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FINITENESS PROPERTIES AND ABELIAN QUOTIENTS OF GRAPH GROUPS

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Abstract. We describe the homological and homotopical Σ-invariants of graph groups in terms of topological properties of sub-flag complexes of finite flag complexes. Bestvina and Brady have recently established the existence of FP groups which are not finitely presented; their examples arise as kernels of maps from graph groups to $\mathbb{Z}$. Since the Σ-invariants of a group G determine the finiteness properties of all normal subgroups above the commutator of G, our Main Theorem extends the work of Bestvina and Brady. That is, our Theorem determines the finiteness properties of kernels of maps from graph groups to abelian groups. Applications of this result are indicated.

Given a finite simplicial graph $\mathcal{G}$ the corresponding *graph group*, or 'right-angled Artin group', has generators corresponding to the vertices of $\mathcal{G}$, where two generators commute if and only if they are adjacent in $\mathcal{G}$. The graph $\mathcal{G}$ is the *defining graph* and the corresponding graph group is denoted $G\mathcal{G}$. For example, if the defining graph $\mathcal{G}$ is the 1-skeleton of an octahedron, then the graph group $G\mathcal{G}$ is the direct product of three copies of F_2. (The class of graph groups includes all finite direct products of free groups.) Graph groups have finite $K(\pi, 1)$'s which can be constructed by glueing tori associated to cliques in $\mathcal{G}$ [MV1]. We denote the $K(G\mathcal{G}, 1)$ by $K\mathcal{G}$, and mention that its universal cover $\widetilde{K\mathcal{G}}$ is CAT(0).

Subgroups of graph groups provide examples of infinite groups exhibiting one kind of 'finiteness' but not another. Recall that a group G is $\mathcal{F}_m$ if and only if there is a $K(G, 1)$ with finite m-skeleton. The properties $\mathcal{F}_1$ and $\mathcal{F}_2$ are topological reformulations of the two most common finiteness conditions, finite generation and finite presentation. On the other hand, a group G is FP_m if $\mathbb{Z}$, thought of as a trivial $\mathbb{Z}G$-module, admits a projective resolution with finitely generated m-skeleton. (See Chapter VIII of [Br] for background on finiteness properties of infinite groups.)

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Let G_m be the direct product of m copies of a finitely generated non-abelian free group F. Let $\chi : G_m \to \mathbb{Z}$ be the map where each generator is carried to $1 \in \mathbb{Z}$. Then Stallings ($m = 3$) and Bieri ($m > 3$) have shown that the kernel of χ is $\mathcal{F}_{m-1}$ but not $\mathcal{F}_m$ ([S] and [Bi]). Using covering spaces it is easy to show that $\mathcal{F}_m \Rightarrow \mathrm{FP}_m$; Bestvina and Brady have shown that $\mathrm{FP}_m \not\Rightarrow \mathcal{F}_m$ for any $m \geq 2$ [BB]. Their work is a natural outgrowth of the work of Stallings and Bieri in that they discovered their groups by examining kernels of maps $\chi : G\mathcal{G} \to \mathbb{Z}$, each generator again going to $1 \in \mathbb{Z}$.

Here we announce a complete description of the Σ-invariants of graph groups. Among other things, these Σ-invariants determine the finiteness properties of normal subgroups $N \triangleleft G$ with G/N abelian. In particular, one recovers the examples of Stallings, Bieri, Bestvina and Brady as special cases of our Main Theorem. Of course, this previous work provided insight into the correct characterization of the Σ-invariants, and the work of Bestvina and Brady simplifies a key step in the proof of our Main Theorem.

Partial computations of the Σ-invariants of graph groups have already appeared. The structure of the Σ-invariants was known in the case when $G\mathcal{G}$ is a direct product of finitely generated free groups [Mn1]. Complete descriptions of the invariants $\Sigma^1(G\mathcal{G})$ and $\Sigma^2(G\mathcal{G})$ for arbitrary graph groups were also known ([MV1], [Mn2], and [MV2]). The work in [MV2] leads naturally to the statement of our Main Theorem, and the arguments there have been directly extended by the first and third authors to establish this characterization using the 'Σ^n-criterion'. (See Appendix B of [BS] or §4 of [BR] for criteria establishing that a map is in $\Sigma^n(G)$ or $\Sigma^n(G, \mathbb{Z})$.) Instead of working with the action of $G\mathcal{G}$ on $\widetilde{K\mathcal{G}}$, the second author established our characterization using a non-free action of $G\mathcal{G}$ on a cubical complex. (The construction of this $G\mathcal{G}$ action is a special case of the work in [Mr] and/or [HM] which work with the more general graph *products*.) This version of a proof proceeds by using techniques developed in [HM]. We are currently combining our approaches in [MMV].

Given a group G (which for our purposes we may assume has a $K(G, 1)$ that is finite in all dimensions), there are two sequences of geometric invariants: The homotopical invariants $\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots$, and the homological invariants $\Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^2(G, \mathbb{Z}) \supseteq \cdots$. The first invariants in these sequences, $\Sigma^1(G)$ and $\Sigma^1(G, \mathbb{Z})$, are the same, and were introduced in a paper by Bieri, Neumann, and Strebel [BNS]. The higher invariants are not always equal, and were introduced by Bieri and Renz [BR].

The set of all *characters* is the complement of the zero map in the real vector space $\mathrm{Hom}(G, \mathbb{R})$. For any character χ let $[\chi] = \{r\chi \mid 0 < r \in \mathbb{R}\}$ be a ray in $\mathrm{Hom}(G, \mathbb{R})$; the set of all such rays is denoted $S(G)$ and should be thought of as a 'sphere' inside the real vector space $\mathrm{Hom}(G, \mathbb{R})$. Since any character of a graph group $G\mathcal{G}$ must factor through the abelianization of $G\mathcal{G}$, $S(G\mathcal{G}) \simeq S^{|V(\mathcal{G})|-1}$.

The Bieri-Neumann-Strebel-Renz invariants $\Sigma^n(G, \mathbb{Z})$ and $\Sigma^n(G)$ are subsets of $S(G)$. These Σ-invariants have fairly geometric descriptions which are quite concrete in the case of graph groups. Since the complex $K\mathcal{G}$ contains a single vertex, there is a one-to-one correspondence between vertices in $\widetilde{K\mathcal{G}}$ and elements in $G\mathcal{G}$. Thus corresponding to any character $\chi : G\mathcal{G} \to \mathbb{R}$ one can define a map $\widetilde{\chi} : \widetilde{K\mathcal{G}} \to \mathbb{R}$; the map $\widetilde{\chi}$ is defined on the vertices of $\widetilde{K\mathcal{G}}$ by $\widetilde{\chi}(v) = \chi(g)$ if $v = b \cdot g$ for some fixed base vertex b, and is extended linearly and $G\mathcal{G}$-equivariantly from the vertices to the entire universal cover. Let $\widetilde{K\mathcal{G}}_\chi^{[a,\infty)}$ denote the maximal subcomplex in $\widetilde{K\mathcal{G}} \cap \widetilde{\chi}^{-1}[a, \infty)$. For any non-negative constant d, the inclusion $\widetilde{K\mathcal{G}}_\chi^{[0,\infty)} \hookrightarrow \widetilde{K\mathcal{G}}_\chi^{[-d,\infty)}$ induces a map between reduced homology groups $\widetilde{H}_i(\widetilde{K\mathcal{G}}_\chi^{[0,\infty)}) \to \widetilde{H}_i(\widetilde{K\mathcal{G}}_\chi^{[-d,\infty)})$ and a map between homotopy groups $\pi_i(\widetilde{K\mathcal{G}}_\chi^{[0,\infty)}) \to \pi_i(\widetilde{K\mathcal{G}}_\chi^{[-d,\infty)})$. A character χ represents a point in $\Sigma^n(G\mathcal{G}, \mathbb{Z})$ [resp. $\Sigma^n(G\mathcal{G})$] if and only if there exists a non-negative constant d such that the induced map on the reduced homology groups [resp. homotopy groups] is zero for $i < n$. (See [BS] for a complete introduction to the Σ-invariants; for more widely accessible sources, see [BNS] and [BR].)

Given any normal subgroup N, with $G/N = A$ abelian, one can look at the subsphere

$$S(G, N) = \{[\chi \circ \phi] \mid \chi \in \mathrm{Hom}(A, \mathbb{R})\} = \{[\chi] \in S(G) \mid \chi(N) = 0\}$$

corresponding to N, in $S(G)$. One particularly convincing reason to study the Σ-invariants of a group is the following result due to Bieri and Renz ($m \geq 2$), building from work of Bieri, Neumann and Strebel ($m = 1$).

Theorem 1. ([BNS] and [BR]) *Let G be an* FP_m [*resp.* $\mathcal{F}_m$] *group, and let N be a normal subgroup of a group G with abelian quotient group G/N. Then N is* FP_m [*resp.* $\mathcal{F}_m$] *if and only if* $S(G, N) \subseteq \Sigma^m(G, \mathbb{Z})$ [*resp.* $S(G, N) \subseteq \Sigma^m(G)$].

The flag complex $\widehat{\mathcal{G}}$ induced by a simple graph $\mathcal{G}$ is the simplicial complex formed by filling in each complete subgraph of $\mathcal{G}$ by a simplex. If $\chi : G\mathcal{G} \to \mathbb{R}$, the *living vertices* with respect to χ are those vertices v where $\chi(v) \neq 0$. We let $\widehat{\mathcal{L}}_\chi$ denote the flag sub-complex of $\widehat{\mathcal{G}}$ induced by the living vertices. The topology of the *living subcomplex* $\widehat{\mathcal{L}}_\chi \subseteq \widehat{\mathcal{G}}$ determines whether or not $[\chi] \in \Sigma^n(G\mathcal{G})$ or $[\chi] \in \Sigma^n(G\mathcal{G}, \mathbb{Z})$.

Similar to being n-connected, a complex is *n-acyclic* if its reduced homology groups up to and including dimension n, are all trivial.

Definition. A subcomplex L of a simplicial complex K is (-1)-*acyclic-dominating* if it is non-empty, or equivalently, (-1)-acyclic. For $n \geq 0$, L is n-*acyclic-dominating (in K)*, if for any vertex $v \in K - L$, the 'restricted link' $lk_L(v) = lk(v) \cap L$ is $(n-1)$-acyclic and an $(n-1)$-acyclic-dominating subcomplex of the 'entire link' $lk(v)$ of v in K. (When $L = \widehat{\mathcal{L}}_\chi \subseteq \widehat{\mathcal{G}} = K$ is the living subcomplex induced by a map $\chi : G\mathcal{G} \to \mathbb{R}$, $lk_L(v)$ is refered to as the 'living link' of v.)

Main Theorem. *Let $\mathcal{G}$ be a simplicial graph, let $\widehat{\mathcal{G}}$ be the induced flag complex based on $\mathcal{G}$, and let $\chi : G\mathcal{G} \to \mathbb{R}$ be a character. Then*

- (i) $[\chi] \in \Sigma^n(G\mathcal{G}, \mathbb{Z})$ *if and only if the subcomplex $\widehat{\mathcal{L}}_\chi$ is $(n-1)$-acyclic and an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$;*
- (ii) $[\chi] \in \Sigma^n(G\mathcal{G})$ *if and only if the subcomplex $\widehat{\mathcal{L}}_\chi$ is $(n-1)$-connected and an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$.*

The proof of our Main Theorem will appear in [MMV].

Because $\Sigma^n(G)^c$ and $\Sigma^n(G, \mathbb{Z})^c$ are closed for any choice of n and G, and the collection of characters χ where $\widehat{\mathcal{L}}_\chi = \widehat{\mathcal{G}}$ are dense in $S(G\mathcal{G})$, it follows from the Main Theorem that if $\widehat{\mathcal{G}}$ is not $(n-1)$-connected [resp. $(n-1)$-acyclic] then $\Sigma^n(G\mathcal{G}) = \emptyset$ [resp. $\Sigma^n(G\mathcal{G}, \mathbb{Z}) = \emptyset$]. In particular, the Main Theorem generates examples illustrating $\Sigma^n(G\mathcal{G}, \mathbb{Z}) \neq \Sigma^n(G\mathcal{G})$ for $n > 1$; for example, if $\widehat{\mathcal{G}}$ is a flag complex with $\pi_1(\widehat{\mathcal{G}}) \simeq A_5$, then $\Sigma^2(G\mathcal{G}) = \emptyset$ while $\Sigma^2(G\mathcal{G}, \mathbb{Z})$ is dense in $S(G)$.

We believe that the description of the Σ-invariants of graph groups given in the Main Theorem is the first complete computation of the invariants for a large class of groups where the higher dimensional invariants are not — explicitely or implicitely — determined by Σ^1.

Using Theorem 1 and the symmetry present in the Σ-invariants of graph groups, the Main Theorem immediately implies

Corollary. *Let $\mathcal{G}$ be a simplicial graph, let $\widehat{\mathcal{G}}$ be the induced flag complex based on $\mathcal{G}$, and let $\chi : G\mathcal{G} \to \mathbb{Z}$ be a non-zero homomorphism. Then the kernel of χ is*

- (i) FP_n *if and only if $\widehat{\mathcal{L}}_\chi$ is $(n-1)$-acyclic and an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$;*
- (ii) $\mathcal{F}_n$ *if and only if $\widehat{\mathcal{L}}_\chi$ is $(n-1)$-connected and an $(n-1)$-acyclic-dominating subcomplex of $\widehat{\mathcal{G}}$.*

Example 1. In [KKM], Kiralis, Krstić and McCool establish that various groups are finitely presentable. They are primarily interested in groups arising in a K-theoretic context; in particular they study the matrix groups $GL_n(\mathbb{Z}G)$ and their non-abelian analogues $\Phi_n(G)$, as well as the elementary subgroups $E_n(\mathbb{Z}G)$, the Steinberg groups $St_n(\mathbb{Z}G)$ and the corresponding groups $E_n^\Phi(G)$ and $St_n^\Phi(G)$. They prove

Theorem 2. ([KKM]) *If G is finitely presented and $n \geq 4$ then the groups $E_n^\Phi(G), St_n^\Phi(G)$ and $St_n(\mathbb{Z}G)$ are finitely presented.*

and then immediately state: "The hypothesis $n \geq 4$ here is probably too strong; we will show that for a large class of groups it can be replaced with $n \geq 3$."

The key technical step in establishing this theorem is a lemma showing that, given a direct product of n copies of a finitely generated non-abelian free group F, the kernel of the map $F \times F \times \cdots \times F \longrightarrow F^{\mathrm{a\,b}}$ is finitely presented. They establish this when there are four or more copies, but their techniques seemingly can't be pushed through when there are only three copies.

As we remarked at the beginning, the Σ-invariants of finite direct products of free groups were computed by the second author in [Mn1]. Either our Main Theorem, or these previous techniques, quickly establishes the following result.

Proposition 3. ([Mn1]) *Let $G = G_1 \times \ldots \times G_n = F \times \ldots \times F$ be a direct product of $n \geq 1$ copies of a finitely generated non-abelian free group F. Let $\phi : G \to F^{\mathrm{a\,b}}$ be the epimorphism induced by the canonical projections $G_i \to G_i^{\mathrm{a\,b}} \cong F^{\mathrm{a\,b}}$. Then the kernel of ϕ is $\mathcal{F}_{n-1}$ but not* FP_n.

Proof. First we indicate how one applies the techniques of [Mn1] to establish this result. The normal subgroup generated by the kernel K of ϕ together with one direct factor G_i equals G. Thus the *depth $\vartheta(K)$ of K* as defined in [Mn1] is 1. Because $\vartheta(K) = 1$, the Corollary of [Mn1] implies that K is $\mathcal{F}_{n-1}$ but not FP_n.

It is also quite easy to apply the Main Theorem announced here. In order to keep notation to a minimum, we'll assume we are in the case not covered in [KKM], namely, that G is the direct sum of three free groups of rank n. By Theorem 1, in order to show that K is $\mathcal{F}_2$, we need to show that $[\chi \circ \phi] \in \Sigma^2(G)$ for any non-zero homomorphism $\chi : \mathbb{Z}^n \to \mathbb{R}$. The defining graph for $F_n \times F_n$ is simply the complete bipartite graph $K_{n,n}$; hence $\widehat{\mathcal{G}}^{(2)}$ is the union of n cones over $K_{n,n}$ identified along their common copies of $K_{n,n}$. For any $\chi : \mathbb{Z}^n \to \mathbb{R}$ the living subcomplex $\widehat{\mathcal{L}}_{\chi\circ\phi}$ is the union of m cones over $K_{m,m}$ $(0 < m \leq n)$, identified along their common copies of $K_{m,m}$. This is 1-connected. Further, if $v \in \mathcal{G}$ is a dead vertex, then its living link ($\simeq K_{m,m}$) is 0-connected and a 0-acyclic-dominating subcomplex of its entire link ($\simeq K_{n,n}$); hence by our Main Theorem, $[\chi \circ \phi] \in \Sigma^2(G)$. □

Using the work in [KKM], Theorem 2 can be extended to

Theorem 4. (Kiralis, Krstić, McCool) *If G is finitely presented and $n \geq 3$ then the groups $E_n^\Phi(G), St_n^\Phi(G)$ and $St_n(\mathbb{Z}G)$ are finitely presented.*

Example 2. Recently Mihalik has explored behaviour which can occur 'at infinity' in finitely generated groups. In particular, he shows

Proposition 5. (Mihalik) *Let $n \geq 2$, let*

$$G_n = (\mathbb{Z}^n \star \mathbb{Z}) \times (\mathbb{Z}^n \star \mathbb{Z}) = ((\langle u_1\rangle \times \cdots \times \langle u_n\rangle) \star \langle x\rangle) \times ((\langle v_1\rangle \times \cdots \times \langle v_n\rangle) \star \langle y\rangle)$$

and let $\phi : G_n \longrightarrow \mathbb{Z}^n = \langle z_1\rangle \times \cdots \times \langle z_n\rangle$ via $\phi(u_i) = \phi(v_i) = z_i$ and $\phi(x) = \phi(y) = 0$. Then the kernel K_n of ϕ is finitely generated and one-ended, but is not finitely presentable.

Thus the extension of a one-ended, finitely *generated* group by an "arbitrarily well-connected at infinity" group (namely $\mathbb{Z}^n$) is not necessarily simply connected at infinity. This is quite striking, since Jackson proved that if

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1$$

is an exact sequence of infinite, finitely *presented* groups, and either A or B is one-ended, then G is simply connected at infinity. (See [Mi] for background and references.)

Sketch of Proof. First note that G_n is a graph group where the flag complex $\widehat{\mathcal{G}}$, associated to the defining graph $\mathcal{G}$ of G has non-trivial π_1; it follows that $\Sigma^2(G_n) = \emptyset$, hence K_n is not finitely presented. A quick computation using our Main Theorem shows that K_n is finitely generated.

To establish that K_n is one-ended, we note that the map ϕ factors through the natural map $\varphi : G_n \to \mathbb{Z}^{n+1}$ where $\varphi(u_i) = \varphi(v_i) = z_i$ and $\varphi(x) = \varphi(y) = z_{n+1}$. Again, the kernel K_n' of φ is finitely generated, and fits into the short exact sequence

$$1 \longrightarrow K_n' \longrightarrow G_n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow 1.$$

Notice that $\phi = \varepsilon \circ \varphi$ where $\varepsilon : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ via $\varepsilon(z_i) = z_i$ for $1 \leq i \leq n$, but $\varepsilon(z_{n+1}) = 0$. Hence $K_n' \triangleleft K_n$, and

$$1 \longrightarrow K_n' \longrightarrow K_n \longrightarrow \mathbb{Z} \longrightarrow 1.$$

Because K_n is an extension of the finitely generated infinite group K_n' by $\mathbb{Z}$, K_n is one-ended. □

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