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EQUIVARIANT RESOLUTION OF SINGULARITIES IN CHARACTERISTIC 0

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0. Introduction

We work over an algebraically closed field k of characteristic 0.

0.1. Statement. In this paper, we use techniques of toric geometry to reprove the following theorem:

Theorem 0.1. *Let X be a projective variety of finite type over k, and let $Z \subset X$ be a proper closed subset. Let $G \subset \mathrm{Aut}_k(Z \subset X)$ be a finite group. Then there is a G-equivariant modification $r : X_1 \to X$ such that X_1 is nonsingular projective variety, and $r^{-1}(Z_{red})$ is a G-strict divisor of normal crossings.*

This theorem is a weak version of the equivariant case of Hironaka's well known theorem on resolution of singularities. It was announced by Hironaka, but a complete proof was not easily accessible for a long time. The situation was remedied by E. Bierstone and P. Milman [B-M2], and by O. Villamayor [V]. They gave constructions of completely canonical resolution of singularities. These constructions are based on a thorough understanding of the effect of blowing up - one carefully build up an invariant pointing to the next blowup.

The proof we give in this paper takes a completely different approach. It uses two ingredients: first, we assume that we know the existence of resolution of singularities without group actions. The method of resolution is not important: any of [H], [B-M1], [V] [ℵ-dJ] or [B-P] would do. Second, we use equivariant toroidal resolution of singularities. Unfortunately, in [KKMS] the authors do not treat the equivariant case. But proving this turns out to be straightforward given the methods of [KKMS]. (For a similar argument in the toric case see [B].)

To this end, section 2 of this paper is devoted to proving the following:

Theorem 0.2. *Let $U \subset X$ be a strict toroidal embedding, and let $G \subset \mathrm{Aut}(U \subset X)$ be a finite group acting toroidally. Then there is a G-equivariant toroidal ideal sheaf $\mathcal{I}$ such that the normalized blowup of X along $\mathcal{I}$ is a nonsingular G-strict toroidal embedding.*

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1. Preliminaries

First recall some definitions. We restrict ourselves to the case of varieties over k. A large portion of the terminology is borrowed from [ℵ-dJ].

A *modification* is a proper birational morphism of irreducible varieties.

Let a finite group G act on a (possibly reducible) variety Z. Let $Z = \cup Z_i$ be the decomposition of Z into irreducible components. We say that Z *is G-strict* if the union of translates $\cup_{g\in G} g(Z_i)$ of each component Z_i is a normal variety. We simply say that Z is *strict* if it is G-strict for the trivial group, namely every Z_i is normal.

A divisor $D \subset X$ is called a *divisor of normal crossings* if étale locally at every point it is the zero set of $u_1 \cdots u_k$ where $u_1, \dots, u_k$ is part of a regular system of parameters. Thus, in a strict divisor of normal crossings D, all components of D are nonsingular.

An open embedding $U \hookrightarrow X$ is called a *toroidal embedding* if locally in the étale topology it is isomorphic to a torus embedding $T \hookrightarrow V$, (see [KKMS], II §1). One may replace "étale locally" by "complex analytically" in case $k = \mathbb{C}$, or "formally", obtaining the same class of embeddings. Let $E_i, i \in I$ be the irreducible components of $X \setminus U$. A finite group action $G \subset \mathrm{Aut}(U \hookrightarrow X)$ is said to be *toroidal* if the stabilizer of every point can be identified on the appropriate neighborhood with a subgroup of the torus T. We say that a toroidal action is *G-strict* if $X \setminus U$ is G-strict. In particular the toroidal embedding itself is said to be strict if $X \setminus U$ is strict. This is the same as the notion of *toroidal embedding without self-intersections* in [KKMS]. For any subset J of I, the components of the sets $\cap_{i\in J} E_i - \cup_{i\notin J} E_i$ define a stratification of X. Each component is called a *stratum*.

Recall that in [KKMS], p. 69-70 one defines the notion of a *conical polyhedral complex* with *integral structure*. As in [KKMS], p. 71, to every strict toroidal embedding $U \subset X$ one canonically associates a conical polyhedral complex with integral structure. In the sequel, when we refer to a conical polhedral complex, it is understood that it is endowed with an integral structure.

In [KKMS], p. 86 (Definition 2) one defines *a rational finite partial polyhedral decomposition* Δ' of a conical polyhedral complex Δ. We will restrict attention to the case where $|\Delta'| = |\Delta|$, and we will call this simply a *polyhedral decomposition* or *subdivision*.

The utility of polyhedral decompositions is given in Theorem 6* of [KKMS] (page 90), which establishes a correspondence between allowable modifications of a given strict toroidal embedding (which in our terminology are proper), and polyhedral decompositions of the associated conical polyhedral complex.

In order to guarantee that a modification is projective, one needs a bit more. Following [KKMS], p. 91, a function $\mathrm{ord} : \Delta \to \mathbb{R}$ defined on a conical polyhedral complex with integral structure is called an *order function* if:

(∗)
(1) $\operatorname{ord}(\lambda x) = \lambda \cdot \operatorname{ord}(x), \lambda \in \mathbb{R}^+$
(2) ord is continuous, piecewise-linear
(3) $\operatorname{ord}(N^Y \cap \sigma^Y) \subset \mathbb{Z}$ for all strata Y.
(4) ord is convex on each cone $\sigma \subset \Delta$

For an order function on the conical polyhedral complex coresponding to X, we can define canonically a coherent sheaf of fractional ideals on X, and vice versa (see [KKMS], I §2). The order function is positive if and only if the corresponding sheaf is a a genuine ideal sheaf. We have the following important theorem [KKMS]:

Theorem 1.1. *Let F be a coherent sheaf of ideals corresponding to a positive order function* ord_F*, and let $B_F(X)$ be the normalized blowup of X along F. Then $B_F(X) \to X$ is an allowable modification of X, described by the decompostion of $|\Delta|$ obtained by subdividing the cones into the biggest subcones on which* ord_F *is linear.*

A polyhedral decomposition is said to be *projective* if it is obtained in such a way from an order function. The corresponding modification is indeed a projective morphism.

Given a cone σ and a rational ray $\tau \subset \sigma$, it is natural to define the decomposition of σ centered at τ, whose cones are of the form $\sigma' + \tau$, where σ' runs over faces of σ disjoint from τ. Given a polyhedral complex Δ and a rational ray τ, we can take the subdivision of all cones containing τ centered at τ, and again call the resulting decompositionion of Δ, the subdivision centered at τ.

From [KKMS] I §2, lemmas 1-3, p. 33-35 it follows that the subdivision centered at τ is projective.

A very important decomposition is the barycentric subdivision. Let σ be a cone with integral structure, $e_1, \dots, e_k$ the integral generators of its edges. The *barycenter* of σ is the ray $b(\sigma) = \mathbb{R}_{\geq 0} \sum e_i$. The *barycentric subdivision* of a polyhedral complex Δ of dimension m is the minimal subdivision $B(\Delta)$ in which the barycenters of all cones in Δ appear as cones in $B(\Delta)$. It may be obtained by first taking the subdivision centered at the barycenters of m dimensional cones, then the decomposition of the resulting complex centered at the barycenters of the cones of dimension $m - 1$ of the *original* complex Δ, and so on. From the discussion above (or [KKMS] III §2 lemma 2.2), we have that the barycentric subdivision is projective.

One can also obtain the barycentric subdivision inductively the other way: the barycentric subdivision of an m-dimensional cone δ is formed by first taking the barycentric subdivision of all its faces, and for each one of the resulting cones σ, including also the cone $\sigma + b(\delta)$. This way it is clear that $B(\Delta)$ is a simplicial subdivision.

2. Equivariant toroidal modifications

Lemma 2.1. *Let $U \subset X$ be a strict toroidal embedding, $G \subset Aut(U \subset X)$ a finite group action. Then*

(1) *The group G acts linearly on $\Delta(X)$.*
(2) *Assume that the action of G is strict toroidal. Let $g \in G$, and let $\delta \subset \Delta(X)$ be a cone, such that $g(\delta) = \delta$. Then $g_{|\delta} = id$.*

Proof.

(1) Clearly, G acts on the stratification of $U \subset X$. Note that, from Definition 3 of [KKMS], page 59, $\Delta(X)$ is built up from the groups M^Y of Cartier divisors on $Star(Y)$ supported on $Star(Y) \setminus U$, as Y runs through the strata. As $g \in G$ canonically transforms M^Y to $M^{g^{-1}Y}$ linearly, our claim follows.
(2) Assume $g : \delta \to \delta$, and $g_{|\delta} \neq id$, then there exists an edge $e_1 \in \delta$, s.t $g(e_1) \neq e_1$. Denote $g(e_1) = e_2$. Assume e_1 corresponds to a divisor E_1, and e_2 corresponds to a divisor E_2. Since $g(e_1) = e_2$ we have $g(E_1) = E_2$. As e_1, e_2 are both edges of δ, $E_1 \cap E_2 \neq \phi$. So $\cup g(E_1)$ can not be normal since it has two intersecting components. This is a contradiction to the fact that G acts strictly on X. □

Lemma 2.2. *Let $G \subset \mathrm{Aut}(U \subset X)$ act toroidally. Let Δ_1 be a G-equivariant subdivision of Δ, with corresponding modification $X_1 \to X$. Then G acts toroidally on X_1. Moreover, if G acts strictly on X, it also acts strictly on X_1.*

Proof. The fact that G acts on X_1 follows from the canonical manner in which X_1 is costructed from the decomposition Δ_1, see Theorems 6* and 7* of [KKMS], II §2, p. 90.

Now for any point $a \in X_1$ and $g \in Stab_a$, we have $g \circ f(a) = f \circ g(a) = f(a)$ hence $g \in Stab_{f(a)}$, Thus $Stab_a$ is a subgroup of $Stab_{f(a)}$, which is identified with a subgroup of the torus in a neigbourhood of $f(a)$. This proved that $Stab_a$ is identified with a subgroup of the torus.

We are left with showing that if G acts strictly on X, then it acts strictly on X_1. Assume it is not the case. There exist two edges τ_1,τ_2 in Δ_1, which are both edges of a cone, δ', and $g(\tau_1) = \tau_2$. We choose the cone δ' of minimal dimension. Clearly, τ_1 and τ_2 cannot be both edges in Δ, since G acts strictly on X. Let us assume τ_2 is not an edge in Δ. So τ_2 must be in the interior of a cone δ in Δ, which contains δ'. Now since $\delta' \cap g(\delta') \supset \tau_2 \subset$ interior of δ, we conclude: interior of $\delta \cap g(\delta) \neq \phi$, which means that $g(\delta) = \delta$. From the previous lemma, $g_{|\delta} = id$, so $g_{|\delta'} = id$ too, contradiction. □

Proposition 2.3.

(1) *There is a one to one correspondence between edges τ_i in the barycentric subdivision $B(\Delta)$ and positive dimensional cones δ_i in Δ. We denote this by $\tau \mapsto \delta_\tau$.*
(2) *Let $\tau_i \neq \tau_j$ be edges of a cone $\hat{\delta} \in B(\Delta)$. Then dim $\delta_{\tau_i} \neq \delta_{\tau_j}$.*

(3) *If G is a finite group acting toroidally on a strict toroidal embedding $U \subset X$ with corresponding polyhedral complex Δ, then the action of G on $X_{B(\Delta)}$ is strict.*

Remark. Using this proposition, the argument at the end of [ℵ-dJ] can be significantly simplified: there is no need to show G-strictness of the toroidal embedding obtained there, since the barycentric subdivision automatically gives a G strict modification.

Proof.

1. Define a map b : positive dimensional cones in $\Delta \to$ edges in $B(\Delta)$ by
$$b(\delta) = \text{ the barycenter of } (\delta)$$
and define δ : edges in $B(\Delta) \to$ cones in Δ by
$$\delta_\tau = \text{ the unique cone whose interior contains } \tau$$
then it is easy to see that b and δ are invereses of each other.
2. We proceed by induction on $\dim \Delta$. The cone δ spanned by τ_i and τ_j must lie in some cone of Δ, say δ^*, which we may take of minimal dimension. We follow the second construction of the barycentric subdivision described in the preliminaries. Either $\dim \delta^* \leq m-1$, so δ is in the barycentric subdivision of the $m-1$-skeleton of Δ, in which case the statement follows by the inductive assumption, or $\dim \delta^* = m$, in which case only one of τ_1 and τ_2 can be its barycenter, and the other is again a barycenter of a cone in the $m-1$ skeleton.
3. From lemma 2.2, since the decomposition $B(\Delta)$ of Δ is equivariant, G acts toroidally on $X_B(\Delta)$. Let $E_1, E_2 = g(E_1) \subset X_B(\Delta) \setminus U$ be divisors corresponding to edges e_1, e_2 in $B(\Delta)$. If $E_1 \cap E_2 \neq \emptyset$, there is a cone in $B(\Delta)$ containing e_1, e_2 as edges. From part (2), $\dim \delta_{e_1} \neq \dim \delta_{e_2}$, so $g(e_1)$ can not equal to e_2. This contradicts the fact that the morphism is equivariant and $g(E_1) = E_2$.

Proposition 2.4. *There is a positive G-equivariant order function on $B(\Delta)$ such that the associated ideal $\mathcal{I}$ induces a blowing up $B_{\mathcal{I}} X_{B(\Delta)}$, which is a non-singular G-strict toroidal embedding, on which G acts toroidally.*

Proof. By the previous proposition, we know that G acts toridally and strictly on $X_{B(\Delta)}$. It follows from Lemma 2.1 that the quotient $B(\Delta)/G$ is a conical polyhedral complex, since no cone has two distinct edges in $B(\Delta)$ which are identified in the quotient. We can use the argument of [KKMS], I §2, lemmas 1-3, to get an order function ord : $B(\Delta)/G \to \mathbb{R}$ which induces a simplicial subdivision with every cell of index 1. Denote by $\pi : B(\Delta) \to B(\Delta)/G$ the quotient map. Then $ord \circ \pi$ is an order function subdividing $B(\Delta)$ into simplicial cones of index 1. Let $\mathcal{I}$ be the corresponding ideal sheaf. The blow up of $X_{B(\Delta)}$ along $\mathcal{I}$ is a nonsingular strict toroidal embedding $U \subset B_{\mathcal{I}} X_{B(\Delta)}$. By lemma 2.2, G acts on $B_\Gamma X_{B(\Delta)}$ strictly and toroidally. □

Proof of Theorem 0.2. Let $G \subset \mathrm{Aut}(U \subset X)$ be as in the theorem. The morphism $X_{B(\Delta)} \to X$ is projective, and by the last two propositions there is a projective, toroidal G-equivariant morphism $Y \to X$ where Y is nonsingular and such that G acts strictly and toroidally on Y. □

Remark. With a little more work we can obtain a *canonical* choice of a toroidal equivariant resolution of singularities. One observes that the cones in the barycentric subdivision have canonically ordered coordinates, which agree on intersecting cones: for a cone δ choose the unit coordinate vectors e_i to be primitive lattice vectors generating the edges τ, where $i = \dim \delta_\tau$, the dimension of the cone of which τ is a barycenter. Recall that in order to resolve singularities, one successively takes the subdivisions centered at lattice points w_j which are not integrally generated by the vectors e_i. These w_j are partially ordered according to the lexicographic ordering of their canonical coordinates, in such a way that if $w_j \neq w_k$ have the same coordinates (e.g. if $g(w_1) = w_2$), they do not lie in a the same cone, and therefore we can take the centered subdivision simultaneosely.

We conclude this section with a simple proposition about quotients:

Proposition 2.5. *Let $U \subset X$ be a strict toroidal embedding, and let $G \subset \mathrm{Aut}(U \subset X)$ be a finite group acting strictly and toroidally. Then $(X/G, U/G)$ is a strict toroidal embedding.*

Proof. Since the quotient of a toric variety by a finite subgroup of the torus is toric, we conclude that X/G is still a toroidal embedding, by the definition of toroidal embedding. We need to show that it is strict. Let $q : X \to X/G$ be the quotient map. Let $Z \subset X \setminus U$ be an irreducible component. Then $q(Z) = q(\cup_g g(Z))$. Since the action is strict, we have $q(\cup_g g(Z)) \simeq Z/Stab(Z)$, which is normal.

3. Proof of theorem 0.1

Given Z, X with G action, G finite, we may blow up Z and therefore we might as well assume that Z is a divisor. Let $Y = X/G$, Z/G be the quotient, B the branch locus. Define $W = B \cup Z/G$. Let $(Y', W') \to (Y, W)$ be a resolution of singularities of Y with W' a strict divisor of normal crossings. Let X' be the normalization of Y' in $K(X)$, and Z' the inverse image of W'. Let $U = X' \setminus Z'$. By Abhyankar's lemma, clearly $U \subset X'$ is a strict toroidal embedding, on which G acts toroidally (moreover, it is G-strict). Applying theorem 0.2 we obtain a nonsingular strict toroidal embedding $U \subset X_1 \to X'$ as required. □

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