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LAUMON'S RESOLUTION OF DRINFELD'S COMPACTIFICATION IS SMALL

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Let C be a smooth projective curve of genus 0. Let $\mathcal{B}$ be the variety of complete flags in an n-dimensional vector space V. Given an $(n-1)$-tuple α of positive integers one can consider the space $\mathcal{Q}_\alpha$ of algebraic maps of degree α from C to $\mathcal{B}$. This space has drawn much attention recently in connection with Quantum Cohomology (see e.g. [Giv], [Kon]). The space $\mathcal{Q}_\alpha$ is smooth but not compact (see e.g. [Kon]). The problem of compactification of $\mathcal{Q}_\alpha$ proved very important. One compactification $\mathcal{Q}_\alpha^K$ was constructed in loc. cit. (the space of *stable maps*). Another compactification $\mathcal{Q}_\alpha^L$ (the space of *quasiflags*), was constructed in [Lau]. However, historically the first and most economical compactification $\mathcal{Q}_\alpha^D$ (the space of *quasimaps*) was constructed by Drinfeld (early 80-s, unpublished). The latter compactification is singular, while the former ones are smooth. Drinfeld has conjectured that the natural map $\pi : \mathcal{Q}_\alpha^L \to \mathcal{Q}_\alpha^D$ is a small resolution of singularities. In the present note we prove this conjecture after the necessary recollections. The arguments in the proof are rather similar to those of [Lau], 3.3.2. In fact, the proof gives some additional information about the fibers of π. It appears that every fiber has a cell decomposition, i.e. roughly speaking, is a disjoint union of affine spaces. This permits to compute not only the stalks of IC sheaf on $\mathcal{Q}_\alpha^D$ but, moreover, the Hodge structure in these stalks. Namely, the Hodge structure is a pure Tate one, and the generating function for the IC stalks is just the Lusztig's q-analogue of Kostant's partition function (see [Lus]).

In conclusion, let us mention that the Drinfeld compactifications are defined for the space of maps into flag manifolds of arbitrary semisimple group, and it would be very interesting to construct their small resolutions.

1. The space of maps into flag variety

1.1. Notations. Let G be a complex semisimple simply-connected Lie group, $H \subset B$ its Cartan and Borel subgroups, N the unipotent radical of B, Y the lattice of coroots of G (with respect to H), l the rank of Y, $I = \{i_1, i_2, \dots i_l\}$ the set of simple coroots, R^+ the set of positive coroots, X the lattice of weights, X^+ the cone of dominant weights, $\Omega = \{\omega_1, \omega_2 \dots \omega_l\}$ the set of fundamental weights ($\langle \omega_k, i_{k'} \rangle = \delta_{kk'}$, where $\langle \bullet, \bullet \rangle$ stands for the natural pairing $X \otimes Y \to \mathbb{Z}$),

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$\mathcal{B} = G/B$ the flag variety and C a smooth projective curve of genus 0. Recall that there are canonical isomorphisms

$$\mathrm{H}_2(\mathcal{B}, \mathbb{Z}) \cong Y \qquad \mathrm{H}^2(\mathcal{B}, \mathbb{Z}) \cong X.$$

For $\lambda \in X$ let $\mathbf{L}_\lambda$ denote the corresponding G-equivariant line bundle on $\mathcal{B}$.

The map $\varphi : C \to \mathcal{B}$ has degree $\alpha \in \mathbb{N}[I] \subset Y$ if the following equivalent conditions hold:

(1) $\varphi_*([C]) = \alpha$;
(2) for any $\lambda \in X$ we have $\deg(\varphi^* \mathbf{L}_\lambda) = \langle \lambda, \alpha \rangle$.

We denote by $\mathcal{Q}_\alpha$ the space of algebraic maps from C to $\mathcal{B}$ of degree α. It is known that $\mathcal{Q}_\alpha$ is smooth variety and $\dim \mathcal{Q}_\alpha = 2|\alpha| + \dim \mathcal{B}$. In this paper we compare two natural compactifications of the space $\mathcal{Q}_\alpha$, which we presently describe.

1.2. Drinfeld's compactification. The Plücker embedding of the flag variety $\mathcal{B}$ gives rise to the following interpretation of $\mathcal{Q}_\alpha$.

For any irreducible representation V_λ ($\lambda \in X^+$) of G we consider the trivial vector bundle $\mathcal{V}_\lambda = V_\lambda \otimes \mathcal{O}_C$ over C.

For any G-morphism $\psi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ we denote by the same letter the induced morphism $\psi : \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$.

Then $\mathcal{Q}_\alpha$ is the space of collections of line subbundles $\mathcal{L}_\lambda \subset \mathcal{V}_\lambda,\ \lambda \in X^+$ such that:

a) $\deg \mathcal{L}_\lambda = -\langle \lambda, \alpha \rangle$;
b) For any nonzero G-morphism $\psi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have $\psi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = \mathcal{L}_\nu$;
c) For any G-morphism $\psi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\psi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = 0$.

Remark 1.2.1. Certainly, the property b) guarantees that in order to specify such a collection it suffices to give $\mathcal{L}_{\omega_k}$ for the set Ω of fundamental weights.

If we replace the curve C by a point, we get the Plücker description of the flag variety $\mathcal{B}$ as the space of collections of lines $L_\lambda \subset V_\lambda$ satisfying conditions of type (b) and (c) (thus $\mathcal{B}$ is embedded into $\prod_{\lambda \in X^+} \mathbb{P}(V_\lambda)$). Here, a Borel subgroup $\mathbf{B}$ in $\mathcal{B}$ corresponds to a system of lines $(L_\lambda, \lambda \in X^+)$ if lines are the fixed points of the unipotent radical of $\mathbf{B}$, $L_\lambda = (V_\lambda)^{\mathbf{N}}$, or equivalently, if $\mathbf{N}$ is the common stabilizer for all lines $\mathbf{N} = \bigcap_{\lambda \in X^+} G_{L_\lambda}$.

The following definition in case $G = SL_2$ appeared in [Dri].

Definition 1.2.2 (V. Drinfeld). The space $\mathcal{Q}^D_\alpha$ of *quasimaps* of degree α from C to $\mathcal{B}$ is the space of collections of invertible subsheaves $\mathcal{L}_\lambda \subset \mathcal{V}_\lambda,\ \lambda \in X^+$ such that:

a) $\deg \mathcal{L}_\lambda = -\langle \lambda, \alpha \rangle$;

b) For any nonzero G-morphism $\psi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have $\psi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = \mathcal{L}_\nu$;

c) For any G-morphism $\psi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\psi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = 0$.

Remark 1.2.3. Here is another version of the Definition, also due to V.Drinfeld. The principal affine space $\mathcal{A} = G/N$ is an H-torsor over $\mathcal{B}$. We consider its affine closure $\mathfrak{A}$, that is, the spectrum of the ring of functions on $\mathcal{A}$. The action of H extends to $\mathfrak{A}$ but it is not free anymore. Consider the quotient stack $\tilde{\mathcal{B}} = \mathfrak{A}/H$. The flag variety $\mathcal{B}$ is an open substack in $\tilde{\mathcal{B}}$. A map $\tilde{\phi}: C \to \tilde{\mathcal{B}}$ is nothing else than an H-torsor Φ over C along with an H-equivariant morphism $f: \Phi \to \mathfrak{A}$. The degree of this map is defined as follows.

Let $\chi_\lambda: H \to \mathbb{C}^*$ be the character of H corresponding to a weight $\lambda \in X$. Let $H_\lambda \subset H$ be the kernel of the morphism χ_λ. Consider the induced $\mathbb{C}^*$-torsor $\Phi_\lambda = \Phi/H_\lambda$ over C. The map $\tilde{\phi}$ has degree $\alpha \in \mathbb{N}[I]$ if

$$\text{for any } \lambda \in X \quad \text{we have} \quad \deg(\Phi_\lambda) = \langle \lambda, \alpha \rangle.$$

Definition 1.2.4. The space $\mathcal{Q}_\alpha^D$ is the space of maps $\tilde{\phi}: C \to \tilde{\mathcal{B}}$ of degree α such that the generic point of C maps into $\mathcal{B} \subset \tilde{\mathcal{B}}$.

The equivalence of 1.2.2 and 1.2.4 follows immediately from the Plücker embedding of $\mathfrak{A}$ into $\prod_{\lambda \in X^+} V_\lambda$.

Proposition 1.2.5. *$\mathcal{Q}_\alpha^D$ is a projective variety.*

Proof. The space $\mathcal{Q}_\alpha^D$ is naturally embedded into the space

$$\prod_{k=1}^{l} \mathbb{P}(\mathrm{Hom}(\mathcal{O}_C(-\langle \omega_k, \alpha \rangle), \mathcal{V}_{\omega_k}))$$

and is closed in it. □

1.3. The stratification of the Drinfeld's compactification. In this subsection we will introduce the stratification of the space of quasimaps.

Configurations of I-colored divisors.

Let us fix $\alpha \in \mathbb{N}[I] \subset Y$, $\alpha = \sum_{k=1}^{l} a_k i_k$. Consider the configuration space C^α of colored effective divisors of multidegree α (the set of colors is I). The dimension of C^α is equal to the length $|\alpha| = \sum_{k=1}^{l} a_k$.

Multisubsets of a set S are defined as elements of some symmetric power $S^{(m)}$ and we denote the image of $(s_1, \ldots, s_m) \in S^m$ by $\{\{s_1, \ldots, s_m\}\}$. We denote by $\Gamma(\alpha)$ the set of all partitions of α, i.e. multisubsets $\Gamma = \{\{\gamma_1, \ldots, \gamma_m\}\}$ of $\mathbb{N}[I]$ with $\sum_{r=1}^{m} \gamma_r = \alpha$, $\gamma_r > 0$.

For $\Gamma \in \Gamma(\alpha)$ the corresponding stratum C^α_Γ is defined as follows. It is formed by configurations which can be subdivided into m groups of points, the r-th group containing γ_r points; all the points in one group equal to each other, the different groups being disjoint. For example, the main diagonal in C^α is the closed stratum given by partition $\alpha = \alpha$, while the complement to all diagonals in C^α is the open stratum given by partition

$$\alpha = \sum_{k=1}^{l} (\underbrace{i_k + i_k + \ldots + i_k}_{a_k \text{ times}}).$$

Evidently, $C^\alpha = \bigsqcup_{\Gamma \in \Gamma(\alpha)} C^\alpha_\Gamma$, for $\Gamma = \{\{\gamma_1, \ldots, \gamma_m\}\}$ we have $\dim C^\alpha_\Gamma = m$.

Normalization and defect of subsheaves.

Let F be a vector bundle on the curve C and let E be a subsheaf in F. Let $F/E = T(E) \oplus L$ be the decomposition of the quotient sheaf F/E into the sum of its torsion subsheaf and a locally free sheaf, and let $\tilde{E} = \operatorname{Ker}(F \to L)$ be the kernel of the natural map $F \to L$. Then $\tilde{E}$ is a vector subbundle in F which contains E and has the following universal property:

for any subbundle $\mathcal{E}' \subset F$ if $\mathcal{E}'$ contains E then $\mathcal{E}'$ contains also $\tilde{E}$.

Moreover, $\operatorname{rank} \tilde{E} = \operatorname{rank} E$, $\tilde{E}/E \cong T(E)$ and $c_1(\tilde{E}) = c_1(E) + \ell(T(E))$ (for any torsion sheaf on C we denote by $\ell(T)$ its length).

Definition 1.3.1. We will call $\tilde{E}$ the *normalization* of E in F and $T(E)$ the *defect* of E.

Remark 1.3.2. If $\tilde{E}$ is the normalization of E in F then $\Lambda^k(\tilde{E})$ is the normalization of $\Lambda^k E$ in $\Lambda^k F$.

For any $x \in C$ and torsion sheaf T on C we will denote by $\ell_x(T)$ the length of the localization of T in the point x.

Definition 1.3.3. For any quasimap $\varphi = (\mathcal{L}_\lambda \subset \mathcal{V}_\lambda)_{\lambda \in X^+} \in \mathcal{Q}^D_\alpha$ we define the *normalization* of φ as follows:

$$\tilde{\varphi} = (\tilde{\mathcal{L}}_\lambda \subset \mathcal{V}_\lambda),$$

and the *defect* of φ as follows:

$$\operatorname{def}(\varphi) = (T(\mathcal{L}_\lambda)),$$

(the defect of φ is a collection of torsion sheaves).

Proposition 1.3.4. *For any $\varphi \in \mathcal{Q}^D_\alpha$ there exists $\beta \leq \alpha \in \mathbb{N}[I]$, partition $\Gamma = (\gamma_1, \ldots, \gamma_m) \in \Gamma(\alpha - \beta)$ and a divisor $D = \sum_{r=1}^{m} \gamma_r x_r \in C^{\alpha-\beta}_\Gamma$ such that*

$$\tilde{\varphi} \in \mathcal{Q}_\beta, \qquad \ell_x(\operatorname{def}(\varphi)_\lambda) = \begin{cases} \langle \lambda, \gamma_r \rangle, & \text{if } x = x_r \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Clear. □

Definition 1.3.5. The pair (β, Γ) will be called the *type of degeneration* of φ. We denote by $\mathfrak{D}_{\beta,\Gamma}$ the subspace of $\mathcal{Q}_\alpha^D$ consisting of all quasimaps φ with the given type of degeneration.

Remark 1.3.6. Note that $\mathfrak{D}_{\alpha,\emptyset} = \mathcal{Q}_\alpha$.

We have

$$\mathcal{Q}_\alpha^D = \bigsqcup_{\substack{\beta \leq \alpha \\ \Gamma \in \Gamma(\alpha-\beta)}} \mathfrak{D}_{\beta,\Gamma}. \tag{1}$$

The map $d_{\beta,\Gamma} : \mathfrak{D}_{\beta,\Gamma} \to \mathcal{Q}_\beta \times C_\Gamma^{\alpha-\beta}$ which sends φ to $(\tilde{\varphi}, D)$ (see 1.3.4) is an isomorphism. The inverse map $\sigma_{\beta,\Gamma}$ can be constructed as follows. Let $\varphi = (\mathcal{L}_\lambda) \in \mathcal{Q}_\beta$. Then

$$\sigma_{\beta,\Gamma}(\varphi, D) \overset{\text{def}}{=} (\mathcal{L}'_\lambda) \qquad \mathcal{L}'_\lambda \overset{\text{def}}{=} \bigcap_{r=1}^{m} \mathfrak{m}_{x_r}^{\langle \lambda, \gamma_r \rangle} \cdot \mathcal{L}_\lambda,$$

where $\mathfrak{m}_x$ denotes the sheaf of ideals of the point $x \in C$.

1.4. Laumon's compactification. Let V be an n-dimensional vector space. From now on we will assume that $G = SL(V)$ (in this case certainly $l = n-1$). In this case there is the Grassmann embedding of the flag variety, namely

$$\mathcal{B} = \{(U_1, U_2, \ldots, U_{n-1}) \in G_1(V) \times G_2(V) \times \cdots \times G_{n-1}(V) \mid U_1 \subset U_2 \subset \cdots \subset U_{n-1}\},$$

where $G_k(V)$ is the Grassmann variety of k-dimensional subspaces in V. This embedding gives rise to another interpretation of $\mathcal{Q}_\alpha$.

We will denote by $\mathcal{V}$ the trivial vector bundle $V \otimes \mathcal{O}_C$ over C. Let $\alpha = \sum_{k=1}^{n-1} a_k i_k$, where i_k is the simple coroot dual to the highest weight ω_k of representation G in $\Lambda^k V$.

Then $\mathcal{Q}_\alpha$ is the space of complete flags of vector subbundles

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_{n-1} \subset \mathcal{V} \text{ such that } \operatorname{rank} \mathcal{E}_k = k \\ \text{and } c_1(\mathcal{E}_k) = -\langle \omega_k, \alpha \rangle = -a_k.$$

Definition 1.4.1 (Laumon, [Lau, 4.2]**).** The space $\mathcal{Q}_\alpha^L$ of *quasiflags* of degree α is the space of complete flags of locally free subsheaves

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_{n-1} \subset \mathcal{V} \text{ such that } \operatorname{rank} \mathcal{E}_k = k \\ \text{and } c_1(E_k) = -\langle \omega_k, \alpha \rangle = -a_k.$$

It is known that $\mathcal{Q}_\alpha^L$ is a smooth projective variety of dimension $2|\alpha| + \dim \mathcal{B}$ (see loc. cit., Lemma 4.2.3).

1.5. The stratification of the Laumon's compactification. There is a stratification of the space $\mathcal{Q}^L_\alpha$ similar to the above stratification of $\mathcal{Q}^D_\alpha$.

Definition 1.5.1. For any quasiflag $E_\bullet = (E_1, \dots, E_{n-1})$ we define its *normalization* as

$$\tilde{E}_\bullet = (\tilde{E}_1, \dots, \tilde{E}_{n-1}), \text{ where } \tilde{E}_k \text{ is the normalization of } E_k \text{ in } \mathcal{V}$$

and *defect*

$$\operatorname{def}(E_\bullet) = (\tilde{E}_1/E_1, \dots, \tilde{E}_{n-1}/E_{n-1}).$$

Thus, the defect of $E_\bullet$ is a collection of torsion sheaves.

Proposition 1.5.2. *For any $E_\bullet \in \mathcal{Q}^L_\alpha$ there exist $\beta \le \alpha \in \mathbb{N}[I]$, partition $\Gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma(\alpha - \beta)$ and a divisor $D = \sum\limits_{r=1}^{m} \gamma_r x_r \in C_\Gamma^{\alpha-\beta}$ such that*

$$\tilde{E}_\bullet \in \mathcal{Q}_\beta, \qquad \ell_x(\operatorname{def}(E_k)) = \begin{cases} \langle \omega_k, \gamma_r \rangle, & \text{if } x = x_r \\ 0, & \text{otherwise} \end{cases}.$$

Definition 1.5.3. The pair (β, Γ) will be called the *type of degeneration* of $E_\bullet$. We denote by $\mathfrak{L}_{\beta,\Gamma}$ the subspace in $\mathcal{Q}^L_\alpha$ consisting of all quasiflags $E_\bullet$ with the given type of degeneration.

Remark 1.5.4. Note that $\mathfrak{L}_{\alpha,\emptyset} = \mathcal{Q}_\alpha$.

We have

$$\mathcal{Q}^L_\alpha = \bigsqcup_{\substack{\beta \le \alpha \\ \Gamma \in \Gamma(\alpha-\beta)}} \mathfrak{L}_{\beta,\Gamma}. \tag{2}$$

1.6. The map from $\mathcal{Q}^L_\alpha$ to $\mathcal{Q}^D_\alpha$. Consider the map $\pi : \mathcal{Q}^L_\alpha \to \mathcal{Q}^D_\alpha$ which sends a quasiflag of degree α $E_\bullet \in \mathcal{Q}^L_\alpha$ to a quasimap given by the collection $(\mathcal{L}_{\omega_k})_{k=1}^{n-1}$ (see Remark 1.2.1) where $\mathcal{L}_{\omega_k} = \Lambda^k E_k \subset \Lambda^k \mathcal{V} = \mathcal{V}_{\omega_k}$.

Proposition 1.6.1. *Let $E_\bullet$ be a quasiflag of degree α and let (β, Γ) be its type of degeneration. Then $\pi(E_\bullet)$ is a quasimap of degree α and its type of degeneration is (β, Γ).*

Proof. Obviously we have $\deg \mathcal{L}_{\omega_k} = \deg \Lambda^k E_k = c_1(E_k) = -\langle \omega_k, \alpha \rangle$ which means that $\pi(E_\bullet) \in \mathcal{Q}^D_\alpha$. According to the Remark 1.3.2, $\tilde{\mathcal{L}}_{\omega_k} = \Lambda^k \tilde{E}_k$ (i.e. $(\tilde{\mathcal{L}}_{\omega_k}) \in \mathcal{Q}_\beta$), hence

$$\ell_x(\tilde{\mathcal{L}}_{\omega_k}/\mathcal{L}_{\omega_k}) = \ell_x(\tilde{E}_k/E_k). \tag{3}$$

This proves the Proposition. $\square$

Remark 1.6.2. Note that (3) implies that π preserves not only β and Γ but also D (see 1.3.4, 1.5.2).

Recall that a proper birational map $f : \mathcal{X} \to \mathcal{Y}$ is called *small* if the following condition holds: let $\mathcal{Y}_m$ be the set of all points $y \in \mathcal{Y}$ such that $\dim f^{-1}(y) \geq m$. Then for $m > 0$ we have

$$\operatorname{codim} \mathcal{Y}_m > 2m. \tag{4}$$

Main Theorem. The map π is a small resolution of singularities.

2. The fibers of π

2.1. We fix $\mathcal{E}_\bullet \in \mathcal{Q}_\beta$, a partition $\Gamma \in \Gamma(\alpha - \beta)$, and a divisor $D \in C_\Gamma^{\alpha-\beta}$. Then $(\mathcal{E}_\bullet, D) \in \mathfrak{D}_{\beta,\Gamma}$. We define $F(\mathcal{E}_\bullet, D)$ as $\pi^{-1}(\mathcal{E}_\bullet, D)$.

Let $D = \sum_{r=1}^m \gamma_r x_r$. We define the space $\mathcal{F}(\mathcal{E}_\bullet, D)$ of commutative diagrams

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r] \arrow[d, "\varepsilon_1"'] & \mathcal{E}_2 \arrow[r] \arrow[d, "\varepsilon_2"'] & \dots \arrow[r] & \mathcal{E}_{n-1} \arrow[d, "\varepsilon_{n-1}"'] \\
T_1 \arrow[r, "\tau_1"] & T_2 \arrow[r, "\tau_2"] & \dots \arrow[r, "\tau_{n-2}"] & T_{n-1}
\end{tikzcd}$$

such that

a) ε_k is surjective,

b) T_k is torsion,

c) $\ell_x(T_k) = \begin{cases} \langle \omega_k, \gamma_r \rangle, & \text{if } x = x_r \\ 0, & \text{otherwise} \end{cases}.$

Lemma 2.1.1. *We have an isomorphism*

$$F(\mathcal{E}_\bullet, D) \cong \mathcal{F}(\mathcal{E}_\bullet, D).$$

Proof. If $E_\bullet \in F(\mathcal{E}_\bullet, D)$ then by the 1.5.2 the collection $(T_1, \dots, T_{n-1}) = \operatorname{def}(E_\bullet)$ satisfies the above conditions.

Vice versa, if the collection $(T_1, \dots, T_k)$ satisfies the above conditions, then consider

$$E_k = \operatorname{Ker}(\mathcal{E}_k \xrightarrow{\varepsilon_k} T_k).$$

Since the square

$$\begin{tikzcd}
\mathcal{E}_k \arrow[r, "\varepsilon_k"] \arrow[d] & T_k \arrow[d, "\tau_k"'] \\
\mathcal{E}_{k+1} \arrow[r, "\varepsilon_{k+1}"] & T_{k+1}
\end{tikzcd}$$

commutes, we can extend it to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_k \arrow[r] \arrow[d] & \mathcal{E}_k \arrow[r, "\varepsilon_k"] \arrow[d] & T_k \arrow[r] \arrow[d, "\tau_k"'] & 0 \\
0 \arrow[r] & E_{k+1} \arrow[r] & \mathcal{E}_{k+1} \arrow[r, "\varepsilon_{k+1}"] & T_{k+1} \arrow[r] & 0
\end{tikzcd}$$

The induced morphism $E_k \to E_{k+1}$ is injective because $\mathcal{E}_k \to \mathcal{E}_{k+1}$ is, and

$$c_1(E_k) = c_1(\mathcal{E}_k) - \ell(T_k) = -\langle \omega_k, \beta \rangle - \sum_{x \in C} \ell_x(T_k) =$$
$$= -\langle \omega_k, \beta \rangle - \sum_{r=1}^{m} \langle \omega_k, \gamma_r \rangle = -\langle \omega_k, \beta + (\alpha - \beta) \rangle = -\langle \omega_k, \alpha \rangle.$$

This means that $E_\bullet \in F(\mathcal{E}_\bullet, D)$. □

Proposition 2.1.2. *If* $D = \sum_{r=1}^{m} \gamma_r x_r$ *is a decomposition into disjoint divisors then*

$$F(\mathcal{E}_\bullet, D) \cong \prod_{r=1}^{m} F(\mathcal{E}_\bullet, \gamma_r x_r). \tag{5}$$

Proof. Recall that if T is a torsion sheaf on the curve C then

$$T = \bigoplus_{x \in C} T_x,$$

where T_x is the localization of T in the point x. Given a locally free sheaf $\mathcal{E}$ we have $\operatorname{Hom}(\mathcal{E}, T) = \bigoplus_{x \in C} \operatorname{Hom}(\mathcal{E}, T_x)$ and a morphism $\varepsilon \in \operatorname{Hom}(\mathcal{E}, T)$ is surjective iff all its components $\varepsilon_x \in \operatorname{Hom}(\mathcal{E}, T_x)$ are. This remark together with Lemma 2.1.1 proves the Proposition. □

The above Proposition implies, that in order to describe general fiber $F(\mathcal{E}_\bullet, D)$ it is enough to have a description of the fibers $F(\mathcal{E}_\bullet, \gamma x)$, which we will call *simple fibers.*

2.2. The stratification of a simple fiber. We will need the following obvious Lemma.

Lemma 2.2.1. *Let* $\mathcal{E}$ *be a vector bundle on* C. *Let* $\mathcal{E}' \subset \mathcal{E}$ *be a vector subbundle, and let* $E \subset \mathcal{E}$ *be a (necessarily locally free) subsheaf. Then* $E' = \mathcal{E}' \cap E$ *is a vector subbundle in* E.

Moreover, the commutative square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E} \end{array}$$

can be extended to the commutative diagram

$$\begin{array}{ccccc}
E' & \longrightarrow & E & \longrightarrow & E/E' \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/\mathcal{E}' \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{E}'/E' & \longrightarrow & \mathcal{E}/E & \longrightarrow & \dfrac{\mathcal{E}/E}{\mathcal{E}'/E'} \cong \dfrac{\mathcal{E}/\mathcal{E}'}{E/E'}
\end{array}$$

in which both the rows and the columns form the short exact sequences.

The sheaf in the lower-right corner of the diagram will be called *cointersection* of E and $\mathcal{E}'$ inside $\mathcal{E}$ and denoted by $\nabla_{\mathcal{E}}\left(E, \mathcal{E}'\right)$.

Let

$$\gamma = \sum_{k=1}^{n-1} c_k i_k.$$

For every $E_\bullet \in F(\mathcal{E}_\bullet, \gamma x)$ we define

$$\mu_{pq}(E_\bullet) \overset{\text{def}}{=} \ell\left(\frac{\mathcal{E}_q}{E_p \cap \mathcal{E}_q}\right) \qquad (1 \le q \le p \le n-1), \tag{6}$$

$$\nu_{pq}(E_\bullet) = \begin{cases} \mu_{pq}(E_\bullet) - \mu_{p+1,q}(E_\bullet), & \text{if } 1 \le q \le p < n-1 \\ \mu_{pq}(E_\bullet), & \text{if } 1 \le q \le p = n-1 \end{cases} \tag{7}$$

$$\kappa_{pq}(E_\bullet) = \begin{cases} \nu_{pq}(E_\bullet) - \nu_{p,q-1}(E_\bullet), & \text{if } 1 < q \le p \le n-1 \\ \nu_{pq}(E_\bullet), & \text{if } 1 = q \le p \le n-1 \end{cases} \tag{8}$$

Remark 2.2.2. The transformations (7) and (8) are invertible, so the integers μ_{pq} can be uniquely reconstructed from ν_{pq} or κ_{pq}. Namely,

$$\nu_{pq} = \sum_{r=1}^{q} \kappa_{pr}; \qquad \mu_{pq} = \sum_{s=p}^{n-1} \nu_{sq} = \sum_{r \le q \le p \le s} \kappa_{sr}. \tag{9}$$

Lemma 2.2.3. *We have*

$$\nu_{pq}(E_\bullet) = \ell\left(\frac{\mathcal{E}_q \cap E_{p+1}}{\mathcal{E}_q \cap E_p}\right). \tag{10}$$

$$\kappa_{pq}(E_\bullet) = \ell\left(\nabla_{\mathcal{E}_q \cap E_{p+1}}\left(\mathcal{E}_q \cap E_p, \mathcal{E}_{q-1} \cap E_{p+1}\right)\right). \tag{11}$$

Proof. The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_q \cap E_p & \longrightarrow & \mathcal{E}_q & \longrightarrow & \dfrac{\mathcal{E}_q}{\mathcal{E}_q \cap E_p} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{E}_q \cap E_{p+1} & \longrightarrow & \mathcal{E}_q & \longrightarrow & \dfrac{\mathcal{E}_q}{\mathcal{E}_q \cap E_{p+1}} & \longrightarrow & 0
\end{array}$$

implies (10). In order to prove (11) note that

$$\mathcal{E}_{q-1} \cap E_p = (\mathcal{E}_q \cap E_p) \cap (\mathcal{E}_{q-1} \cap E_{p+1}),$$

and apply Lemma 2.2.1 and (10). □

Corollary 2.2.4. *Integers μ_{pq}, ν_{pq} and κ_{pq} satisfy the following inequalities:*

$$0 \le \kappa_{pq}, \tag{12}$$

$$0 \le \nu_{p1} \le \nu_{p2} \le \cdots \le \nu_{pp}, \tag{13}$$

$$0 \le \mu_{n-1,q} \le \mu_{n-2,q} \le \cdots \le \mu_{qq} = c_q. \tag{14}$$

Proof. See (11),(8),(7) and compare the definition of μ_{qq} with 2.1.1. □

We will denote by $[p, q]$ the positive coroot

$$[p, q] \overset{\text{def}}{=} \sum_{k=q}^{p} i_k \in R^+. \tag{15}$$

Lemma 2.2.5. *For any $E_\bullet \in F(\mathcal{E}_\bullet, \gamma x)$ we have*

$$\sum_{1 \le q \le p \le n-1} \kappa_{pq}(E_\bullet)[p, q] = \gamma.$$

Proof. Applying (8), (15) and (9) we get

$$\sum_{1 \le q \le p \le n-1} \kappa_{pq}[p, q] = \sum_{1 \le q \le p \le n-1} (\nu_{pq} - \nu_{p,q-1})[p, q] =$$

$$\sum_{1 \le q \le p \le n-1} \nu_{pq}([p, q] - [p, q+1]) = \sum_{1 \le q \le p \le n-1} \nu_{pq} i_q = \sum_{q=1}^{n-1} \left(\sum_{p=q}^{n-1} \nu_{pq} \right) i_q = \sum_{q=1}^{n-1} \mu_{qq} i_q.$$

Now Lemma follows from (14). □

Let $\mathfrak{K}(\gamma)$ be the set of all partitions of $\gamma \in \mathbb{N}[I]$ into the sum of positive coroots: $\gamma = \sum_{s=1}^{t} \delta_s$, where $\delta_s \in R^+$ (note that $\mathfrak{K}(\gamma) \ne \Gamma(\gamma)$). In other words, since every positive coroot for $G = SL(V)$ is equal to $[p, q]$ for some p, q,

$$\mathfrak{K}(\gamma) = \{(\kappa_{pq})_{1 \le q \le p \le n-1} \mid \kappa_{pq} \ge 0 \quad \text{and} \quad \sum_{1 \le q \le p \le n-1} \kappa_{pq}[p, q] = \gamma\}.$$

Let $\mathfrak{M}(\gamma)$ denote the set of all collections (μ_{pq}) which can be produced from some $(\kappa_{pq}) \in \mathfrak{K}(\gamma)$ as in (9).

The Lemma 2.2.5 implies that for any $E_\bullet \in F(\mathcal{E}_\bullet, \gamma x)$ we have $(\mu_{pq}(E_\bullet)) \in \mathfrak{M}(\gamma)$. Define the stratum $\mathfrak{S}((\mu_{pq})_{1\le q\le p\le n-1}, (\mathcal{E}_k)_{k=1}^{n-1})$ as follows:

$$\mathfrak{S}((\mu_{pq})_{1\le q\le p\le n-1}, (\mathcal{E}_k)_{k=1}^{n-1}) = \{E_\bullet \in F(\mathcal{E}_\bullet, \gamma x) \mid \mu_{pq}(E_\bullet) = \mu_{pq}\}.$$

To unburden the notations in the cases when it is clear which flag $\mathcal{E}_\bullet$ is used we will write just $\mathfrak{S}_\mu$. We have obviously

$$F(\mathcal{E}_\bullet, \gamma x) = \bigsqcup_{\mu\in\mathfrak{M}(\gamma)} \mathfrak{S}_\mu. \tag{16}$$

Remark 2.2.6. We will also use the similar varieties $\mathfrak{S}((\mu_{pq})_{1\le q\le p\le N}, (\mathcal{E}_k)_{k=1}^{N})$ that can be defined in the same way for any short flag $(\mathcal{E}_k)_{k=1}^{N}$ (that is a flag of subbundles $\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_N$ with $\operatorname{rank}\mathcal{E}_k = k$: we want to emphasize that though in all short flags appearing in this paper $\mathcal{E}_N$ is a subsheaf in $\mathcal{V}$, but it is *not* a subbundle, nevertheless all $\mathcal{E}_k$ are *subbundles* in $\mathcal{E}_N$).

2.3. The strata $\mathfrak{S}_\mu$. In order to study $\mathfrak{S}_\mu$ we will introduce some more varieties.

For every $1 \le N \le n-1$, a short flag of subbundles $(\mathcal{E}_k)_{k=1}^{N}$ (see Remark 2.2.6) and a collection of integers $(\nu_k)_{k=1}^{N}$ such that $0 \le \nu_1 \le \cdots \le \nu_N$, we define the space $\mathfrak{T}((\nu_k)_{k=1}^{N}, (\mathcal{E}_k)_{k=1}^{N})$ as follows:

$$\begin{aligned}\mathfrak{T}((\nu_k)_{k=1}^{N}, (\mathcal{E}_k)_{k=1}^{N}) = \{E \subset \mathcal{E}_N \mid \operatorname{rank}(E) = N \quad &\text{and} \\ \operatorname{supp}\left(\frac{\mathcal{E}_k}{\mathcal{E}_k\cap E}\right) = \{x\}, \quad &\ell\left(\frac{\mathcal{E}_k}{\mathcal{E}_k\cap E}\right) = \nu_k\}.\end{aligned} \tag{17}$$

We define *pseudoaffine* spaces by induction in dimension. First, the affine line $\mathbb{A}^1$ is a pseudoaffine space. Now a space A is called pseudoaffine if it admits a fibration $A \to B$ with pseudoaffine fibers and pseudoaffine B.

Theorem 2.3.1. *The space $\mathfrak{T}((\nu_k)_{k=1}^{N},(\mathcal{E}_k)_{k=1}^{N})$ is pseudoaffine of dimension $\sum_{k=1}^{N-1}\nu_k$.*

Proof. We use induction in N. The case $N = 1$ is trivial. There is only one subsheaf E in the line bundle $\mathcal{E}_1$ with $\operatorname{supp}(\mathcal{E}_1/E) = \{x\}$ and $\ell(\mathcal{E}_1/E) = \nu_1$, namely $E = \mathfrak{m}_x^{\nu_1} \cdot \mathcal{E}_1$. This means that $\mathfrak{T}(\nu_1, \mathcal{E}_1)$ is a point and the base of induction follows.

If $N > 1$ then consider the map

$$\tau : \mathfrak{T}((\nu_k)_{k=1}^{N}, (\mathcal{E}_k)_{k=1}^{N}) \to \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1}),$$

which sends $E \in \mathfrak{T}((\nu_k)_{k=1}^{N}, (\mathcal{E}_k)_{k=1}^{N})$ to $E' = E \cap \mathcal{E}_{N-1} \in \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1})$.

Lemma 2.3.2. *Let* $L = \mathfrak{m}_x^{\nu_N - \nu_{N-1}} \cdot \left(\dfrac{\mathcal{E}_N}{\mathcal{E}_{N-1}} \right)$. *For any* $E' \in \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1})$ *there is an isomorphism*

$$\tau^{-1}(E') \cong \mathrm{Hom}(L, \mathcal{E}_{N-1}/E') \cong \mathbb{A}^{\ell(\mathcal{E}_{N-1}/E')} = \mathbb{A}^{\nu_{N-1}}.$$

Thus, the space $\mathfrak{T}((\nu_k)_{k=1}^{N}, (\mathcal{E}_k)_{k=1}^{N})$ is affine fibration over a pseudoaffine space, hence it is pseudoaffine and its dimension is equal to

$$\dim \left(\mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1}) \right) + \nu_{N-1} = \sum_{k=1}^{N-2} \nu_k + \nu_{N-1} = \sum_{k=1}^{N-1} \nu_k.$$

The Theorem is proved. □

Proof of Lemma 2.3.2. Let $E \in \tau^{-1}(E')$. Since $E' = E \cap \mathcal{E}_{N-1}$ we can apply Lemma 2.2.1 which gives the following commutative diagram:

$$\begin{array}{ccccc}
E' & \longrightarrow & E & \longrightarrow & L \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_{N-1} & \xrightarrow{j} & \mathcal{E}_N & \xrightarrow{\psi} & \mathcal{E}_N/\mathcal{E}_{N-1} \\
\downarrow & & \downarrow & & \downarrow \\
T_{N-1} & \longrightarrow & T_N & \longrightarrow & T_N/T_{N-1}
\end{array}$$

(Note that since $\mathcal{E}_N/\mathcal{E}_{N-1}$ is a line bundle and $\ell(T_N/T_{N-1}) = \ell(T_N) - \ell(T_{N-1}) = \nu_N - \nu_{N-1}$ the kernel of natural map $\mathcal{E}_N/\mathcal{E}_{N-1} \to T_N/T_{N-1}$ is isomorphic to L.) Let $\tilde{\mathcal{E}}_N = \psi^{-1}(L)$. Then E is contained in $\tilde{\mathcal{E}}_N$ and we have the following commutative diagram:

$$\begin{array}{ccccc}
E' & \longrightarrow & E & \longrightarrow & L \\
\downarrow & & \downarrow & & \| \\
\mathcal{E}_{N-1} & \xrightarrow{j} & \tilde{\mathcal{E}}_N & \xrightarrow{\psi} & L \\
\downarrow & & {\scriptstyle\varepsilon}\downarrow & & \\
T_{N-1} & = & T_{N-1} & &
\end{array}$$

This means that the points of $\tau^{-1}(E')$ are in one-to-one correspondence with maps $\varepsilon : \tilde{\mathcal{E}}_N \to T_{N-1}$ such that $\varepsilon \cdot j$ is equal to the canonical projection from $\mathcal{E}_{N-1}$ to T_{N-1}. Applying the functor $\mathrm{Hom}(\bullet, T_{N-1})$ to the middle row of the above diagram we get an exact sequence:

$$0 \to \mathrm{Hom}(L, T_{N-1}) \to \mathrm{Hom}(\tilde{\mathcal{E}}_N, T_{N-1}) \xrightarrow{j^*} \mathrm{Hom}(\mathcal{E}_{N-1}, T_{N-1}) \to \mathrm{Ext}^1(L, T_{N-1}).$$

The last term in this sequence is zero because L is locally free and T_{N-1} is torsion. This means that the space of maps ε which we need to describe is a torsor over the group $\mathrm{Hom}(L, T_{N-1})$. Hence this space can be identified with the group. Thus, we have proved that $\tau^{-1}(E') \cong \mathrm{Hom}(L, T_{N-1})$ is an affine space.

Now,

$$\dim(\tau^{-1}(E')) = \dim \mathrm{Hom}(L, T_{N-1}) = \dim \mathrm{H}^0(T_{N-1}) = \ell(T_{N-1}) = \nu_{N-1}.$$

The Lemma is proved. □

Theorem 2.3.3. *The space $\mathfrak{S}((\mu_{pq})_{1\le q\le p\le N}, (\mathcal{E}_k)_{k=1}^N)$ is a pseudoaffine space of dimension $\mu_{21} + \mu_{32} + \cdots + \mu_{N,N-1}$.*

Proof. We use induction in N. If $N = 1$ then $\mathfrak{S}_\mu$ is a point and the base of induction follows.

If $N > 1$ consider the map

$$\sigma : \mathfrak{S}((\mu_{pq})_{1\le q\le p\le N}, (\mathcal{E}_k)_{k=1}^N) \to \mathfrak{T}((\mu_{N,k})_{k=1}^N, (\mathcal{E}_k)_{k=1}^N),$$

which sends $(E_k)_{k=1}^N$ to $E_N \subset \mathcal{E}_N$.

Lemma 2.3.4. *Let $E \in \mathfrak{T}((\mu_{N,k})_{k=1}^N, (\mathcal{E}_k)_{k=1}^N)$. Consider $\tilde{\mathcal{E}}_k = \mathcal{E}_k \cap E \quad (1 \le k \le N-1)$ and set $\tilde{\mu}_{pq} = \mu_{pq} - \mu_{Nq} \quad (1 \le q \le p \le N-1)$. Then $(\tilde{\mathcal{E}}_k)_{k=1}^{N-1}$ is a short flag of subbundles and for any $E \in \mathfrak{T}((\mu_{N,k})_{k=1}^N, (\mathcal{E}_k)_{k=1}^N)$ we have*

$$\sigma^{-1}(E) \cong \mathfrak{S}((\tilde{\mu}_{pq})_{1\le q\le p\le N-1}), (\tilde{\mathcal{E}}_k)_{k=1}^{N-1}). \tag{18}$$

Thus $\mathfrak{S}((\mu_{pq})_{1\le q\le p\le N}, (\mathcal{E}_k)_{k=1}^N)$ is a fiber space with pseudoaffine base and fiber, therefore it is pseudoaffine.

Now, the calculation of the dimension

$$\dim\left(\mathfrak{S}((\mu_{pq})_{1\le q\le p\le N}, (\mathcal{E}_k)_{k=1}^N)\right) = \sum_{k=1}^{N-1} \mu_{N,k} + \sum_{k=1}^{N-2} \tilde{\mu}_{k+1,k} =$$
$$= \sum_{k=1}^{N-1} \mu_{N,k} + \sum_{k=1}^{N-2} (\mu_{k+1,k} - \mu_{N,k}) = \mu_{N,N-1} + \sum_{k=1}^{N-2} \mu_{k+1,k} = \sum_{k=1}^{N-1} \mu_{k+1,k},$$

finishes the proof of the Theorem. □

Proof of Lemma 2.3.4. Assume that $(E_k)_{k=1}^N \in \mathfrak{S}_\mu$ and $E_N = E$. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_q \cap E_p \arrow[r] \arrow[d] & \mathcal{E}_q \arrow[r] \arrow[d, equal] & \dfrac{\mathcal{E}_q}{\mathcal{E}_q \cap E_p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_q \cap E \arrow[r] & \mathcal{E}_q \arrow[r] & \dfrac{\mathcal{E}_q}{\mathcal{E}_q \cap E} \arrow[r] & 0
\end{tikzcd}$$

implies that

$$l\left(\frac{\mathcal{E}_q \cap E}{\mathcal{E}_q \cap E_p}\right) = l\left(\frac{\mathcal{E}_q}{\mathcal{E}_q \cap E_p}\right) - l\left(\frac{\mathcal{E}_q}{\mathcal{E}_q \cap E}\right) = \mu_{pq} - \mu_{Nq} = \tilde{\mu}_{pq}, \tag{19}$$

hence $(E_k)_{k=1}^{N-1} \in \mathfrak{S}_{\tilde\mu}$.

Vice versa, if $(E_k)_{k=1}^{N-1} \in \mathfrak{S}_{\tilde\mu}$ then the above commutative diagram along with (19) implies that $(E_k)_{k=1}^{N} \in \mathfrak{S}_{\mu}$, where we have put $E_N = E$. □

2.4. The cohomology of the simple fiber. Now we will compute the dimension of the strata $\mathfrak{S}_\mu$ in terms of the partition κ.

Definition 2.4.1. A space $\mathcal{X}$ is called *cellular* if it admits a stratification with pseudoaffine strata.

Suppose $\mathcal{X} = \bigsqcup_{\xi\in\Xi} S_\xi$ is a pseudoaffine stratification of a cellular space $\mathcal{X}$. For a positive integer j we define $\chi(j) \stackrel{\text{def}}{=} \#\{\xi \in \Xi \mid \dim S_\xi = j\}$.

Lemma 2.4.2. *The Hodge structure* $\mathrm{H}^\bullet(\mathcal{X},\mathbb{Q})$ *is a direct sum of Tate structures, and* $\mathbb{Q}(j)$ *appears with multiplicity* $\chi(j)$. *In other words,*

$$\mathrm{H}^\bullet(\mathcal{X},\mathbb{Q}) = \oplus_{j\in\mathbb{N}}\mathbb{Q}(j)^{\chi(j)}.$$

Proof. Evident. □

Given a Tate structure $\mathcal{H} = \oplus_{j\in\mathbb{N}}\mathbb{Q}(j)^{\chi(j)}$ we consider a *generating function*

$$P(\mathcal{H},t) = \sum_{j\in\mathbb{N}} \chi(j)t^j \in \mathbb{N}[t].$$

For $\kappa \in \mathfrak{K}(\gamma)$ we define $K(\kappa) \stackrel{\text{def}}{=} \sum_{1\le q\le p\le n-1} \kappa_{pq}$ as the number of summands in the partition κ. For $\gamma \in \mathbb{N}[I]$ the following q-analog of the Kostant's partition function was introduced in [Lus]:

$$\mathcal{K}_\gamma(t) = t^{|\gamma|} \sum_{\kappa\in\mathfrak{K}(\gamma)} t^{-K(\kappa)}. \tag{20}$$

Lemma 2.4.3. *Let* $\kappa \in \mathfrak{K}(\gamma)$ *and* $\mu \in \mathfrak{M}(\gamma)$ *be defined as in (9). Then*

$$\dim \mathfrak{S}_\mu = \sum_{k=1}^{n-2} \mu_{k+1,k} = |\gamma| - K(\kappa), \tag{21}$$

Proof. The first equality follows from 2.3.3. Applying (9) we get

$$\sum_{k=1}^{n-2} \mu_{k+1,k} = \sum_{k=1}^{n-2} \left(\sum_{\substack{1\le q\le k\\ k+1\le p\le n-1}} \kappa_{pq} \right) = \sum_{1\le q\le p\le n-1} (p-q)\kappa_{pq} =$$

$$\sum_{1\le q\le p\le n-1} (|[p,q]|-1)\kappa_{pq} = |\gamma| - \sum_{1\le q\le p\le n-1} \kappa_{pq} = |\gamma| - K(\kappa).$$

□

Corollary 2.4.4. *For any $\gamma \in \mathbb{N}[I]$, $x \in C$, the simple fiber $F(\mathcal{E}_\bullet, \gamma x)$ is a cellular space, and the generating function of its cohomology is equal to the Lusztig–Kostant polynomial*

$$P(\mathrm{H}^\bullet(F(\mathcal{E}_\bullet, \gamma x)), t) = \mathcal{K}_\gamma(t).$$

Proof. Apply (16), 2.3.3, 2.4.2 and 2.4.3. □

Corollary 2.4.5. *Let $D = \sum_{r=1}^{m} \gamma_r x_r \in C_\Gamma^{\alpha-\beta}$. The fiber $F(\mathcal{E}_\bullet, D)$ is a cellular space and*

$$P(\mathrm{H}^\bullet(F(\mathcal{E}_\bullet, D)), t) = \mathcal{K}_\Gamma(t) \stackrel{def}{=} \prod_{r=1}^{m} \mathcal{K}_{\gamma_r}(t). \tag{22}$$

Proof. Apply 2.1.2, 2.4.2 and 2.4.4. □

Lemma 2.4.6. *Let $D = \sum_{r=1}^{m} \gamma_r x_r$. We have*

$$\dim F(\mathcal{E}_\bullet, D) \le \left| \sum_{r=1}^{m} \gamma_r \right| - m.$$

Proof. Note that for any $\kappa \in \mathfrak{K}(\gamma_r)$ we have $K(\kappa) \ge 1$, hence $\deg \mathcal{K}_{\gamma_r} \le |\gamma_r| - 1$. Now, the Lemma follows from 2.4.5. □

Proof of Main Theorem. Consider the stratum $\mathfrak{D}_{\beta,\Gamma}$ of $\mathcal{Q}_\alpha^D$. Its dimension is $2|\beta| + \dim \mathcal{B} + m$ and codimension is $2|\alpha - \beta| - m$. The Lemma 2.4.6 implies that the dimension of the fiber of π over the stratum $\mathfrak{D}_{\beta,\Gamma}$ is less than or equal to $|\alpha - \beta| - m$, which is strictly less then the half codimension of the stratum. □

2.5. Applications. Let $\underline{\mathbb{Q}}$ denote the smooth constant irreducible Hodge module on $\mathcal{Q}_\alpha^L$ (as a constructible complex it lives in cohomological degree $-2|\alpha| - \dim \mathcal{B}$). Let IC denote the minimal extension of a smooth constant irreducible Hodge module from $\mathcal{Q}_\alpha$ to $\mathcal{Q}_\alpha^D$. It is well known that the smallness of π implies the following corollary.

Corollary 2.5.1.

$$IC = \pi_* \underline{\mathbb{Q}}.$$

Now we can compute the stalks of IC as cohomology of fibers of π: for $\varphi \in \mathcal{Q}_\alpha^D$ we have

$$IC^\bullet_{(\varphi)} = \mathrm{H}^\bullet(\pi^{-1}(\varphi), \underline{\mathbb{Q}})$$

as graded Hodge structures.

Corollary 2.5.2 (Parity vanishing).

$$IC^j_{(\varphi)} = 0 \quad \textit{if } j - \dim \mathcal{B} \textit{ is odd.}$$

Proof. Use 2.4.5. □

Corollary 2.5.3. *For $\varphi \in \mathfrak{D}_{\beta,\Gamma}$ we have*

$$IC_{(\varphi)}^{-2|\alpha|-\dim \mathcal{B}+2j} = \mathbb{Q}(j)^{\mathfrak{k}_\Gamma(j)},$$

where $\mathfrak{k}_\Gamma(j)$ is the coefficient of t^j in $\mathcal{K}_\Gamma(t)$.

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