Mathematical Research Letters **4**, 79–89 (1997)

SATAKE AND MARTIN COMPACTIFICATIONS OF SYMMETRIC SPACES ARE TOPOLOGICAL BALLS

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1. Introduction

Symmetric spaces of noncompact type form a very important class of simply connected nonpositively curved Riemannian manifolds with connection to Lie group theory and harmonic analysis. For many applications, it is important to compactify the symmetric spaces.

Several compactifications of symmetric spaces have been defined from different points of view. For example, in [17], in order to understand the boundaries which arise in the study of automorphic forms, Satake defined finitely many compactifications of a symmetric space by embedding it into the space of positive definite Hermitian matrices of determinant 1 and then the associated projective space. Another important compactification is the Martin compactification from potential theory. In [10], the Martin compactification of the symmetric space is described in terms of geodesics and the maximal Satake compactification (see Theorem 2.5 below for a precise statement; and see the book [10] for definitions of other compactifications and relations between the various compactifications.)

Both the Satake compactifications and the Martin compactification of a symmetric space are topological compactifications. In this note, we prove that they are topologically a closed ball (2.4 and 2.6). In the proof (§4), we use the convexity result of Atiyah [3] that the image of the moment map of a Hamiltonian torus action is a convex polytope (3.1) in order to identify the closure of a flat in the compactifications of the symmetric space (4.1). We also formulate a conjecture on the Martin compactification of simply connected and nonpositively curved Riemannian manifolds (5.1).

2. Statement of results

Let G be a connected semisimple Lie group of noncompact type with finite center, $K \subset G$ a maximal compact subgroup. Let $\mathfrak{g}$ be the Lie algebra of G. Then the Killing form on $\mathfrak{g}$ defines a G-invariant metric on $X = G/K$. With respect to this metric, X is a Riemannian symmetric space of noncompact type; in particular, X is simply connected and nonpositively curved.

For any irreducible faithful projective representation $\tau : X \to \mathrm{PSL}(n, \mathbb{C})$ satisfying $\tau(\theta(g)) = \overline{\tau(g)}^t$, where θ is the Cartan involution of G associated with K,

Received July 11, 1996.
Partially supported by NSF postdoctoral fellowship DMS-9407427.

$\tau(K) \subset \mathrm{PSU}(n)$. Let $\mathcal{P}_n = \mathrm{PSL}(n, \mathbb{C})/\mathrm{PSU}(n)$ be the space of positive definite Hermitian matrixes of determinant one. Then the representation τ defines a map $X \to \mathcal{P}_n : gK \to \tau(g)\overline{\tau(g)}^t$, which is still denoted by τ.

Lemma 2.1. *The map $\tau : X \to \mathcal{P}_n$ is an embedding.*

Proof. If not, there exists $g \notin K$ such that $\tau(gK) = \tau(g)\overline{\tau(g)}^t = Id$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ induced by θ. Then $g = e^p k$ for some $p \in \mathfrak{p}$, $p \neq 0$, $k \in K$. By the assumption on τ, $\overline{\tau(e^p)}^t = \tau(e^p)$ and hence $\tau(e^p)$ is a positive definite Hermitian matrix. On the other hand, $\tau(g)\overline{\tau(g)}^t = \tau(e^p)\overline{\tau(e^p)}^t = \tau(e^p)^2 = Id$. This implies that $\tau(e^p) = Id$. Since τ is faithful, this contradicts that $p \neq 0$.

Remark 2.2. In [17, §2.2], Satake proved that the image $\tau(X)$ is a totally geodesic submanifold in $\mathcal{P}_n$ and any realization of X as a totally geodesic submanifold of $\mathcal{P}_n$ is of this form.

Let $\mathcal{H}_n$ be the real vector space of Hermitian matrixes of size $n \times n$, and $P(\mathcal{H}_n)$ the real projective space. Then $\mathcal{P}_n$ can clearly be embedded in $P(\mathcal{H}_n)$, and the map $\tau : X \to \mathcal{P}_n$ defines an embedding $\tau : X \to P(\mathcal{H}_n)$.

Definition 2.3. The closure of $\tau(X)$ in $P(\mathcal{H}_n)$ is called the Satake compactification of X associated to the representation τ and denoted by $\overline{X}^S_\tau$.

Let $x_0 = K \in X$ be the basepoint. The first result in this note is the following:

Theorem 2.4 (§4). *Every Satake compactification $\overline{X}^S_\tau$ is homeomorphic to the closed unit ball in the tangent space $T_{x_0}X$, in particular, $\overline{X}^S_\tau$ is a topological manifold with boundary.*

Using the exponential map, the unit ball in $T_{x_0}X$ can be identified with the unit geodesic ball in X with center x_0.

Even though there are infinitely many such representations τ of G, there are only finitely many different Satake compactifications $\overline{X}^S_\tau$ of X. Among them, there is a unique maximal Satake compactification, denoted by $\overline{X}^S_{\max}$ (for details, see [17] and [10, Chap 4]. See also Remark 4.3 below).

Let Δ be the Laplace operator of X, where Δ is normalized to be non-negative. Denote the bottom of the spectrum of Δ by $\lambda_0(X)$. Then for any $\lambda \leq \lambda_0$, Green's function $G_\lambda(x, y)$ of $\Delta - \lambda$ exists [12, Theorem 16.6.1]. Define $K_\lambda(x, y) = G_\lambda(x, y)/G_\lambda(x_0, y)$. Then there exists a unique metrizable compactification $X \cup \partial_\lambda X$ such that

(1) For any $x \in X$, the function $y \to K_\lambda(x, y)$ extends continuously to the boundary of $X \cup \partial_\lambda X$.
(2) These continuous extensions separate points on the boundary $\partial_\lambda X$.

This compactification $X \cup \partial_\lambda X$ is called the Martin compactification of X. Originally, the Martin compactification was defined only for $\lambda = 0$ and domains in $\mathbb{R}^n$ in [16]; this definition works also for general complete manifolds (see [1, §4] for example).

Each boundary point $\xi \in \partial_\lambda X$ corresponds to a unique positive solution $K_\lambda(x,\xi)$ of $\Delta u = \lambda u$ and these functions $K_\lambda(x,\xi)$ generate the cone of positive solutions of $\Delta u = \lambda u$. So it is an important problem to describe the Martin compactification in terms of the geometry of the manifold. For the symmetric space X, the Martin compactification $X \cup \partial_\lambda X$ is studied in detail in [10].

Let $X(\infty)$ be the set of equivalence classes of unit speed geodesics in X, where two geodesics $\gamma_1(t), \gamma_2(t)$ are defined to be equivalent if $\lim_{t\to+\infty} \sup d(\gamma_1(t), \gamma_2(t)) < +\infty$. The space $X(\infty)$ can naturally be identified with the unit sphere in $T_{x_0}X$ and hence called the sphere at infinity. Then X can be compactified by adding the sphere at infinity, and the compactification $X \cup X(\infty)$ is called the conic compactification (see [5, §3] and [10, Chap 4] for details). The closure of the diagonal embedding $X \to X \cup X(\infty) \times \overline{X}^S_{\max}$ is also a compactification of X, denoted by $X \cup X(\infty) \vee \overline{X}^S_{\max}$.

One of the main results in [10] is the following:

Theorem 2.5. *With the notation as above, for $\lambda_0 = \lambda_0(X)$, $X \cup \partial_{\lambda_0} X$ is homeomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$ as a G-space; and for $\lambda < \lambda_0(X)$, $X \cup \partial_\lambda X$ is homeomorphic to $X \cup X(\infty) \vee \overline{X}^S_{\max}$ as a G-space.*

The G-action on X extends to the compactifications $\overline{X}^S_{\max}$, $X \cup \partial_\lambda X$ and $X \cup X(\infty) \vee \overline{X}^S_{\max}$. In Theorem 2.5, the homeomorphisms restrict to the identity map on X and are equivariant with respect to the G-actions.

As a corollary of Theorem 2.5 and the proof of Theorem 2.4, we get the following.

Corollary 2.6 (§4, §5). *For any $\lambda \le \lambda_0(X)$, the Martin compactification $X \cup \partial_\lambda X$ is homeomorphic to the closed unit ball in the tangent space $T_{x_0}X$.*

This corollary is one of the motivations of this note (see §5 below for some discussions about the Martin compactification of more general manifolds). In Theorem 2.4 and Corollary 2.6, the symmetric space X is mapped to the interior of the unit ball in $T_{x_0}X$. If the rank of X is greater than or equal to 2, these two maps from X to the open unit ball in $T_{x_0}X$ are not the inverse of the map obtained by retracting the exponential map along the radial direction into the unit ball.

As far as we know, the only known similar result is a theorem of Kusner [15] that the Karpelevic compactification of X defined in [12] is homeomorphic to the closed unit ball in $T_{x_0}X$. The original definition of the Karpelevic compactification is inductive on the rank and very complicated, and a more direct characterization of the Karpelevic compactification $\overline{X}^K$ is given in [10, Chap 4]. Based on this description, it is proved there that $\overline{X}^K$ always dominates $X\cup\partial_\lambda X$.

These results might suggest that any nontrivial (i.e., not one point) G-compactification of X should be homeomorphic to the closed unit ball. This turns out to be not true if the rank of X is greater than or equal to 2. In fact, we can construct a G-compactification of X whose boundary is a simplex of dimension rank$(X)-1$, the so-called Weyl chamber at infinity. This compactification is clearly not a ball since the codimension of the boundary is greater than 1.

This compactification can roughly be constructed as follows. Let $\mathfrak{a}^+$ be a positive Weyl chamber, and $\overline{\mathfrak{a}^+}$ its closure. Then X has polar coordinates decomposition $X = K\exp(\overline{\mathfrak{a}^+})x_0$. The subspace $\exp(\overline{\mathfrak{a}^+})x_0$ can be compactified by adding the Weyl chamber at infinity $\overline{\mathfrak{a}^+}(\infty)$, which represents all the directions in $\exp(\overline{\mathfrak{a}^+})x_0$ going to infinity. Let K act trivially on this boundary component. Then we get the compactification $K(\exp(\overline{\mathfrak{a}^+})x_0\cup\overline{\mathfrak{a}^+}(\infty))$.

On the other hand, it is conceivable that any G-compactification of X which is bigger than either $X\cup X(\infty)$ or any Satake compactification $\overline{X}^S_\tau$, or more generally any G-compactification such that the closure of a flat is a topological ball, should be homeomorphic to the closed unit ball in $T_{x_0}X$.

3. The convexity of the moment map

The proof of Theorem 2.4 depends crucially on a convexity result of Atiyah (see Theorem 3.1 below). To state and apply this result, we need some preparation.

An even dimensional manifold M is a symplectic manifold if there exists a closed nondegenerate 2 form ω on M. An important class of symplectic manifolds are Kähler manifolds, whose Kähler form is closed and positive definite (in particular, nondegenerate).

Let $T = (S^1)^n$ be a compact torus, $n \geq 1$. Assume that T acts sympletically on M, i.e., preserves the sympletic form ω, and that M is simply connected. For any $v \in \mathfrak{t}$, the Lie algebra of T, denote also by v the vector field on M induced by v. Then $\omega(v,\cdot)$ is a closed 1 form on M, and hence there exists a function $\varphi^v \in C^\infty(M)$ such that $d\varphi^v = \omega(v,\cdot)$.

The functions φ^v, $v \in \mathfrak{t}$, can be chosen up to a constant so that the map $v \to \varphi^v$ becomes a Lie algebra homomorphism when $C^\infty(M)$ is given the Poisson structure. These functions define the moment map $\Phi : M \to \mathfrak{t}^*$ of the T-action as follows: For any $v \in \mathfrak{t}, \varphi^v = \Phi(v)$. (see [13, §2] [4] for details of symplectic manifolds and the moment map).

Assume now that M is a Kähler manifold and T acts on M symplectically and holomorphically. Then the T action extends to the complex torus $T_{\mathbb{C}}$, the complexification of T. Let Y be an orbit of the complex torus $T_{\mathbb{C}}$, and $\overline{Y}$ its closure in M. The moment map Φ restricts to a map $\Phi : \overline{Y} \to \mathfrak{t}^*$. Then we have the following theorem of Atiyah [3, Theorem 2].

Theorem 3.1. *The image $\Phi(\overline{Y})$ is a bounded convex polytope in $\mathfrak{t}^*$ whose vertices are the image of the fixed points of T in $\overline{Y}$; and Φ is a homeomorphism from the quotient $\overline{Y}/T$ to the convex polytope $\Phi(\overline{Y})$.*

Since $T_{\mathbb{C}}/T \cong \mathbb{R}^n$, $\overline{Y}/T$ is the closure of an orbit of the noncompact part of the complex torus. This theorem shows that this closure is homeomorphic to a convex polytope in a canonical way.

To apply this result to study the Satake compactifications, we need to compute the related moment map.

We start with the example $M = P^{n-1}(\mathbb{C})$, $T = (S^1)^n$ [4, §4]. The compact torus T acts on $P^{n-1}(\mathbb{C})$ as follows: Let $(z_1, \cdot, z_n)$ be the homogeneous coordinates of $P^{n-1}(\mathbb{C})$. Then for any $(e^{i\theta_1}, \cdots, e^{i\theta_n}) \in T$, $(z_1, \cdot, z_n) \to (e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n)$. Clearly T acts holomorphically and preserves the Kähler form. Identify $P^{n-1}(\mathbb{C})$ with $\{(z_1, \cdots, z_n) \mid \sum_1^n \frac{1}{2}|z_j|^2 = 1\}/ \sim$, where $(z_1, \cdot, z_n) \sim (z_1', \cdot, z_n')$ if there exists $\theta \in \mathbb{R}$ such that $(z_1, \cdot, z_n) = e^{i\theta}(z_1', \cdot, z_n')$. Then the moment map $\Phi : P^{n-1}(\mathbb{C}) \to \mathfrak{t}^* = \mathbb{R}^n$ is given by $\Phi((z_1, \cdot, z_n)) = (\frac{1}{2}|z_1|^2, \cdots, \frac{1}{2}|z_n|^2)$, and the image $\Phi(P^{n-1}(\mathbb{C}))$ is the standard simplex of dimension $n-1$ with vertices $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1) \in \mathbb{R}^n$.

Lemma 3.2. *Let $\tau : T \to \mathrm{SL}(N, \mathbb{C})$ be a representation with image $\tau(T) \subset \mathrm{SU}(n)$. Denote the weights of τ by $\mu_1, \cdots, \mu_N$. Then T acts on $P^{N-1}(\mathbb{C})$ through τ, and the image of the moment map $\Phi(P^{N-1}(\mathbb{C}))$ is the convex hull of the weights $\mu_1, \cdots, \mu_n$ in $\mathfrak{t}^*$.*

Proof. Decompose $\mathbb{C}^N$ into irreducible subspaces $V_1 + \cdots + V_N$, where V_j has weight μ_j. Let $(S^1)^N$ act on $\mathbb{C}^N$ by $(v_1, \cdots, v_N) \to (e^{i\theta_1} v_1, \cdots, e^{i\theta_N} v_N)$, where $(e^{i\theta_1}, \cdots, e^{i\theta_N}) \in (S^1)^N$. Then the T action on $P^{N-1}(\mathbb{C})$ factors through the map $T \to (S^1)^N : e^X \to (e^{\mu_1(X)}, \cdots, e^{\mu_N(X)})$, for any $X \in \mathfrak{t}$. Therefore the moment map Φ of T is obtained by composing the moment map for $(S^1)^N$ with the projection $\mathbb{R}^N \to \mathfrak{t}^*$. It is clear that the vertices $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$ in $\mathbb{R}^N$ are projected to $\mu_1, \cdots, \mu_N$ in $\mathfrak{t}^*$. Then it follows from the above example that the image $\Phi(P^{N-1}(\mathbb{C}))$ is the convex hull of $\mu_1, \cdots, \mu_N$ in $\mathfrak{t}^*$.

4. Proof of Theorem 2.4 and Corollary 2.6

In this section, we first study the closure of a flat of X in the Satake compactifications. Then we use it to prove Theorem 2.4 and Corollary 2.6.

Recall that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Then $e^{\mathfrak{a}} x_0 \subset X$ is a totally geodesic flat submanifold in X of maximal dimension, a so-called flat in X. Conversely, any flat in X containing x_0 is of this form.

It follows from the Cartan decomposition that $X = K e^{\mathfrak{a}} x_0$. For any G-compactification $\overline{X}$ of X, let $\overline{e^{\mathfrak{a}} x_0}$ be the closure of the flat $e^{\mathfrak{a}} x_0$. Then $\overline{X} = K\overline{e^{\mathfrak{a}} x_0}$. So it is important to understand the closure $\overline{e^{\mathfrak{a}} x_0}$ in order to study the compactification $\overline{X}$.

Let $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$ be an irreducible faithful projective representation as in §2. Denote the weights of τ in $\mathfrak{a}^*$ by $\mu_1, \cdots, \mu_n$.

Proposition 4.1. *The closure of the flat $e^{\mathfrak{a}} x_0$ in the Satake compactification $\overline{X}^S_\tau$ is homeomorphic to the convex hull of $2\mu_1, \cdots, 2\mu_n$ in $\mathfrak{a}^*$. Furthermore, for any two flats $e^{\mathfrak{a}} x_0$, $e^{\mathfrak{a}'} x_0$, where $\mathfrak{a}'$ is another maximal abelian subalgebra of $\mathfrak{p}$, their homeomorphisms have the same restriction to their intersection $e^{\mathfrak{a}} x_0 \cap e^{\mathfrak{a}'} x_0$.*

Proof. Let $\mathcal{H}_n \otimes \mathbb{C}$ be the complex vector space spanned by the real vector space of Hermitian matrices, and $P(\mathcal{H}_n \otimes \mathbb{C})$ its complex projective space. Then the real projective space $P(\mathcal{H}_n)$ can be identified canonically with a subspace of $P(\mathcal{H}_n \otimes \mathbb{C})$.

Choose a basis of $\mathbb{C}^n$ such that for any $H \in \mathfrak{a}$, $\tau(e^H)$ is a diagonal matrix $(e^{\mu_1(H)}, \cdots, e^{\mu_n(H)})$. Then $e^{\mathfrak{a}}$ acts on $\mathcal{H}_n$ by multiplying from both sides: For $H \in \mathfrak{a}$ and $M \in \mathcal{H}_n$,

$$M \to \tau(e^H) M \tau(e^H).$$

This action extends to a holomorphic action of the complex torus $e^{\mathfrak{a}+i\mathfrak{a}}$ on $\mathcal{H}_n \otimes \mathbb{C}$ and hence on $P(\mathcal{H}_n \otimes \mathbb{C})$. The compact torus $e^{i\mathfrak{a}}$, which is contained in the compact dual of G, acts on $\mathcal{H}_n \otimes \mathbb{C}$ as follows: For $e^{iH} \in e^{i\mathfrak{a}}$ and $M = (m_{jk}) \in \mathcal{H}_n \otimes \mathbb{C}$, $(m_{jk}) \to (e^{i(\mu_j(H)+\mu_k(H))} m_{jk})$. Hence the compact torus $e^{i\mathfrak{a}}$ preserves the Kähler form of $P(\mathcal{H}_n \otimes \mathbb{C})$. Denote the moment of this action by $\Phi : P(\mathcal{H}_n \otimes \mathbb{C}) \to \mathfrak{a}^*$.

Let Id be the point in $P(\mathcal{H}_n \otimes \mathbb{C})$ corresponding to the line which contains the identity matrix. Then the closure of the orbit $e^{\mathfrak{a}} Id$ in $P(\mathcal{H}_n \otimes \mathbb{C})$ is the closure of the flat $e^{\mathfrak{a}} x_0$ in the Satake compactification $\overline{X}^S_\tau$ and will be identified using the moment map Φ.

Let Y be the orbit $e^{\mathfrak{a}+i\mathfrak{a}} Id$ of the complex torus $e^{\mathfrak{a}+i\mathfrak{a}}$ in $P(\mathcal{H}_n \otimes \mathbb{C})$, and $\overline{Y}$ its closure. Then the closure of the orbit $e^{\mathfrak{a}} Id$ can be identified with quotient of $\overline{Y}$ by the compact torus $e^{i\mathfrak{a}}$. Therefore, it follows from Theorem 3.2 that the closure of the flat $e^{\mathfrak{a}} x_0$ in $\overline{X}^S_\tau$ is homeomorphic to the convex polytope $\Phi(\overline{Y})$.

We claim that $\Phi(\overline{Y})$ is the convex hull of $2\mu_1, \cdots, 2\mu_n$. To prove the claim, let D_n be the complex subspace of diagonal matrices in $\mathcal{H}_n \otimes \mathbb{C}$ and $P(D_n)$ its associated complex projective space. Then the complex torus $e^{\mathfrak{a}+i\mathfrak{a}}$ preserves D_n and the orbit $e^{\mathfrak{a}+i\mathfrak{a}} Id$ is contained in $P(D_n)$. The weights of the representation $e^{\mathfrak{a}+i\mathfrak{a}}$ on D_n are $2\mu_1, \cdots, 2\mu_n$. So it follows from Lemma 3.2 that $\Phi(P(D_n))$ is the convex hull of $2\mu_1, \cdots, 2\mu_n$; in particular, $\Phi(\overline{Y})$ is contained in this convex hull.

Let $2\mu_j$ be a vertex of this convex hull. For simplicity, assume $j = 1$. Then $2\mu_1$ is the image of $(1, 0, \cdots) \in P(D_n)$ (see the proof of Lemma 3.2). To complete the proof of the claim, it suffices to prove that $(1, 0, \cdots, 0) \in \overline{Y}$.

Since $2\mu_1$ is an extremal weight, there exists a positive Weyl chamber $\mathfrak{a}^+$ such that $2\mu_1$ is the highest weight of the representation on D_n. Then for any $j \geq 2$,

$$2\mu_1 - 2\mu_j = \sum_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})} c_{\alpha,j}\alpha, \ \ c_{\alpha,j} \geq 0, \quad \sum_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})} c_{\alpha,j} > 0,$$

where $\Delta(\mathfrak{g},\mathfrak{a})$ is the set of simple roots with respect to the choice of the positive Weyl chamber $\mathfrak{a}^+$. For any $H \in \mathfrak{a}^+$, as $t \to +\infty$,

$$\begin{aligned}\tau(e^{tH})\cdot Id &= (e^{2\mu_1(tH)}, e^{2\mu_2(tH)}, \cdots, e^{2\mu_n(tH)})\\ &= (1, e^{2\mu_1(tH)-2\mu_2(tH)}, \cdots, e^{2\mu_1(tH)-2\mu_n(tH)})\\ &\to (1,0,\cdots,0) \ \text{ in } \ P(D_n).\end{aligned}$$

Therefore, $2\mu_1 \in \Phi(\overline{Y})$ and the claim is proved. This proves the first statement of Proposition 4.1.

For any two flats $e^{\mathfrak{a}}x_0$ and $e^{\mathfrak{a}'}x_0$, their intersection $e^{\mathfrak{a}}x_0 \cap e^{\mathfrak{a}'}x_0 = e^{\mathfrak{a}_1}x_0$, where $\mathfrak{a}_1 = \mathfrak{a}\cap\mathfrak{a}'$ is a linear subspace of the form $\{H \in \mathfrak{a} \mid \alpha(H) = 0, \alpha \in I\}$, where I is a subset of $\Delta(\mathfrak{g},\mathfrak{a})$. The restrictions to the intersection $e^{\mathfrak{a}}x_0 \cap e^{\mathfrak{a}'}x_0$ of the two moment maps associated with the tori $e^{i\mathfrak{a}}$ and $e^{i\mathfrak{a}'}$ are equal to the moment map for the subtorus $e^{i\mathfrak{a}_1}$, and hence are the same. The proof of Proposition 4.1 is complete.

Remark 4.2. This trick of using the complexification is due to Atiyah [3, p.13]. The closure of $\mathfrak{a}$ in some Grassmannians is first studied in [9, Theorem 2.3.4] without explicitly using the Hamiltonian action of a compact torus.

Remark 4.3. Let W be the Weyl group of $\mathfrak{a}$ and μ_τ be the highest weight of the representation $\tau : G \to \mathrm{PSL}(n,\mathbb{C})$ with respect to a positive Weyl chamber. Then the convex hull of the weights $2\mu_1, \cdots, 2\mu_n$ is the same as the convex hull of the orbit $W(2\mu_\tau)$ (see [8, p.204]). This suggests that the Satake compactification be determined by the degeneracy of the highest weight μ_τ, i.e., the face of the Weyl chamber which contains μ_τ as an interior point, and hence there are only finitely many different Satake compactifications (see [17, Theorem 2 on p. 102] for details). If μ_τ is generic, i.e., an interior point of the positive Weyl chamber, then $\overline{X}^S_\tau$ is the maximal Satake compactification $\overline{X}^S_{\max}$.

Proof of Theorem 2.4. For any maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, identify $\mathfrak{a}^*$ with $\mathfrak{a}$ using the Killing form. Then it follows from Proposition 4.1 that the closure of every flat $e^{\mathfrak{a}}x_0$ in $\overline{X}^S_\tau$ can be mapped homeomorphically to the convex hull of $2\mu_1, \cdots, 2\mu_n$ in $\mathfrak{a}$. This convex hull is a bounded convex polytope which contains the origin as an interior point.

Any ray from the origin to a boundary point of the polytope can be scaled to have length one. This scaling defines a homeomorphism from this bounded convex polytope to the closed unit ball in $\mathfrak{a}$. By composition, we get a homeomorphism from the closure of $e^{\mathfrak{a}}x_0$ in $\overline{X}^S_\tau$ to the closed unit ball in $\mathfrak{a}$. See Figure 1 for $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $\dim\mathfrak{a} = 2$.

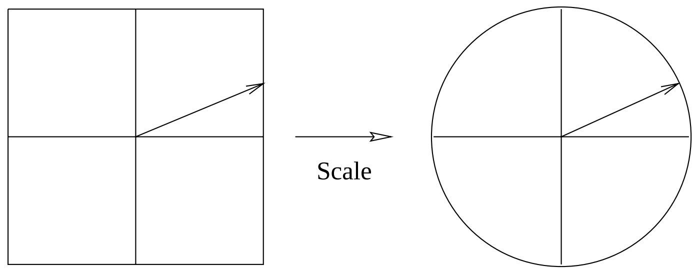

Figure 1

Since $X = Ke^{\mathfrak{a}}x_0 = \cup_{k\in K} k(e^{\mathfrak{a}}x_0)$, a union of flats, and these homeomorphisms on the flats coincide on their intersection, they glue together and define a homeomorphism from $\overline{X}^S_\tau$ to the closed unit ball in $\mathfrak{p} = K \cdot \mathfrak{a}$. Since $\mathfrak{p}$ can be identified with the tangent space $T_{x_0}X$, Theorem 2.4 is proved.

Proof of Corollary 2.6. Let τ be a representation whose highest weight is an interior point of the positive Weyl chamber $\mathfrak{a}^{*+}$. Then $\overline{X}^S_\tau$ is the maximal Satake compactification $\overline{X}^S_{\max}$ (See Remark 4.3 above). According to Proposition 4.1, the moment map Φ defines a homeomorphism from the closure of the flat $e^{\mathfrak{a}}x_0$ in $\overline{X}^S_\tau$ to the convex hull of $2\mu_1, \cdots, 2\mu_n$ in $\mathfrak{a}^*$.

On the other hand, the closure of the flat $e^{\mathfrak{a}}x_0$ in $X \cup X(\infty)$ is homeomorphic to the closed unit ball in $\mathfrak{a}$ under the map

$$C : e^H x_0 \to H/(1 + |H|),$$

where $|H|$ is the norm defined by the Killing form.

According to Theorem 2.5, if $\lambda = \lambda_0(X)$, the Martin compactification $X \cup \partial_\lambda X$ is homeomorphic to the maximal Satake compactification $\overline{X}^S_\tau$ and hence homeomorphic to the closed unit ball in $T_{x_0}X$, by Theorem 2.4.

On the other hand, if $\lambda < \lambda_0(X)$, the closure $\overline{e^{\mathfrak{a}}x_0}$ of the flat $e^{\mathfrak{a}}x_0$ in $X \cup \partial_\lambda X$ is homeomorphic to the closure of the flat $e^{\mathfrak{a}}x_0$ in $X \cup X(\infty) \times \overline{X}^S_\tau$ under the diagonal embedding. Identify $\mathfrak{a}^*$ with $\mathfrak{a}$ using the Killing form as above. Then this closure $\overline{e^{\mathfrak{a}}x_0}$ is mapped homeomorphically into a bounded subset of $\mathfrak{a} \times \mathfrak{a}$ under the map

$$\Phi \times C : e^H x_0 \to (\Phi(e^H x_0), C(e^H x_0)).$$

Denote by π the orthogonal projection from $\mathfrak{a} \times \mathfrak{a}$ to the diagonal subspace $\{(H, H) \in \mathfrak{a} \times \mathfrak{a}\} = \mathfrak{a}$. Then $\pi(\Phi \times C(\overline{e^{\mathfrak{a}}x_0}))$ is a bounded convex domain in $\mathfrak{a}$ around the origin. Intuitively, this domain can be obtained by blowing up the vertices and some faces of the convex polytope $\overline{\Phi(e^{\mathfrak{a}}x_0)}$ to accommodate for directions approaching them. See Figure 2 for $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$.

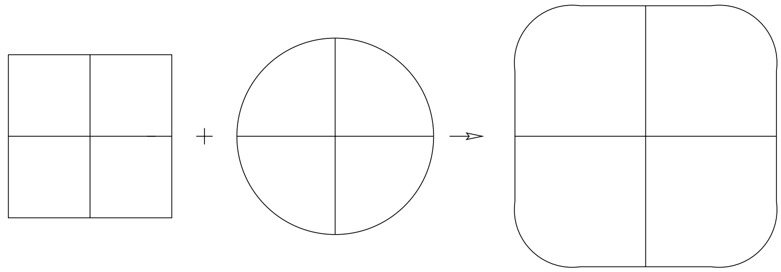

Figure 2

We claim that π defines a homeomorphism from $\Phi \times C(\overline{e^{\mathfrak{a}} x_0})$ to this bounded convex domain in $\mathfrak{a}$. Note that $\pi(\Phi \times C(e^H x_0)) = 1/\sqrt{2}(\Phi(e^H x_0) + C(e^H x_0))$. Since both Φ and C are homeomorphisms on the interior $e^{\mathfrak{a}} x_0$ and have positive definite Jacobian matrices, π is a homeomorphism on the interior $\Phi \times C(e^{\mathfrak{a}} x_0)$ of $\Phi \times C(\overline{e^{\mathfrak{a}} x_0})$. On the other hand, the boundary $\Phi \times C(\partial \overline{e^{\mathfrak{a}} x_0})$ is locally of the form $F \times S$, where F is a face of the convex polytope $\overline{\Phi(e^{\mathfrak{a}} x_0)}$ and S is a spherical simplex on the unit sphere in $\mathfrak{a}$ of dimension $\dim \mathfrak{a} - \dim F - 1$. Since π is a homeomorphism on such a set, π is also a homeomorphism on the boundary $\Phi \times C(\partial \overline{e^{\mathfrak{a}} x_0})$.

Therefore we have a homeomorphism from the closure $\overline{e^{\mathfrak{a}} x_0}$ of the flat $e^{\mathfrak{a}} x_0$ in $X \cup \partial_\lambda X$ to a bounded convex domain in $\mathfrak{a}$ around the origin. The homeomorphisms for different flats coincide on their intersection. Then the same arguments as in the proof of Theorem 2.4 finish the proof of Corollary 2.6.

5. Comments on Martin compactifications

Let M be a complete noncompact Riemannian manifold and Δ the Laplace operator. As mentioned earlier, one of the motivations of the Martin compactification of M is to parametrize a set of generators of the cone of positive solutions of $\Delta u + \lambda u = 0$ on M in terms of the Martin boundary $\partial_\lambda M$.

If M is the unit ball with the Poincare hyperbolic metric, then the Martin boundary is the unit sphere and the Martin compactification is the closed unit ball.

In his ICM talk [7, p.21], Dynkin raised a general question concerning the connection between the geometry of the manifold M and structure of the Martin boundary $\partial_\lambda M$. Based on the last few paragraphs of [7] and recent developments on Martin compactifications, in particular, Corollary 2.6, it seems appropriate to formulate the following:

Conjecture 5.1. *If M is simply connected and nonpositively curved, i.e., a Hadamard manifold, then for any $\lambda < \lambda_0(M)$, the Martin compactification $M \cup \partial_\lambda M$ is homeomorphic to the closed unit ball in $T_{x_0} M$, and the Martin boundary $\partial_\lambda M$ is homeomorphic to the unit sphere and hence has codimension 1.*

In general, the Martin compactification $M \cup \partial_\lambda M$ is only a metrizable topological space. This conjecture says that for a Hadamard manifold M, $M \cup \partial_\lambda M$ is a topological manifold with boundary.

If the sectional curvature K_M of M is negatively pinched, i.e., $-b^2 \leq K_M \leq a^2 < 0$ for some constants a and b, then Ancona [1] proved that for any $\lambda < \lambda_0(M)$, $M \cup \partial_\lambda M$ is canonically homeomorphic to the conic compactification $M \cup M(\infty)$, which is homeomorphic to the closed unit ball in $T_{x_0}M$; in particular, Conjecture 5.1 is true.

If M is the Euclidean space, then $\lambda_0(M) = 0$, and for any $\lambda < 0$, $M \cup \partial_\lambda M$ is also canonically homeomorphic to the conic compactification $M \cup M(\infty)$ (see [14, §2]), and hence Conjecture 5.1 holds.

If M is a symmetric space of noncompact type of rank greater than 1, Corollary 2.6 shows that the above conjecture holds even though $M \cup \partial_\lambda M$ is strictly bigger than $M \cup M(\infty)$.

For other nonpositively curved and simply connected manifolds, in particular, the rank one manifolds [1], the problem of identifying the Martin compactification is open (see [18, Problem 47] and the introduction of [10]).

Remark 5.2. If $M = \mathbb{R}^n$ and $\lambda = \lambda_0(M) = 0$, then $M \cup \partial_\lambda M$ is a one point compactification. Ancona [2, §3] constructed a Hadamard manifold with negatively pinched sectional curvature such that $M \cup \partial_{\lambda_0} M$ is one point compactification. This is the reason why $\lambda = \lambda_0(M)$ is excluded from Conjecture 5.1.

Remark 5.3. The assumption that M is nonnegatively curved is necessary for the Martin boundary $\partial_\lambda M$ to have high dimension. For example, take $M = S^n \times \mathbb{R}$, where S^n is the a sphere of dimension $n \geq 2$ and $\mathbb{R}$ is the line. Then for any $\lambda < \lambda_0(M) = 0$, the cone of positive solutions of $\Delta u - \lambda u = 0$ has two only generators and hence $\dim \partial_\lambda M \leq 1$.

Remark 5.4. The assumption that M is simply connected is also important. For example, let M be a finite volume quotient of the 3 dimensional hyperbolic space $\mathbb{H}^3$. Then $\lambda_0(M) = 0$ and for any $\lambda < 0$, $M \cup \partial_\lambda M$ is obtained by adding a point to every end of M. In this example, the injectivity radius goes to zero near infinity.

Another example is $\mathbb{R}^n / \mathbb{Z}^{n-1} \cong (S^1)^{n-1} \times \mathbb{R}$, where $n \geq 2$ and $\mathbb{Z}^{n-1}$ acts by translation on the first $n-1$ coordinates. In this example, the injectivity radius is bounded away from zero, but the Martin boundary is of dimension less than 1.

Acknowledgements

I would like to thank Professor R. MacPherson for pointing out Proposition 4.1. I would also like to thank Professors A. Korányi and J. C. Taylor for conversations on connection between Satake compactifications and polyhedral

[1] A simply connected and nonpositively curved manifold is called a rank one manifold if it is not a product or symmetric space and admits a finite volume quotient.

cone decomposition of flats [11] and Y. Guivarch for the reference [7] and the formulation of the problem considered in this paper. After this paper is finished, I found a related result in a preprint of Casselman [6] that the closure of a flat in the Satake compactification is combinatorially equivalent to the convex polytope in Proposition 4.1.

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