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COMPLETE SYSTEMS OF TOPOLOGICAL AND ANALYTICAL INVARIANTS FOR A GENERIC FOLIATION OF $(\mathbb{C}^2, 0)$

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Abstract. In this announcement, we describe results on topological and analytical local classifications of germs of singular holomorphic foliations at the origin in the complex plane.

The analytical and topological classifications of holomorphic differential 1-forms of $(\mathbb{C}^2, 0)$, singular at the origin and having linear part in Poincaré's domain, or resonant is now well known. The analytical classification follows from Poincaré's linearization Theorem ; and in the resonant case it is due to S. Voronin [18], J. Ecalle [6], J. Martinet and J. P. Ramis [9], and B. Malgrange [8]. The topological classification was given by C. Camacho and P. Sad in [3]. In this announcement we give under generic conditions, a complete list of local analytical or topological invariants for a singularity of any *algebraic mutiplicity*[1]. By *local analytical, resp. topological invariant* we mean an object associated to the 1-form which for any holomorphic deformation of ω, is constant when the deformation is trivial for the relation of analytical, resp. topological conjugation of foliations. The details of our results will appear in two forthcoming papers [13] and [14]. Except when explicitly stated, all the objects considered will be holomorphic. In particular, a holomorphic differential 1-form will be called a 1-form.

1. Quasi hyperbolic foliations

We recall that a germ at the origin of $\mathbb{C}^2$ of a holomorphic foliation $\mathcal{F}_\omega$ given by a germ of a 1-form $\omega = a(x,y)\,dx + b(x,y)\,dy$; $a, b \in \mathcal{O}_{\mathbb{C}^2,0}$ with isolated singularity $\operatorname{Sing}(\mathcal{F}_\omega) := \{a(x,y) = b(x,y) = 0\} = \{0\}$ admits a canonical *reduction* (see [17] or [12]). More precisely, there is a holomorphic map $E_\omega : \mathcal{M}_\omega \longrightarrow \mathbb{C}^2$, obtained as a composition of a finite number of blowing ups at points (called the *centres*) over $\{0\}$, such that

- at each point m of the *exceptional divisor* $\mathcal{D}_\omega := E_\omega^{-1}(0)$, the germ of the foliation with isolated singularities $\widetilde{\mathcal{F}}_\omega$ constructed from $E_\omega^*(\omega)$ is

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[1] The algebraic multiplicity at the origin of a holomorphic differential 1-form ω is the smallest integer k such that the k-jet of ω is nonzero.

reduced : there exist local coordinates (u, v) at m such that the germ of $\widetilde{\mathcal{F}}_\omega$ is defined by a 1-form of one of the following types :

(1) $\widetilde{w}_m = g(u,v)\,du$, with $g \in \mathcal{O}_{\mathbb{C}^2,0}$, $g(0,0) \neq 0$,

(2) $\widetilde{w}_m = \lambda u\,dv + \mu v\,du + \cdots$ with $\mu \neq 0$, $\lambda/\mu \in \mathbb{C} - \mathbb{Q}_{<0}$, where $+\cdots$ means higher order terms,

- the map E_ω is "minimal" for these properties.

The foliation $\widetilde{\mathcal{F}}_\omega$ is called the *reduced foliation associated to* $\mathcal{F}_\omega$. If m is a singular point of $\mathcal{F}_\omega$ of type (2) with $\lambda/\mu \in \mathbb{C} - \mathbb{R}_{\leq 0}$, we say that m is of *saddle type.*

Definition 1.0.1. *A 1-form ω or a foliation $\mathcal{F}_\omega$ is quasi hyperbolic* (*QH for short*) *if* :

(1) *the exceptional divisor $\mathcal{D}_\omega$ is an invariant set of the foliation $\widetilde{\mathcal{F}}_\omega$, i.e. $\mathcal{D}_\omega$ is a finite union of leaves and singular points of $\widetilde{\mathcal{F}}_\omega$,*

(2) *all the singular points $m \in \mathrm{Sing}(\widetilde{\mathcal{F}}_\omega) \subset \mathcal{D}_\omega$ are of saddle type.*

Such a foliation has the following property : on any neighbourhood U of the origin, the saturated set by $\mathcal{F}_\omega$ of any analytical noninvariant curve is the complement in U of a finite (nonempty by [4]) union $\mathrm{Sep}(\mathcal{F}_\omega)$ of irreducible analytic invariant curves through 0 (the *separatrices of the foliation*).

Let us now introduce two notions of genericity that will be useful to obtain finite dimensional moduli spaces. These notions describe properties of $\widetilde{\mathcal{F}}_\omega$ on neighbourhoods of the irreducible components D of $\mathcal{D}_\omega$.

The *valence of a component* D of $\mathcal{D}_\omega$ is the number $v(D)$ of singular points of $\widetilde{\mathcal{F}}_\omega$ on D. A *chain of* $\mathcal{D}_\omega$ is either:

- a connected component $\mathcal{C}$ of the complement $\mathcal{D}_\omega - \cup_{v(D)\geq 3} D$ of all components of $\mathcal{D}_\omega$ with valence ≥ 3, such that $\mathcal{C}$ joins two components of valence ≥ 3, or
- an intersection point of two components of $\mathcal{D}_\omega$ with valence ≥ 3.

Definition 1.0.2. *A quasi hyperbolic 1-form ω or foliation $\mathcal{F}_\omega$ is generic* (*QHG for short*) *if it satisfies the following conditions* :

(1) *for any chain $\mathcal{C}$ of $\mathcal{D}_\omega$, the restriction of $\widetilde{\mathcal{F}}_\omega$ to any open neighbourhood of $\mathcal{C}$ does not admit a holomorphic first integral,*

(2) *for any component D of $\mathcal{D}_\omega$ with valence ≥ 3, the holonomy group of the leaf $D - \mathrm{Sing}(\widetilde{\mathcal{F}}_\omega \cap D)$ of $\widetilde{\mathcal{F}}_\omega$ is nonabelian,*

(3) *there is a component D of $\mathcal{D}_\omega$ with valence ≥ 3 and nonsolvable holonomy group.*

The quasi hyperbolicity property can be checked on a finite order jet $j^k(\omega)$ of ω : if ω is QH, any 1-form ω' with the same k-jet as ω is also QH. We say that k is an *order of quasi hyperbolicity.* The term "generic" is justified by the following proposition

Proposition 1.0.3. *Let ω be a quasi hyperbolic generic 1-form of order k of quasi hyperbolicity, and let $\Lambda(\omega; k)$ be the set of 1-forms having the same k-jet as ω. Then the set of all $\omega' \in \Lambda(\omega; k)$ which do not satisfy one of the conditions* (1), (2) *or* (3) *of* (1.0.2) *is contained in the intersection of $\Lambda(\omega; k)$ with a pro algebraic set of infinite codimension in the space of infinite order jets of 1-forms. In particular, let ω be a quasi hyperbolic generic 1-form of order k of quasi hyperbolicity, then for any positive integer r there exists an integer $q \geq r$ and a 1-form η with $j^r(\eta) = j^r(\omega)$, such that any 1-form η' satisfying $j^q(\eta) = j^q(\eta')$ is quasi hyperbolic generic.*

The topological classification will be given for QHG 1-forms. To get the analytical classification we need weaker conditions.

Definition 1.0.4. *A quasi hyperbolic 1-form ω or foliation $\mathcal{F}_\omega$ is nondegenerate if it satisfies conditions* (1) *and* (2) *of* (1.0.2).

2. Equireduction

A *deformation of a quasi hyperbolic 1-form $\omega = a(x,y)\,dx + b(x,y)\,dy$; $a, b \in \mathcal{O}_{\mathbb{C}^2,0}$ with base space* $(\mathbb{C}^p, 0)$ is a 1-form on $\big(\mathbb{C}^2 \times \mathbb{C}^p, 0\big)$

$$\eta := A(x,y;t)\,dx + B(x,y;t)\,dy\,, \quad A, B \in \mathcal{O}_{\mathbb{C}^2\times\mathbb{C}^p,0} \quad \text{with}$$

$$A(x,y;0) = a(x,y), \quad B(x,y;0) = b(x,y), \quad \text{and} \quad A(0,0;t) = B(0,0;t) = 0\,.$$

To a deformation one can associate the "family" of reductions $E_{\eta_t} : \mathcal{M}_{\eta_t} \longrightarrow (\mathbb{C}^2, 0)$ of the restrictions η_t of η to $\mathbb{C}^2 \times \{t\}$. We shall often denote the deformation by $(\eta_t)_{t\in\mathbb{C}^p,0}$. The property of equireduction characterizes "analytical dependence" and "regularity" of E_{η_t} with respect to the parameter t.

Definition 2.0.5. *The deformation η is equireducible if there is a holomorphic map $E_\eta : \mathcal{M}_\eta \longrightarrow \big(\mathbb{C}^2 \times \mathbb{C}^p, 0\big)$ obtained as a finite composition of blowing ups whose centres are smooth manifolds of dimension p, whose irreducible components project biholomorphically to* $(\mathbb{C}^p, 0)$, *and such that*

(1) *if t is sufficiently small, the restriction of E_η to ${E_\eta}^{-1}\big(\mathbb{C}^2 \times \{t\}\big)$ is the reduction map of η_t ,*
(2) *at every point m of the exceptional divisor $\mathcal{D}_\eta := {E_\eta}^{-1}(\{0\} \times \mathbb{C}^p)$, the 1-form ${E_\eta}^*(\eta)$ can be written as $f \cdot \widetilde{\eta}_m$, where f is a local equation of $\mathcal{D}_\eta$ and $\widetilde{\eta}_m$ is a germ at m of a 1-form with singular locus of codimension ≥ 2 in $\mathcal{M}_\eta$,*
(3) *if $\widetilde{\mathcal{F}}_\eta$ denotes the holomorphic* (*singular*) *foliation of dimension 1 defined by the germs $\widetilde{\eta}_m$, then the restriction of E_η to any irreducible component of the singular locus of $\widetilde{\mathcal{F}}_\eta$ is a biholomorphism to $(\{0\} \times \mathbb{C}^p, 0)$.*

This definition is rather heavy. We give an equivalent definition, using combinatorial data obtained from the reduction of the 1-form.

Definition 2.0.6. *Let ω be a quasi hyperbolic 1-form. The dual tree associated to ω is the weighted graph with arrows $\mathbb{A}^*(\omega)$ constructed in the following way* :

- *there is a 1-1 correspondence between vertices of $\mathbb{A}^*(\omega)$ and irreducible components of $\mathcal{D}_\omega$,*
- *there is an edge between two vertices of $\mathbb{A}^*(\omega)$ if the corresponding components of $\mathcal{D}_\omega$ intersect,*
- *we attach an arrow to a vertex of $\mathbb{A}^*(\omega)$ for each regular point on the corresponding component of $\mathcal{D}_\omega$, which is a singular point of $\widetilde{\mathcal{F}}_\omega$,*
- *the weight at a vertex of $\mathbb{A}^*(\omega)$ is the Chern class of the normal bundle to the corresponding component.*

One can prove, using the multiplicity formulaes given in [2] and [11] :

Proposition 2.0.7. *A deformation η of a quasi hyperbolic 1-form ω is equireducible if and only if for any sufficiently small value of the parameter t one has : $\mathbb{A}^*(\eta_t) \equiv \mathbb{A}^*(\omega)$.*

By [2], a QH foliation $\mathcal{F}_\omega$ has same reduction of its singularities as its set $\mathrm{Sep}(\mathcal{F}_\omega)$ of separatrices. We prove :

Proposition 2.0.8. *A topologically trivial deformation of a quasi hyperbolic foliation is equireducible.*

3. Semi local invariants

They are natural analytical invariants of $\mathcal{F}_\omega$ obtained from the reduction of a quasi hyperbolic 1-form ω. They describe the behaviour of the reduced associated foliation in the neighbourhood of its "critical elements".

Definition 3.0.9. *A* critical element *α of a quasi hyperbolic foliation $\mathcal{F}_\omega$ is either*

- *a singular point c of the reduced foliation $\widetilde{\mathcal{F}}_\omega$ associated to $\mathcal{F}_\omega$, or*
- *a connected component L of $\mathcal{D}_\omega - \mathrm{Sing}(\widetilde{\mathcal{F}}_\omega)$.*

We denote the set of critical elements (resp. of dimension j, $j = 0, 1$) of $\mathcal{F}_\omega$ by $\mathrm{Crit}(\omega)$ (resp. $\mathrm{Crit}_j(\omega)$). Two critical elements are *adjacent* if their closures intersect. The set of all critical elements adjacent to α is denoted by $Ad(\alpha)$.

To every critical element c of dimension 0 we can associate the analytical type of $\widetilde{\mathcal{F}}_\omega$ at c. More precisely, let MS_2 be the quotient of the set of 1-forms of saddle type $S_2 = \{x_1(\lambda_1 + \cdots)\, dx_2 + x_2(\lambda_2 + \cdots)\, dx_1; \lambda_1/\lambda_2 \in \mathbb{C} - \mathbb{R}_{\leq 0}\}$ by the action of the group Diff_2 of germs of holomorphic diffeomorphisms of $(\mathbb{C}^2, 0)$ fixing each axis (the description of MS_2 is now classical, see [9]). At every singular point $c \in \mathrm{Crit}_0(\omega)$ we choose a germ of a diffeomorphism $\Phi_c : (\mathcal{M}_\omega, c) \longrightarrow (\mathbb{C}^2, 0)$ which conjugates the strict transform $\widetilde{w}_c$ to an element $\widetilde{w}'_c \in S_2$ defining a class $[\widetilde{w}'_c] \in MS_2$.

The *local system associated to* ω is the collection $L(\omega) := ([\widetilde{\omega}'_c]\,;\,\sigma_c)_{c\in \mathrm{Crit}_0(\omega)}$ of the analytical types of $\widetilde{\mathcal{F}}_\omega$ at its singular points; and of maps σ_c from $Ad(c)$ to $\{1,2\}$ describing to which axis $\{x_j=0\}$ the chart Φ_c transforms each element adjacent to c.

Every $L\in \mathrm{Crit}_1(\omega)$ is an invariant manifold of $\widetilde{\mathcal{F}}_\omega$. We can associate to it the class $[\mathcal{H}(L)]$ of its holonomy representation $\mathcal{H}:\pi_1(L)\longrightarrow \mathrm{Diff}(\mathbb{C},0)$ under the action of inner automorphisms (at the source and the target).

Definition 3.0.10. *The* semi local system of invariants associated to ω *is the collection* $SL(\omega)=(\mathbb{A}^*(\omega);\,L(\omega);\,\mathcal{H}(\omega))$ *with* $\mathcal{H}(\omega):=([\mathcal{H}(L)])_{L\in\mathrm{Crit}_1(\omega)}$.

Remark 3.0.11. The above description has some ambiguities:

• when the local model at a singularity c has a symmetry which permutes the coordinate axis, σ_c is not well defined;

• the holonomy representations $\mathcal{H}(L)$ are not combinatorial data. To get combinatorial data, we ought to define $\mathcal{H}(L)$ as a family of diffeomorphisms of $(\mathbb{C},0)$ parametrized by $Ad(L)$, modulo the natural actions of the $v(L)$-th braid group and of $\mathrm{Diff}(\mathbb{C},0)$.

In [13], we give an abstract, purely combinatorial definition of a semi local system. We also prove that we can realize any abstract semi local system as the semi local system associated to a 1-form, generalizing in this way a result of A. Lins Neto [7].

4. Analytical classification

Obviously $SL(\omega)$ is an analytical invariant of ω or $\mathcal{F}_\omega$. To get a complete system of local invariants it is sufficient to classify 1-forms having the same semi local system as ω.

Definition 4.0.12. *A deformation η of ω is SL-equisingular if it is equireducible and if $SL(\eta_t)\equiv SL(\omega)$ for any sufficiently small value of the parameter t.*

We see that an SL-equisingular deformation η of ω with space of parameters $(\mathbb{C},0)$ is "semilocally trivial": for every critical element α of ω we can construct a germ of an analytic family of holomorphic diffeomorphisms $\Psi_{\alpha,t}$ defined in a small neighbourhood U_α of α in $\mathcal{M}_\eta$, which for each t conjugates the foliations $\widetilde{\mathcal{F}}_{\eta_t}$ and $\widetilde{\mathcal{F}}_\omega$. The global nontriviality can be read on the cocycle

$$\Phi_{\alpha\alpha',\,t} := \Psi_{\alpha',\,t}\circ\Psi^{-1}_{\alpha,\,t}\quad \alpha,\alpha'\in\mathrm{Crit}(\omega),\quad \alpha'\in Ad(\alpha)$$

with values in the sheaf $Aut_{\mathbb{C}}(\widetilde{\mathcal{F}}_\omega)$ over $\mathcal{D}_\omega$ of germs of families parametrized by t of automorphisms of $\widetilde{\mathcal{F}}_\omega$ which are the identity for $t=0$. At first order, the "initial speed of deformation" $\left.\dfrac{\partial\Phi_{\alpha\alpha',\,t}}{\partial t}\right|_{t=0}$ is a cocycle with values in the sheaf

$\mathcal{B}_{\widetilde{\mathcal{F}}_\omega}$ over $\mathcal{D}_\omega$ of germs at points in $\mathcal{D}_\omega$ of holomorphic vector fields of $\mathcal{M}_\omega$ which are *basic for* $\widetilde{\mathcal{F}}_\omega$, i.e. their flows leave $\widetilde{\mathcal{F}}_\omega$ invariant. We denote

$$\left[\frac{\partial \mathcal{F}_\eta}{\partial t}\right]_{t=0} := \left[\left.\frac{\partial \Phi_{\alpha\alpha',t}}{\partial t}\right|_{t=0}\right] \in H^1\left(\mathcal{U};\, \mathcal{B}_{\widetilde{\mathcal{F}}_\omega}\right) ,$$

where $\mathcal{U}$ is the covering $(U_\alpha \cap \mathcal{D}_\omega)_{\alpha \in \mathrm{Crit}(\omega)}$ of $\mathcal{D}_\omega$.

Theorem 4.0.13. *Every 1-cocycle in* $Z^1\left(\mathcal{U};\, Aut_{\mathbb{C}}(\widetilde{\mathcal{F}}_\omega)\right)$ *can be realized as the cocycle associated to an* SL*-equisingular deformation of* ω*. Moreover, given* $r \in \mathbb{N}^*$ *and* $X_1, \dots, X_r \in Z^1\left(\mathcal{U};\, \mathcal{B}_{\widetilde{\mathcal{F}}_\omega}\right)$*, there is an* SL*-equisingular deformation* η *of* ω *such that :*

$$\left[\frac{\partial \mathcal{F}_\eta}{\partial t_j}\right]_{t=0} = [X_j] \qquad j = 1, \dots, r\,.$$

Definition 4.0.14. *An active chain of* $\mathcal{D}_\omega$ *is a chain which admits a global nontrivial section of the quotient sheaf* $\mathcal{T}_{\widetilde{\mathcal{F}}_\omega}$ *given by the short exact sequence :*

$$0 \longrightarrow \mathcal{X}_{\widetilde{\mathcal{F}}_\omega} \longrightarrow \mathcal{B}_{\widetilde{\mathcal{F}}_\omega} \longrightarrow \mathcal{T}_{\widetilde{\mathcal{F}}_\omega} \longrightarrow 0 ,$$

where $\mathcal{X}_{\widetilde{\mathcal{F}}_\omega}$ *is the sheaf over* $\mathcal{D}_\omega$ *of germs at points in* $\mathcal{D}_\omega$ *of holomorphic vector fields tangent to* $\widetilde{\mathcal{F}}_\omega$*.*

We can easily check that a chain is active if and only if at one of its singular points, the fibre of $\mathcal{T}_{\widetilde{\mathcal{F}}_\omega}$ is not zero, or also iff at this point $\widetilde{\mathcal{F}}_\omega$ is defined by a reduced 1-form which is analytically normalizable.

Theorem 4.0.15. *Let* ω *be a quasi hyperbolic nondegenerate 1-form. Then* $H^1\left(\mathcal{U};\, \mathcal{B}_{\widetilde{\mathcal{F}}_\omega}\right)$ *is a* $\mathbb{C}$*-vectorial space of finite dimension* $\delta(\omega) + \tau(\omega)$ *with :*

$$\delta(\omega) = \sum_{m \in \varrho(\omega)} \frac{(\nu_m - 1)(\nu_m - 2)}{2} \qquad \text{and} \qquad \tau(\omega) = \#\{\,\text{active chains of } \mathcal{D}_\omega\,\} ,$$

where $\varrho(\omega)$ *is the set of all the singular points (0 included) appearing in the reduction process of* ω *, and* ν_m *is the algebraic multiplicity of the foliation at* m*.*

More precisely, we prove that if we choose the covering $\mathcal{U}$ with trivial 3 by 3 intersections of open sets, then the sequence

$$0 \longrightarrow H^1\left(\mathcal{U};\, \mathcal{X}_{\widetilde{\mathcal{F}}_\omega}\right) \longrightarrow H^1\left(\mathcal{U};\, \mathcal{B}_{\widetilde{\mathcal{F}}_\omega}\right) \longrightarrow H^1\left(\mathcal{U};\, \mathcal{T}_{\widetilde{\mathcal{F}}_\omega}\right) \longrightarrow 0 ,$$

is exact. By [10], we have $\dim_{\mathbb{C}} H^1\left(\mathcal{U};\, \mathcal{X}_{\widetilde{\mathcal{F}}_\omega}\right) = \delta(\omega)$, without any hypothesis on ω. We show that $\dim_{\mathbb{C}} H^1\left(\mathcal{U};\, \mathcal{T}_{\widetilde{\mathcal{F}}_\omega}\right) = \tau(\omega)$ when ω is a quasi hyperbolic nondegenerate 1-form.

A complete list of local analytical invariants is given by the following Theorem :

Theorem A. *Let ω be a quasi hyperbolic nondegenerate 1-form. There exists a deformation $\overline{\omega}$ of ω with space of parameters $Q \simeq (\mathbb{C}^{\delta(\omega)+\tau(\omega)}, 0)$, which is universal SL-equisingular in the following sense :*

• *for any SL-equisingular deformation η with space of parameters $(\mathbb{C}^p,0)$ there exists a germ of an analytical map $\lambda : (\mathbb{C}^p,0) \longrightarrow Q$ and an analytical family $(\Phi_t)_{t\in\mathbb{C}^p,0}$ of germs of holomorphic diffeomorphisms of $\left(\mathbb{C}^2,0\right)$ which for all t sufficiently small, conjugates the foliations on $\left(\mathbb{C}^2\times\{t\},(0,t)\right)$ given by η_t and $\overline{\omega}_{\lambda(t)}$. Moreover, the factorization λ is unique.*

The deformation $\overline{\omega}$ is unique up to holomorphic conjugation, and there is a canonical identification of the tangent space of Q at 0 to $H^1\left(\mathcal{U};\, \mathcal{B}_{\widetilde{\mathcal{F}}_\omega}\right)$. The construction of $\overline{\omega}$ follows from (4.0.13) and from :

Theorem 4.0.16. *A deformation η with space of parameters $(\mathbb{C}^q,\, 0)$ of a quasi hyperbolic nondegenerate 1-form is SL-universal if and only if the $\left[\dfrac{\partial\widetilde{\mathcal{F}}_\eta}{\partial t_j}\right]_{t=0}$, $j = 1, \ldots, q$ form a basis of $H^1\left(\mathcal{U};\, \mathcal{B}_{\widetilde{\mathcal{F}}_\omega}\right)$.*

5. Topological classification

The key point to get topological classification is in interpreting topological triviality in terms of unfolding.

Definition 5.0.17. *A deformation $\eta = A(x,y;t)\,dx + B(x,y;t)\,dy$ with $t = (t_1,\ldots,t_p)$ of ω is induced by an unfolding if there exist germs of holomorphic fonctions $C_j(x,y;t)$, $j=1,\ldots,p$, such that the 1-form of $(\mathbb{C}^{2+p},0)$*

$$\Omega := A(x,y;t)\,dx + B(x,y;t)\,dy + \sum_{j=1}^{p} C_j(x,y;t),\, dt_j$$

satisfies the Frobenius integrability condition $\Omega\wedge d\Omega \equiv 0$.

In [10], a precise definition is given of an equisingular unfolding. For such an unfolding Ω the codimension 1 foliation given by Ω is topologically trivial. Thus the deformation η is equireducible and the family of regular foliations on the complement of the origin given by $(\eta_t)_{t\in\mathbb{C},0}$ has (holomorphically) constant holonomy pseudogroup. Conversely one has :

Theorem B. *A deformation of a generic quasi hyperbolic 1-form is topologically trivial if and only if it is induced by an equisingular unfolding.*

The proof relies on dynamical properties of nonsolvable subgroups of $\mathrm{Diff}(\mathbb{C},0)$. Indeed, we use a Theorem of orbit density of A. A. Scherbakov [16], a construction of I. Nakai [15], and a Theorem of fixed point density ([19] and [1]) to show that in a topologically trivial deformation of a generic quasi hyperbolic 1-form, the topological foliation of codimension 1 given by the trivialization is analytical and defines an equisingular unfolding.

One immediatly deduces from Theorem B :

Corollary 5.0.18. *Every topologically trivial deformation of a generic quasi hyperbolic* 1*-form is SL-equisingular.*

When the reduction of w is obtained after only one blowing up, D. Cerveau and P. Sad have obtained in [5] the same result as (5.0.18), using a rigidity theorem on the holonomy group.

Remark 5.0.19. A universal equisingular unfolding is constructed in [10], i.e. an equisingular unfolding which factorizes every other equisingular unfolding in a unique manner, up to analytical conjugation. Its base space R has dimension $\delta(\omega)$, and T_0R is canonically identified with $H^1\left(\mathcal{D}_\omega;\, \mathcal{X}_{\widetilde{\mathcal{F}}_\omega}\right)$. When ω is QHG, Theorem B gives an interpretation of R as the moduli space of local analytical types for foliations topologically conjugate to $\mathcal{F}_\omega$.

As $SL(\omega)$ is a local topological invariant, to get a complete list of local topological invariants it is sufficient to "recognize" the areas where the deformation $(\overline{\omega}_u)_{u\in Q}$ is topologically trivial, in the universal space Q given by Theorem A. In the space $\mathcal{M}_{\overline{\omega}}$ of equireduction of $\overline{\omega}$ given by $E_{\overline{\omega}} : \mathcal{M}_{\overline{\omega}} \longrightarrow \left(\mathbb{C}^2 \times Q, 0\right)$, let us therefore consider the sheaves $\mathcal{X}_{\overline{\omega}}$, resp. $\mathcal{B}_{\overline{\omega}}$ over $\mathcal{D}_\omega \subset \mathcal{D}_{\overline{\omega}}$ of germs of vertical vector fields of $\mathcal{M}_{\overline{\omega}}$ (for the projection $\pi := pr_Q \circ E_{\overline{\omega}}$) which are tangent, resp. basic for the foliation of dimension one $\widetilde{\mathcal{F}}_{\overline{\omega}}$ on $\mathcal{M}_{\overline{\omega}}$.

With the notations of the previous section, we construct the "Kodaira-Spencer map" $\left[\dfrac{\partial \mathcal{F}_{\overline{\omega}}}{\partial t}\right] := \left[\dfrac{\partial \Phi_{\alpha\alpha',\,t}}{\partial t} \circ \Phi^{-1}_{\alpha\alpha',\,t}\right]$ as a morphism from the space $\mathcal{X}_Q$ of germs of holomorphic vector fields on Q to the space $H^1\left(\mathcal{U};\, \mathcal{B}_{\overline{\omega}}\right)$. One then has :

Lemma 5.0.20. *Let $\overline{H^1}\left(\mathcal{U};\, \mathcal{B}_{\overline{\omega}}\right)$ be the image of the natural morphism from $H^1\left(\mathcal{U};\, \mathcal{X}_{\overline{\omega}}\right)$ to $H^1\left(\mathcal{U};\, \mathcal{B}_{\overline{\omega}}\right)$. Then the preimage of $\overline{H^1}\left(\mathcal{U};\, \mathcal{B}_{\overline{\omega}}\right)$ by the Kodaira-Spencer map $\left[\dfrac{\partial \mathcal{F}_{\overline{\omega}}}{\partial t}\right]$ is a free involutive submodule of $\mathcal{X}_Q$.*

We therefore have a regular holomorphic foliation of dimension $\delta(\omega)$ on $Q \simeq (\mathbb{C}^{\delta(\omega)+\tau(\omega)},\, 0)$ denoted by Top and we show that :

- $\overline{\omega}$ is topologically trivial in the direction of the leaves of Top,
- the restriction of $(\overline{\omega}_u)_{u\in F}$ to each leaf F of Top is induced by an unfolding which is a universal equisingular unfolding,
- a deformation of an element $\overline{\omega}_{u_0}$ of $\overline{\omega}$ is topologically trivial iff it factorizes in the leaf of Top through u_0; moreover this factorization is unique.

It follows that the restriction of the space of parameters $Q \simeq (\mathbb{C}^{\delta(\omega)+\tau(\omega)}, 0)$ to a smooth submanifold of Q of dimension $\tau(\omega)$ strictly transverse to the leaves of Top induces an SL-equisingular deformation of ω whose topological type "varies continuously" (the topological equivalence classes are "discrete"). We thus obtain

Theorem C. *Let ω be a generic quasi hyperbolic 1-form. There exists an SL-equisingular deformation $\overline{\omega}'$ of ω with space of parameters $T \simeq (\mathbb{C}^{\tau(\omega)}, 0)$ which is top-universal in the follwing sense :*

- *for any SL-equisingular deformation η with space of parameters $(\mathbb{C}^p, 0)$ there exists a germ of an analytical map $\mu : (\mathbb{C}^p, 0) \longrightarrow T$ and an analytical family $(\Phi_t)_{t\in\mathbb{C}^p,0}$ of germs of homeomorphisms of $\left(\mathbb{C}^2, 0\right)$ which for t sufficiently small, conjugates the foliations on $\left(\mathbb{C}^2 \times \{t\}, (0,t)\right)$ defined by η_t and $\overline{\omega}'_{\mu(t)}$.*

Moreover, the factorization μ is unique and $\overline{\omega}'$ is unique up to analytical conjugation.

We also have the following infinitesimal criterion :

Theorem 5.0.21. *A deformation η with space of parameters $(\mathbb{C}^q, 0)$ of a generic quasi hyperbolic 1-form is top-universal iff the $\left[\dfrac{\partial\widetilde{\mathcal{F}}_\eta}{\partial t_j}\right]_{t=0}$, $j = 1, \ldots, q$, form a basis of $H^1\left(\mathcal{U};\, \mathcal{T}_{\widetilde{\mathcal{F}}_\omega}\right)$.*

6. The example of the double cusp QHG

Let us consider the "multiform double-cusp" given by the 1-form

$$\omega_1 := (y^2 - x^3)(y^3 - x^2)\left(\alpha\, \frac{d(y^2 - x^3)}{(y^2 - x^3)} + \beta\, \frac{d(y^3 - x^2)}{(y^3 - x^2)}\right), \qquad \alpha,\, \beta\, \in \mathbb{C}\,.$$

It has $(y^2 - x^3)^\alpha (y^3 - x^2)^\beta$ as a multiform first integral. The reduction of ω_1 is the same as the reduction of the curve $X = X' \cup X''$ where X' (resp. X") is given by the equation $y^2 - x^3 = 0$ (resp. $y^3 - x^2 = 0$). It will be denoted by $E : \mathcal{M} \longrightarrow \mathbb{C}^2$. The exceptional divisor is described in Figure 1,

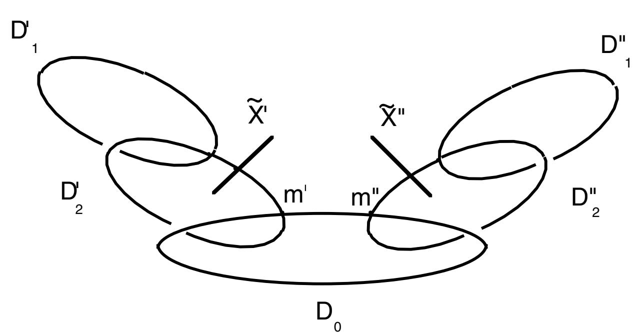

FIGURE 1

where $\widetilde{X'}$ and $\widetilde{X''}$ are the strict transforms of X' and X'' respectively; D_0 is the first component created; D_1' and D_1'' are the second, and D_2' , D_2'' are the last. The dual tree is given in Figure 2.

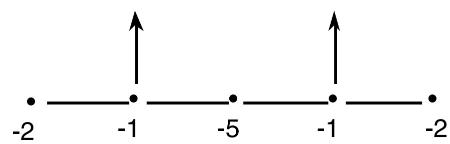

FIGURE 2

The components D_2' and D_2'' which carry the strict transforms of X are the only components with valence ≥ 3. There is a unique chain $\mathcal{C}$ and it is reduced to D_0.

Let m' and m'' be the intersection points of D_0 with D_2' and D_2'' respectively. At the point m', the strict transform of ω_1 can be written in suitable coordinates (u,v) as $\widetilde{w}_{1,m'} = (\lambda + \cdots)u\,dv + (\mu + \cdots)v\,du$ where the eigenvalues λ and μ can be computed from α and β. Clearly, for almost any α and β the quotient λ/μ is not real. We shall therefore suppose that the singularities m' and m'' are of saddle type.

By (1.0.3), for every sufficiently big integer k one can find a 1-form ω_2 with $j^k(\omega_2) = 0$ such that $\omega := \omega_1 + \omega_2$ is QHG and has the same reduction as ω_1.

Let us now choose neighbourhoods W', resp. W'' of $D_1' \cup D_2'$, resp. $D_0 \cup D_2'' \cup D_1''$ in $\mathcal{M}$ such that $W' \cap W''$ is a small polydisc $\{|\, uv \,|< \epsilon\}$ centred on m'. A classical Theorem of Poincaré gives the existence of coordinates (U,V) such that the strict transform of ω at m' is linear : $\widetilde{w}_{m'} = \lambda U\,dV + \mu V\,dU$. The "radial" vector field $R := U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}$ leaves $\widetilde{\mathcal{F}}_\omega$ and $\{UV = 0\}$ invariant, and gives a nonzero element of $\mathcal{T}_{\widetilde{\mathcal{F}}_\omega}$. We obtain a holomorphic manifold $\mathcal{M}_t$ by gluing W' to W'' with the flow of R at time t. It has a (singular) holomorphic foliation $\widetilde{\mathcal{F}}_t$ which leaves a compact divisor $\mathcal{D}_t$ invariant and $\mathcal{D}_t$ has the same dual tree as $\mathcal{F}_\omega$. A classical Theorem of Grauert gives a holomorphic map from $\mathcal{M}_t$ to $(\mathbb{C}^2, 0)$ obtained as a finite composition of blowing downs of $\mathcal{D}_t$.[2] By taking direct images, we get a family of holomorphic foliations $\mathcal{F}_t$ at the origin of $\mathbb{C}^2$ which form an SL-equisingular deformation of ω. Using the infinitesimal criterion (5.0.21), we see that this deformation is the *top*-universal deformation of ω.

In the general case we have a similar construction. We fix a singular point $m_{\mathcal{C}}$ on each active chain $\mathcal{C}$. We construct an adapted covering $\mathcal{W}$ of the exceptional divisor with trivial 3 by 3 intersections of the open sets such that the nontrivial intersections of two open sets are small neighbourhoods of the points $m_{\mathcal{C}}$. On each $U \cap V$, $U, V \in \mathcal{W}$, we take a basic vector field. We thus get a basis of $H^1\left(\mathcal{U}; \mathcal{T}_{\widetilde{\mathcal{F}}_\omega}\right) = H^1\left(\mathcal{W}; \mathcal{T}_{\widetilde{\mathcal{F}}_\omega}\right)$. As in the case of the double cusp, we can then

[2]In the case of the double cusp, $\mathcal{M}_t$ is biholomorphic to $\mathcal{M}$.

construct a deformation of ω with base space $(\mathbb{C}^{\tau(\omega)}, 0)$. We prove that this deformation is *top*-universal by using (5.0.21) as before.

References

1. M. Belliart, I. Liousse and F. Loray, *Sur l'existence de points fixes attractifs pour les sous-groupes de Aut* $(\mathbb{C}, 0)$, to appear in C. R. Acad. Sci. Paris Sér. I Math.
2. C. Camacho, A. Lins Neto and P. Sad, *Topological invariants and equidesingularization for holomorphic vector fields* J. Diff. Geom. **20** (1984), 143–174.
3. C. Camacho and P. Sad, *Topological classification and bifurcations of holomorphic flows with resonance in* $\mathbb{C}^2$, Invent. Math. **67** (1982), 447–472.
4. ____, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. **115** (1982), 579–595.
5. D. Cerveau and P. Sad, *Problèmes de modules pour les formes différentielles dans le plan complexe*, Comment. Math. Helv. **61** (1986), 222–253.
6. J. Ecalle, *Les fonctions résurgentes*, Publ. Math. Orsay, Tomes I-III, 1985.
7. A. Lins Neto, *Construction of singular holomorphic vector fields and foliations in dimension two*, J. Diff. Geom. **26** (1987), 1–31.
8. B. Malgrange, *Travaux d'Ecalle et Martinet-Ramis sur les systèmes dynamiques*, Séminaire Bourbaki **582** (1981).
9. J. Martinet and J.-P. Ramis, *Classification analytique des équations différentielles non linéaires résonnantes du premier ordre*, Ann. Sci. École Norm. Sup. **4** (1983), 571–621.
10. J.-F. Mattei, *Modules de feuilletages holomorphes singuliers : 1 équisingularité*, Invent. Math. **103** (1991), 297–325.
11. ____, *Quasihomogénéité et équiréductibilité de feuilletages holomorphes en dimension 2*, preprint, Univ. of Toulouse, 1996.
12. J.-F. Mattei and R. Moussu, *Holonomie et intégrales premières*, Ann. Sci. École Norm. Sup. **4** (1980), 469–523.
13. J-F. Mattei and E. Salem, *Classification formelle et analytique des feuilletages génériques de* $(\mathbb{C}^2, 0)$, (to appear).
14. ____, *Classification topologique des feuilletages génériques de* $(\mathbb{C}^2, 0)$, in preparation.
15. I. Nakai, *Separatrices for non solvable dynamics on* $(\mathbb{C}, 0)$, Ann. Inst. Fourier (Grenoble) **44** (1994), 569–599.
16. A. A. Scherbakov, *On the density of an orbit of a pseudogroup of conformal mappings and a generalization of the Hudai-Verenov theorem*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **31** (1982), 10–15.
17. A. Seidenberg, *Reduction of singularities of the differentiable equation* $A\,dY = B\,dX$, Amer. J. Math. **90** (1968), 248–269.
18. S. Voronin, *Analytical classification of germs of maps* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, J. Funct. Anal. **15** (1981), 1–17.
19. B. Wirtz, *Fixed points for non solvable dynamics of* $(\mathbb{C}, 0)$, preprint.

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