma). \tag{19}$$

b) *One has* $(\alpha, \beta) = -(\beta, -\alpha)$.

Proof. a) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $g \cdot \{i\infty, 0\} = \{\frac{a}{c}, \frac{b}{d}\} = \xi(c,d)$. The involution $z \to -\bar{z}$ sends the modular symbol $\{\frac{a}{c}, \frac{b}{d}\}$ to $\{-\frac{a}{c}, -\frac{b}{d}\}$. On the other hand

$$\{-\frac{a}{c}, -\frac{b}{d}\} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \{i\infty, 0\} = \xi(-c,d),$$

and we get (18). Notice $g\tau = \begin{pmatrix} b & -a-b \\ d & -c-d \end{pmatrix}$, $\quad g\tau^2 = \begin{pmatrix} -a-b & b \\ -c-d & d \end{pmatrix}$. So the element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms the triangle $< \gamma, \tau\gamma, \tau^2\gamma >$ to the geodesic triangle $< \xi(c,d), \xi(d, -c-d), \xi(-c-d, c) >$. Under the complex conjugation it goes to the geodesic triangle $< \xi(-c,d), \xi(-d, -c-d), \xi(c+d, c) >$, which coinsides with the triangle $< \xi(d,c), \xi(-c-d, d), \xi(c, -c-d) >$ but has a different orientation.

b) Indeed, $\sigma\gamma$ is the geodesic from $i\infty$ to 0 and $\Gamma_1(N) \cdot g\sigma = (\beta, -\alpha)$. The lemma is proved.

The stabilizer of the triangle $< \gamma, \tau\gamma, \tau^2\gamma >$ in the group $PGL_2(\mathbb{Z})$ is the group S_3. So the only relations between the generators (α, β, γ) are those provided by the action of the group $S_3 \times \mathbb{Z}/2\mathbb{Z}$. Under the map g_2 they go precisely to the symmetry relations (14). So g_2 is well defined. Proposition (3.6) is proved.

All relations between the generators (α, β) of $C_1(X_1(p))_+$ are given by (18) and lemma (3.8b). So g_1 is well defined.

Looking at the diagram

$$\begin{array}{ccc} (\alpha,\beta,\gamma) & \xrightarrow{\partial} & (\alpha,\beta)+(\beta,\gamma)+(\gamma,\alpha) \\ g_2 \downarrow & & g_1 \downarrow \\ -\{\zeta_N^\alpha, \zeta_N^\beta, \zeta_N^\gamma\} & \xrightarrow{\delta} & (1-\zeta_N^\alpha)\wedge(1-\zeta_N^\beta) + (1-\zeta_N^\beta)\wedge(1-\zeta_N^\gamma) \\ & & +(1-\zeta_N^\gamma)\wedge(1-\zeta_N^\alpha) \end{array}$$

and comparing formula (15) for δ with formula (17) for ∂ we see the commutativity for $\alpha\beta\gamma \neq 0$. Notice that $(0, \beta, -\beta) = 0$ and $(1-\zeta_N^\beta)\wedge(1-\zeta_N^{-\beta}) = 0$ modulo N-torsion. So the diagram is commutative. Theorem (3.5a) is proved.

5. The case when the level $N = p$ is a prime. We will construct a subcomplex $\tilde{C}_\bullet(X_1(p)) \subset C_\bullet(X_1(p))$. For this we need to recall some facts.

The boundary of Manin's symbols. The cusps of $X_1(N)$ are parametrized by the orbits of the unipotent group $\Gamma_\infty := \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ on the set E_N. If $N = p$ is a prime they are the orbits of the elements $< 0, \beta >$ and $< \alpha, 0 >$ where $\alpha \neq 0, \beta \neq 0$. We will denote these orbits by $[0, \beta]$ and $[\alpha, 0]$.

The boundary of Manin's symbols looks as follows:

$$\partial(\alpha, \beta) = [0, \beta] - [0, \alpha] \quad \text{if } \alpha \neq 0, \beta \neq 0;$$

$$\partial(0,\beta) = -\partial(\beta,0) = [0,\beta] - [\beta,0] \quad \text{if } \beta \neq 0;$$

From this and lemma 3.8 we see that the complex conjugation acts as the identity map on the cusps of $X_1(p)$.

There are just two cusps on $X_0(p)$ called the ∞ and 0 cusps. They are the images of $i\infty$ and 0 under the projection $\mathcal{H}^* \to X_0(p)$. The covering $X_1(p) \to X_0(p)$ is unramified of degree $(p-1)/2$. The cusps on $X_1(p)$ lying over the ∞ cusp on $X_0(p)$ are called the ∞-cusps. They are canonically identified with the set of p-th roots of unity different from 1 considered modulo inversion (i.e. ζ and ζ^{-1} correspond to the same cusp). If we think about the cusps as of the isomorphism classes of pairs,

{*a generalized elliptic curve of Deligne-Rappoport, a point of order p on it such that the subgroup scheme it generates meets every irreducible component of each geometrical fiber*},

then the ∞-cusps correspond to the standard Neron polygon with one side and the choice of a generator of its unique subgroup of order p. In our parametrization this is the cusps $[0,\beta]$. By definition $\tilde{C}_0(X_1(p))$ is the free abelian group generated by the ∞-cusps. So there is a canonical isomorphism modulo p-torsion

$$g_0 : \tilde{C}_0(X_1(p)) \longrightarrow C_p^*, \qquad [0,\beta] \longmapsto 1-\zeta_p^{\beta}$$

group $\tilde{C}_1(X_1(p))$ is generated by the geodesics $\xi(\alpha,\beta)$ connecting the ∞-cusps i.e. $\alpha\beta \neq 0$. The group $\tilde{C}_2(X_1(p))$ is generated by the geodesic triangles (α,β,γ) such that $\alpha\beta\gamma \neq 0$. These groups form a subcomplex $\tilde{C}_\bullet(X_1(p))$ of Manin's complex. *Its + part is canonically quasiisomorphic to the + part of Manin's complex.* Indeed, each 0-cusp is a boundary of just one edge in Manin's complex connecting it with an ∞-cusp. ($[0,\beta]$ is connected with $[\beta,0]$). Notice that $(0,\beta,-\beta) = 0$ in $C_2(X_1(p))_+$.

over ∞ with the p-th roots of unity modulo

Consider a homomorphism

$$\delta' : \Lambda^2 C_p^* \longrightarrow C_p^*, \qquad (1-\zeta_p^{\alpha}) \wedge (1-\zeta_p^{\beta}) \quad \longmapsto \quad \frac{1-\zeta_p^{\beta}}{1-\zeta_p^{\alpha}}.$$

The kernel of this map modulo p-torsion is the group $\Lambda^2 C_p$.

Theorem 3.9. a) *There is a canonical isomorphism of complexes modulo p-torsion*

$$\begin{array}{ccccc} \tilde{C}_2(X_1(p))_+ & \longrightarrow & \tilde{C}_1(X_1(p))_+ & \longrightarrow & \tilde{C}_0(X_1(p))_+ \\ g_2 \downarrow & & g_1 \downarrow & & g_0 \downarrow \\ \bar{\mathcal{C}}(p)_2 & \xrightarrow{\delta} & \Lambda^2 C_p^* & \xrightarrow{\delta'} & C_p^* \end{array}$$

b) *It transmits the action of the Galois group of the covering $X_1(p) \to X_0(p)$ on the complex $\tilde{C}_\bullet(X_1(p))$ to the action of* $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ *on the bottom complex.*

Remark. In the definition of Manin's complex for $X_1(N)$ we have already choosen the generator $e^{2\pi i/N}$ of the group of N-th roots of unity.
Proof. The only relations modulo p-torsion in C_p^* are $1-\zeta_p^{\alpha} = 1-\zeta_p^{-\alpha}$. Thanks to this, lemma (3.8b) and formula (18), the map g_1 is an isomorphism.

The complex conjugation acts on $H_2(X_1(p), \mathbb{Q})$ with the eigenvalue -1. This means that the restriction of ∂ to $C_2(X_1(p))_+$ is injective. Since the diagram is commutative and g_1 is an isomorphism, we conclude that g_2 is injective. So it is an isomorphism by theorem (3.5). The theorem is proved.

6. Corollaries. Set $Z_1(X_1(p))_+ := \quad \mathrm{Ker}\Big(C_1(X_1(p))_+ \stackrel{\partial}{\longrightarrow} C_0(X_1(p))_+\Big)$.

Corollary 3.10. a) *There is an isomorphism of complexes modulo p-torsion*

$$\begin{array}{ccc} C_2(X_1(p))_+ & \stackrel{\partial}{\longrightarrow} & Z_1(X_1(p))_+ \\ g_2 \downarrow & & \downarrow g_1 \\ \bar{\mathcal{C}}(p)_2 & \stackrel{\delta}{\longrightarrow} & \Lambda^2 C_p \end{array} \tag{20}$$

b) $\dim \bar{\mathcal{C}}(p)_2 = \frac{(p-1)(p-5)}{12}$.

Proof. a) follows from theorem (3.9). To check b) notice that $rk\Lambda^2 C_p = \frac{(p-3)(p-5)}{8}$ and $\dim S_2^0(\Gamma_1(p)) = 1 + \frac{p^2-1}{24} - \frac{p-1}{2}$.

Example. $p = 13$, so $\dim\bar{\mathcal{C}}(13)_2 = 8$, $\dim S_2^0\Gamma_1(13) = 2$.

Corollary 3.11.

$$\mathrm{Ker}\Big(\mathcal{C}_2(p) \stackrel{\delta}{\longrightarrow} \Lambda^2 C_p\Big) \otimes \mathbb{Q} = K_3(\mathbb{Z}[\zeta_p]) \otimes \mathbb{Q},$$

$$\mathrm{Coker}\Big(\mathcal{C}_2(p) \stackrel{\delta}{\longrightarrow} \Lambda^2 C_p\Big) = H_1(X_1(p), \mathbb{Z})_+ \qquad \textit{modulo p-torsion.} \tag{21}$$

The analog of the Bass theorem on the cyclotomic units is

Theorem 3.12. *All relations between the motivic depth two polylogarithms at p-th roots of unity are given by the symmetry relations* (8) *and* (9).

The corollary and the theorem follow immediately from theorem (3.9).

7. Motivic picture. Let $\mathcal{M}_T(S)$ be the abelian category of mixed Tate motives over one dimensional arithmetic scheme S. The key property is the fundamental Beilinson formula

$$Ext^1_{\mathcal{M}_T(S)}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(S) \otimes \mathbb{Q}, \quad n \geq 1 \tag{22}$$

and higher Ext's vanish. The corresponding triangulated category was recentely defined by Voevodsky, however the key formula (**??**) is not available yet. So we can define an object of $\mathcal{M}_T(S)$, but can not afford to use the whole formalism. In this section we will assume the formalism, so the results are conditional.

We will say that an equivalence class of n-framed mixed Tate motives is defined over S if one can find a representative in the equivalence class defined over S.

Theorem 3.13. *Let us assume* (??). *Suppose that $a_1^p = a_2^p = 1$ and p is a prime number. Then there exist $M \in \mathbb{Z}$ such that the equivalence class of the 2-framed mixed Tate motive $M \cdot \tilde{L}i_{1,1}(a_1, a_2)$ is defined over $S_p := \mathrm{Spec}\ \mathbb{Z}[\zeta_p]$.*

However, *we do not get all 2-framed mixed Tate motives over S_p this way*! Namely, let $\mathcal{L}(p)_\bullet$ be the Lie coalgebra of the category of mixed Tate motives over S_p. Then it follows from the Tannakian formalism (see s. 2.4) that $H^i_{(n)}(\mathcal{L}(p)_\bullet) = Ext^i_{\mathcal{M}_T(S_p)}(\mathbb{Q}(0), \mathbb{Q}(n))$ where $H^i_{(n)}$ is the degree n part of H^i. So by (**??**) one has $\mathcal{L}(p)_1 = C_p \otimes \mathbb{Q}$,

$$\begin{aligned} \mathrm{Coker}\Big(\mathcal{L}(p)_2 \to \Lambda^2 \mathcal{L}(p)_1\Big) &= Ext^2_{\mathcal{M}_T(S_p)}(\mathbb{Q}(0), \mathbb{Q}(2)) \\ &= K_2(\mathbb{Z}[\zeta_p]) \otimes \mathbb{Q} = 0, \end{aligned} \tag{23}$$

$\mathrm{Ker}\Big(\mathcal{L}(p)_2 \to \Lambda^2 \mathcal{L}(p)_1\Big) = Ext^1_{\mathcal{M}_T(S_p)}(\mathbb{Q}(0), \mathbb{Q}(2)) = K_3(\mathbb{Z}[\zeta_p]) \otimes \mathbb{Q} \subset C(p)_2 \otimes \mathbb{Q}$.

Theorem (**??**) shows that $\mathcal{C}(p)_2 \otimes \mathbb{Q} \subset \mathcal{L}(p)_2$. So δ provides an isomorphism

$$\frac{\mathcal{L}(p)_2}{\mathcal{C}(p)_2 \otimes \mathbb{Q}} = \frac{\Lambda^2 C_p \otimes \mathbb{Q}}{\delta(\mathcal{C}(p)_2)}.$$

Thanks to (**??**) the right hand side is isomorphic to $H_1(X_1(p), \mathbb{Q})_+$. So we get

Theorem 3.14. *Let us assume* (??). *Then there exists a canonical isomorpism*

$$\frac{\mathcal{L}(p)_2}{\mathcal{C}(p)_2 \otimes \mathbb{Q}} = H_1(X_1(p), \mathbb{Q})_+.$$

Proof of theorem (??). It is easy to show that $\tilde{L}i_{1,1}(a_1, a_2)$ is defined over $S_p^* := \mathrm{Spec}\, \mathbb{Z}[\zeta_p][\frac{1}{p}]$. One has $\delta \tilde{L}i_{1,1}(a_1, a_2) \in \Lambda^2 C_p$ modulo p-torsion. So by (**??**) there exists a 2-framed mixed Tate motive X over S_p such that $M \cdot \delta(X - \tilde{L}i_{1,1}(a_1, a_2)) = 0$ for certain $M \in \mathbb{Z}$. So $M(X - \tilde{L}i_{1,1}(a_1, a_2))$ represents a class in $Ext^1_{\mathcal{M}_T(S_p^*)}(\mathbb{Q}(0), \mathbb{Q}(2))$, which is given by $\sum n_a \tilde{L}i_2(\zeta_p^a)$ and thus defined over S_p.

4. Multiple polylogarithms at roots of unity: fragments of the picture

1. Motivic point of view and its consequences.

Theorem 4.1. a) *Suppose that* $a_i^p = 1$. *Then* $\tilde{L}i_{n_1,...,n_l}(a_1,...,a_l)$ *is a mixed Tate motive over the scheme* $S_p^* :=$ Spec $(\mathbb{Z}[x]/(x^p-1)[\frac{1}{p}])$.

b) *In particular* $\tilde{\zeta}(n_1,...,n_l) := \tilde{L}i_{n_1,...,n_l}(1,...,1)$ *is a w-framed mixed Tate motive over* Spec $\mathbb{Z}$.

Let $L(\mathbb{Z})_\bullet$ be the free graded Lie algebra generated by elements $e_{(2n+1)}$ of degree $-(2n+1)$ $(n \geq 1)$ and $UL(\mathbb{Z})_\bullet^*$ be the dual to its universal enveloping algebra (graded by positive integers). Let $f_3, f_5, ...$ be the functionals on the vector space generated by the vectors $e_3, e_5, ...$ such that $< f_i, e_j >= 0$. Then $UL(\mathbb{Z})_\bullet^*$ is isomorphic to the space of noncomutative polynomials on f_{2n+1} with the shuffle product.

Let Z_w be the $\mathbb{Q}$-vector space generated by the numbers $\zeta(n_1,...,n_l)$ of weight w. Then $Z_\bullet := \sum Z_w$ is obviously an algebra.

Conjecture 4.2. a) *The weight provides a grading on the algebra* $Z_\bullet$.

b) *One has an isomorphism of graded algebras*

$$Z_\bullet \quad = \quad \mathbb{Q}[\pi^2] \otimes UL(\mathbb{Z})_\bullet^* \qquad \deg(\pi^2) = 2.$$

In particular $Z_\bullet$ should have a natural structure of a graded Hopf algebra. Part a) means that relations between ζ's of different weight, like $\zeta(5) = \lambda \cdot \zeta(7)$ where $\lambda \in \mathbb{Q}$ are impossible.

Let $\mathcal{F}(2,3)$ be the free graded Lie algebra generated by 2 elements of degree -2 and -3. $U\mathcal{F}(2,3)^*$ is isomorphic as a graded vector space to the space of noncommutative polynomials in 2 variables p and g_3 of degrees 2 and 3. There is canonical isomorphism of graded vector spaces $\mathbb{Q}[\pi^2] \otimes UL_\bullet(\mathbb{Z})^* = U\mathcal{F}(2,3)^*$. The rule is clear from the pattern $(\pi^2)^3 f_3 (f_7)^3 (f_5)^2 \longrightarrow p^3 g_3 (g_3 p^2)^3 (g_3 p)^2$. In particular if $d_k := \dim Z_k$ then one should have $d_k = d_{k-2} + d_{k-3}$. Computer calculations of D.Zagier lead to this formula for $k \leq 12$. Just recently much more extensive calculations, also confirming the formula above, were made by D. Broadhurst.

Remark. Let $\pi_1^{(l)}(\mathbb{P}^1 \backslash \{0,1,\infty\})$ be the l-adic completion of the fundamental group. One has canonical homomorphism

$$\varphi^l : \quad \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}\, \pi_1^{(l)}(\mathbb{P}^1 \backslash \{0,1,\infty\}). \tag{24}$$

It was studied by Deligne, Ihara and others (see the beautiful talk [Ih] delivered by Ihara in ICM-90, Kyoto and references there). Conjecture (**??**) is closely related to some conjectures/questions of Deligne [D] and Ihara about the image of the map (**??**) and Drinfeld [Dr] about the structure of the Grothendiek-Teichmuller group. Namely, conjecture **??** means that the image of the map (**??**) should be as big as possible, i.e. the field stabilized by Ker φ^l is the maximal pro-l extension of the l^∞-cyclotomic extension of $\mathbb{Q}$ unramified outside l, and that the Lie algebra of the Grothendiek-Teichmuller group should be isomorphic to $L(\mathbb{Z})_\bullet$.

Theorem 4.3. *Let us assume* (??). *Then*

$$\dim Z_k \quad \leq \quad \text{the dimension of the degree } k \text{ part of} \quad U\mathcal{F}(2,3)^*. \tag{25}$$

This is a consequence of the part b) of the previous theorem.

2. On a generalization of conjecture ?? to multiple polylogarithms at roots of unity. Let $Z(N)_w$ be the space of $\mathbb{Q}$-span of the numbers

$$\bar{L}i_{n_1,\dots,n_l}(\zeta_N^{\alpha_1},\dots,\zeta_N^{\alpha_m}) := \frac{1}{(2\pi i)^w} Li_{n_1,\dots,n_l}(\zeta_N^{\alpha_1},\dots,\zeta_N^{\alpha_m}).$$

Then $Z(N)_\bullet := \sum Z(N)_w$ is a *bifiltered* by the weight and by the depth.

Let $L(S_N^*)_\bullet$ be the free graded Lie algebra generated by the $\mathbb{Q}$-vector spaces $K_{2n-1}(S_N^*) \otimes \mathbb{Q}$ in degrees $n \geq 1$. It is isomorphic to the Lie algebra of the (unipotent part of) the Galois group of the category $\mathcal{M}_T(S_N^*)$. This follows from (**??**) and the Tannakian formalism (see s.2.4).

Conjecture 4.4. a) *There exists a quotient $C(N)_\bullet$ of the graded Lie algebra $L(S_N^*)_\bullet$ such that one has an isomorphism of filtered* (*by the weight on the left and by the degree on the right*) *graded spaces*

$$Z(N)_{>0}/Z(N)_{>0}^2 = C(N)_\bullet^*.$$

One can deduce from theorem **??** and the motivic formalism the upper bound: $\dim Z(N)_w/\text{products} \leq \dim L(S_N^*)_w$.

Example. Using Borel's computation of K-groups of number fields [Bo] we get the following explicit description of the Lie algebra $L(S_p^*)_\bullet$.

a) If $p = 1$ then $L(\operatorname{Spec}\mathbb{Z})_\bullet$ is a free graded Lie algebra with one generator in all odd degrees starting from 3, (corresponding to $\bar{\zeta}(3), \bar{\zeta}(5), \dots$).

b) Let $p = 2$. Then $L(S_2^*)_\bullet$ is a free graded Lie algebra with one generator in each odd degree (corresponding to $(2\pi i)^{-1}\log 2, \bar{\zeta}(3), \bar{\zeta}(5), \dots$).

c) Let $p > 2$. Then $rkK_{2n-1}(S_p^*) = \frac{p-1}{2}, \quad n \geq 1$ and thus $L_\bullet(S_p^*)$ is a free Lie algebra with $(p-1)/2$ generators in each positive degree (related to $(2\pi i)^{-1}\log(1-\zeta_p^a), \bar{L}i_2(\zeta_p^a), \bar{L}i_2(\zeta_p^a), \dots$, where $1 \leq a \leq (p-1)/2$).

The conjecture **??** claims that this bound is exact for $p = 1$. Nothing like that could happen for any prime $p > 7$. Indeed, the only weight two multiple polylogarithms are the dilogarithm and the double logarithm, so by (**??**) $H^2_{(2)}(C(p)_\bullet) = H_1(X_1(p),\mathbb{Q})_+$. Therefore $H_1(X_1(p),\mathbb{Q})_+$ is isomorphic to the space of the 2-framed mixed motives over Spec $\mathbb{Z}[\zeta_p]$ "missed" by the multiple polylogarithms construction. So in general $C(N)_\bullet$ is not free and thus smaller then $L(S_N^*)_\bullet$. However recent computer calculations of D. Broadhurst [Br] show that for $p = 2$ the upper bound is exact for the weights ≤ 10.

3. Explicit description of the depth two Hopf algebra of the multiple polylogarithms. The framed mixed Hodge structures corresponding to multiple polylogarithms form a Hopf algebra. Its restriction to N-th roots of unity is the Hopf algebra $U(\mathcal{C}(N)_{\bullet})^{\vee}$ whose structure miraculously related with $GL(\mathbb{Z})$. The associated graded quotient with respect to the depth filtration of the Lie coalgebra $\mathcal{C}(N)^{\vee}_{\bullet}$ described explicitely in [G4]. Below we present the formulas in the first nontrivial case of the depth two polylogarithms. The general case will be treated in [G1] (see also [G3]).

Let $*$ be the product in the Hopf algebra. To simplify the formulas below, write $e^{x\cdot t}$ for $exp(\tilde{L}i_1(1-x)\cdot t)$. Recall $\Delta'(X) := \Delta(X) - (X\otimes 1 + 1\otimes X)$.

Let us extend formally the definition of $\tilde{L}i_{n_1,n_2}(x,y)$ to the case when $n_i \in \mathbb{Z}$

$$\tilde{L}i_n(x) = 0, \quad \text{and} \quad \tilde{L}i_{m,n}(x,y) = 0 \quad \text{if} \quad n \le 0,$$
$$\tilde{L}i_{m,n}(x,y) = -\tilde{L}i_{m+n}(xy) \quad \text{if} \quad m \le 0.$$

Set

$$\tilde{L}(x_1,x_2|t_1,t_2) := \sum_{n_1,n_2\in\mathbb{Z}} \tilde{L}i_{n_1,n_2}(x_1,x_2)t_1^{n_1-1}t_2^{n_2-1}.$$

Theorem 4.5. *The coproduct for the framed mixed Hodge structures related to multiple polylogarithms of depths* 1 *and* 2 *is given by the following formulas*:

$$\Delta' : \tilde{L}(x_1|t_1) \longmapsto \tilde{L}(x_1|t_1) \otimes (e^{x_1\cdot t_1} - 1) \tag{26}$$

$$\Delta' : \tilde{L}(x_1,x_2|t_1,t_2) \longmapsto \tag{27}$$

$$\tilde{L}(x_1,x_2|t_1,t_2) \otimes (e^{x_1\cdot t_1 + x_2\cdot t_2} - 1) + \tilde{L}(x_2|t_2) \otimes e^{x_2\cdot t_2} * \tilde{L}(x_1|t_1)$$
$$+ \tilde{L}(x_1x_2|t_1) \otimes e^{x_1x_2\cdot t_1} * \tilde{L}(x_2|t_2-t_1) - \tilde{L}(x_1x_2|t_2) \otimes e^{x_1x_2\cdot t_2} * \tilde{L}(x_1|t_1-t_2).$$

For example

$$\Delta' : \tilde{L}i_{2,1}(x_1,x_2) \;\longmapsto\; \tilde{L}i_{1,1}(x_1,x_2)\otimes x_1 + \tilde{L}i_1(x_2)\otimes\tilde{L}i_2(x_1) + \tilde{L}i_2(x_1x_2)\otimes\tilde{L}i_1(x_2)$$
$$-\tilde{L}i_1(x_1x_2)\otimes\Big(\tilde{L}i_2(x_1) + \tilde{L}i_2(x_2) - \tilde{L}i_1(x_2)*(x_1x_2) + \frac{x_1^2}{2}\Big).$$

4. The motivic complex of the double ζ's and cuspidal cohomology of $SL_2(\mathbb{Z})$ ([G3]). We define the framed Hodge-Tate structures $\tilde{\zeta}(1)$ and $\tilde{\zeta}(n,1)$ (corresponding to the divergent series) as the limiting mixed Hodge structures for $L_1(x)$ and $L_{n,1}(x,y)$ at $x=1$ and $x=y=1$. Then

$$\tilde{\zeta}(1) = 0, \qquad \tilde{\zeta}(1,1) = 0, \qquad \tilde{\zeta}(n,1) = -\sum_{i=2}^{n-1}\tilde{\zeta}(n-i,i) - \tilde{\zeta}(1,n) \quad (n>1).$$

For example $\tilde{\zeta}(3,1) = -\tilde{\zeta}(2,2) - 2\tilde{\zeta}(1,3)$. Setting $x_1 = x_2 = 1$ in (**??**) we get

$$\Delta' : \tilde{\zeta}(t_1,t_2) \longmapsto \tilde{\zeta}(t_2)\otimes\tilde{\zeta}(t_1) + \tilde{\zeta}(t_1)\otimes\tilde{\zeta}(t_2-t_1) - \tilde{\zeta}(t_2)\otimes\tilde{\zeta}(t_1-t_2). \tag{28}$$

One can show that $\tilde{\zeta}(2n) = 0$. Using this we get from (**??**)

$$\delta\tilde{\zeta}(t_1, t_2) = -\sum_{m,n>0} \tilde{\zeta}(m) \wedge \tilde{\zeta}(n) \cdot \Big(I + U + U^2\Big)(t_1^{m-1} t_2^{n-1}). \tag{29}$$

where U is the linear operator $(t_1, t_2) \to (t_1 - t_2, t_1)$.

Let $\mathcal{C}_w^{(2)}$ be the $\mathbb{Q}$-vector space spaned by $\tilde{\zeta}(n_1, n_2)$, $\quad n_1 + n_2 = w$ in the *Lie coalgebra of framed Hodge-Tate structures* (i.e. modulo the products $\tilde{\zeta}(n_1) * \tilde{\zeta}(n_2)$). Set $\mathcal{C}_n^{(1)} :=< \zeta(n) >_{\mathbb{Q}}$ and $\mathcal{C}_\bullet^{(1)} := \oplus \mathcal{C}_n^{(1)}$.

Theorem 4.6. (G3)

a) *One has a canonical isomorphism*

$$\text{Coker}\Big(\mathcal{C}_w^{(2)} \stackrel{\delta}{\longrightarrow} (\Lambda^2 \mathcal{C}_\bullet^{(1)})_w\Big) = H^1_{cusp}(SL_2(\mathbb{Z}), S^{w-2}V_2)^\vee.$$

b) *The kernel of this map is generated by $\tilde{\zeta}(w)$, so it is zero when w is even and one dimensional when w is odd.*

Here V_2 is the standard SL_2-module of dimension 2. Part a) follows immediately comparing (**??**) with the Eihler - Shimura theory of periods(see [La]).

From this we immediately conclude that $(k \geq 1)$

$$\dim \mathcal{C}_{2k+1}^{(2)} = 1, \qquad \dim \mathcal{C}_{2k}^{(2)} = [\frac{k-2}{2}] - \dim S_{2k}^0(SL_2(\mathbb{Z})) = [\frac{k}{3}] - 1. \tag{30}$$

For the l-adic version of the story a result similar to (**??**) was found independentely and simultaniuosly by Y. Ihara and N. Takao (unpublished, see the statement in the appendix to [Ma]). For $2k = 12$ see [Ih2]. (I am grateful to Y. Ihara and M. Matsumoto for this information). Let $d_w^{(2)}$ be the quotient of the $\mathbb{Q}$-vector space generated by $\zeta(n_1, n_2)$ modulo the products $\zeta(n_1)\zeta(n_2)$ where $n_1 + n_2 = w$. L. Euler ([E]) discovered that $d_{2k+1}^{(2)} \leq 1$. Using the double shuffle relations D. Zagier proved that $d_{2k}^{(2)}$ is not bigger than the right hand side of (**??**), and conjectured that this bound is exact. ([Z1-3]).

The Lie algebra $\mathcal{C}(N)_\bullet^\vee \otimes_{\mathbb{Q}} \mathbb{Q}_l$ should be isomorphic to the Lie algebra of the Zariski closure of the image of the canonical homomorphism

$$\varphi_N^l : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Der_{\text{Out}} Lie \pi_1^{(l)}(\mathbb{P}^1 \backslash \{0, \{\zeta_N^\alpha\}, \infty\}). \tag{31}$$

Here $\{\zeta_N^\alpha\}$ is the set of all N-th roots of unity.

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