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EXOTIC 4-MANIFOLDS WITH $b_2^+ = 1$

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1. Introduction

In this paper we present a new family of simply-connected smooth closed 4-manifolds with $b_2^+ = 1$.

The first examples of simply-connected smooth closed 4-manifolds that are homeomorphic but not diffeomorphic were found by Donaldson, see [D1]. Later hordes of such examples were found, see for example [FM1], [D2], [GM], [FS1], [FS3], [Ko1], [Sz1], [MSz]. While the smooth structures of simply-connected smooth closed 4-manifolds turned out to be very rich, we know much less of the $b_2^+ = 1$ case. The previously studied simply-connected smooth closed 4-manifolds with $b_2^+ = 1$ were all Kähler surfaces: $S^2 \times S^2$, $CP^2 \# n\overline{CP}^2$, $B \# n\overline{CP}^2$ and $E_{p,q} \# n\overline{CP}^2$, where B is the Barlow surface, $E_{p,q}$ is an elliptic surface with geometric genus $p_g = 0$ and two multiple fibers with multiplicity p, q, where $p > 1$, $q > 1$ and $(p, q) = 1$. These 4-manifolds all have different smooth structures, see [D1], [FM1], [Ko1], [Ko2], [Fr], [FM2].

Our first result is the following:

Theorem 1.1. *There exists a family of smooth closed simply-connected 4-manifolds Y_n, parametrized by $n \geq 2$, with $b_2^+(Y_n) = 1$, $b_2^-(Y_n) = 9$ such that*

- (i) *Y_n is irreducible.*
- (ii) *If $k \geq 0$ and $n \neq m$ then $Y_n \# k\overline{CP}^2$ is not diffeomorphic to $Y_m \# k\overline{CP}^2$.*
- (iii) *If $k \geq 0$, then $Y_n \# k\overline{CP}^2$ is not diffeomorphic to any Kähler surface.*

It follows that Y_n form a new family of simply-connected smooth closed 4-manifolds with $b_2^+ = 1$. The construction of Y_n is presented in Section 2. We prove Theorem 1.1 in Section 3 by using Seiberg-Witten invariants in the $b_2^+ = 1$ case.

Using results of Taubes on symplectic 4-manifolds, see [T1], [T2], we can strengthen Theorem 1.1:

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Theorem 1.2. *For all* $n \geq 2$ *neither* Y_n *nor* $\overline{Y}_n$ *have symplectic structure.*

It follows that the 4-manifolds Y_n provide new counter-examples to the Minimal Conjecture. Counter-examples with $b_2^+ > 1$ were given in [Sz2] using a related construction.

2. Construction of Y_n

Let us start by recalling the Kodaira-Thurston manifold [Th], which we denote by W. Let $\phi : T^2 \to T^2$ be an orientation preserving self-diffeomorphism satisfying $\phi_*(a_1) = a_1 + a_2$, $\phi_*(a_2) = a_2$, where $a_1, a_2 \in H_1(T^2, Z)$ form a basis. Let Z_ϕ denote the mapping torus of ϕ. Then W is defined as $W = Z_\phi \times S^1$.

The definition of Z_ϕ gives a fibration $T^2 \to Z_\phi \to S^1$. We can assume that ϕ has fixpoints. Let the circle $\gamma \hookrightarrow Z_\phi$ be a section corresponding to a fixpoint. Let us fix another circle $\delta \hookrightarrow Z_\phi$ that lies in a fiber and represents a_1. Now we define smoothly embedded 2-tori $T_1 = \gamma \times S^1 \hookrightarrow W$ and $T_2 = \delta \times S^1 \hookrightarrow W$. The self-intersections of T_1 and T_2 are equal to 0. It follows from [Th] that W has a symplectic structure for which T_1 is a symplectic submanifold. By fixing such a symplectic form on W we get an induced orientation on T_1.

Now take a rational elliptic surface $E(1) = CP^2 \# 9\overline{CP}^2$. Fix a generic fiber $F \hookrightarrow E(1)$ of the elliptic fibration of $E(1)$. Then F is a smoothly embedded torus of self-intersection 0, and the complex structure of $E(1)$ induces an orientation on F. Fix an orientation preserving diffeomorphism $f : F \to T_1$ and lift it to an orientation reversing diffeomorphism g between the closed tubular neighborhoods. Using g we get the fiber sum of $E(1)$ and W:

$$M = (E(1) \setminus nd(F)) \cup_g (W \setminus nd(T_1)),$$

where nd denotes the open tubular neighborhood.

Now $T_2 \hookrightarrow M$ is a smoothly embedded torus of self-intersection 0. We define the family Y_n by performing logarithmic transformations along T_2:

Let us fix a circle $\delta' \hookrightarrow \partial(Z_\phi \setminus nd(\delta))$ that lies in a fiber of Z_ϕ and represents a_1. In other words δ' is a parallel copy of δ. Let $\alpha \in H_1(\partial(M \setminus nd(T_2)), Z)$ be the homology class of $\delta' \times p \hookrightarrow \partial(M \setminus nd(T_2)) = \partial(W \setminus nd(T_2))$, where $p \in S^1$. Let $\beta \in H_1(\partial(M \setminus nd(T_2)), Z)$ represent the homology class of the meridian around T_2.

For each $n \geq 0$ let us fix an orientation reversing diffeomorphism $\phi_n : \partial(D^2 \times T^2) \to \partial(M \setminus nd(T_2))$ that satisfies

$$(\phi_n)_*(e) = \alpha + n\beta,$$

where $e \in H_1(\partial(D^2 \times T^2), Z)$ is defined by $e = [\partial(D^2) \times q]$, where $q \in T^2$.

Now we define Y_n:

$$Y_n = (M \setminus nd(T_2)) \cup_{\phi_n} (D^2 \times T^2).$$

Lemma 2.1. *For all $n \geq 0$ the smooth closed 4-manifolds Y_n are simply-connected, $b_2^+(Y_n) = 1$ and $b_2^-(Y_n) = 9$.*

Proof. First note that

$$\pi_1(Z_\phi) = \langle g_1, g_2, g_3 | [g_1, g_2] = [g_2, g_3] = 1, g_3^{-1} g_1 g_3 = g_1 g_2 \rangle,$$

where g_1, g_2 correspond to a_1, a_2 and g_3 corresponds to γ. Since $\pi_1(E(1) \setminus nd(F)) = 1$, it follows that $\pi_1(M) = \pi_1(Z_\phi)/(g_3 = 1)$. So we get $\pi_1(M) = Z$ where the generator is g_1. It is not hard to see that $\pi_1(M \setminus nd(T_2)) = Z$ and the generator is represented by $\delta' \times p \hookrightarrow \partial(M \setminus nd(T_2))$. Let $i : \partial(M \setminus nd(T_2)) \to M \setminus nd(T_2)$ be the inclusion. Since $i_*(\beta) = 0$ and $\alpha = [\delta' \times p]$, it follows that $H_1(Y_n, Z) = 0$ for all n. On the other hand $\pi_1(M \setminus nd(T_2)) = Z$ shows that $\pi_1(Y_n)$ is abelian. It follows that $\pi_1(Y_n) = 1$. The rest of the lemma is trivial. □

3. Proof of Theorem 1.1 and Theorem 1.2

In this section we use Seiberg-Witten invariants for smooth closed oriented 4-manifolds with $b_2^+ = 1$. Let us recall that the usual Seiberg-Witten invariant for a smooth closed oriented 4-manifold X with $b_2^+(X) > 1$ is an integer valued function defined on the set of *spin*^{c} structures over X. In case $H_1(X, Z)$ has no 2-torsion it is convenient to use the one-to-one correspondence between the set of *spin*^{c} structures over X and set of characteristic elements in $H^2(X, Z)$. After fixing a homology orientation, i.e an orientation on $det H_+^2(X, R) \otimes det H^1(X, R)$, we have

$$SW_X : \{K \in H^2(X, Z) | K \equiv w_2(TX) \ (mod\ 2)\} \to Z.$$

K is called a basic class of X if $SW_X(K) \neq 0$.

In the $b_2^+(X) = 1$ case however SW_X depends on other parameters as well. Let us recall, see [Wi], [KM], [M], that the perturbed Seiberg-Witten moduli space $\mathcal{M}_X(K, g, h)$ is defined as the solution space of the Seiberg-Witten equations

$$F_A^+ = q(\phi) + ih, \quad D_A \phi = 0$$

divided by the gauge-group. Here g is a riemannian metric on X, A is an S^1 connection on the line bundle L with $c_1(L) = K$, ϕ is a section of the positive spin bundle corresponding to the *spin*^{c} structure determined by K, F_A^+ is the self-dual part of the curvature of A, q is a certain quadratic map, D_A is the Dirac operator coupled with A, and h is an arbitrary closed real-valued self-dual 2-form on X.

If $b_2^+(X) \geq 1$ and h is generic then the moduli space $\mathcal{M}_X(K,g,h)$ is a closed manifold with formal dimension $d = (K^2 - 2e(X) - 3sign(X))/4$, where $d < 0$ implies that $\mathcal{M}_X(K,g,h)$ is empty. If $d < 0$ then $SW_X(K) = 0$ by definition. In the $d \geq 0$ case one defines

$$SW_X(K,g,h) = \langle [\mathcal{M}_X(K,g,h)], \mu^{d/2} \rangle,$$

where $\mu \in H^2(\mathcal{M}_X(K,g,h), Z)$ is the Euler-class of the base fibration.

In the $b_2^+(X) = 1$ case $SW_X(K,g,h)$ depends on g and h, since if one varies the metric g and the perturbing 2-form h in a generic one-parameter family then the corresponding cobordism could contain singularities (where $\phi \equiv 0$).

In this paper we work with the $b_2^+(X) = 1$, $H_1(X,Z) = 0$ case, where the dependence is as follows.

Lemma 3.1. (See[KM], [M, p105]) *Let X be a smooth closed oriented 4-manifold with $b_2^+(X) = 1$ and $H_1(X,Z) = 0$. Fix a homology orientation of $H_+^2(X,R)$. For each riemannian metric g let $\omega^+(g)$ be the unique g-harmonic self-dual 2-form that has norm 1 and is compatible with the orientation of $H_+^2(X,R)$. Then for each characteristic elements $K \in H^2(X,Z)$ with $d = (K^2 - 2e(X) - 3sign(X))/4 \geq 0$ we have*

- *If $(2\pi K + h_1) \cdot \omega^+(g_1)$ and $(2\pi K + h_2) \cdot \omega^+(g_2)$ are not zero and have the same signs then*

$$SW_X(K,g_1,h_1) = SW_X(K,g_2,h_2)$$

- *If $(2\pi K + h_1) \cdot \omega^+(g_1) < 0 < (2\pi K + h_2) \cdot \omega^+(g_2)$, then*

$$SW_X(K,g_1,h_1) = SW_X(K,g_2,h_2) + (-1)^{d/2}.$$

It follows that if furthermore $b_2^-(X) \leq 9$ then we have a preferred Seiberg-Witten invariant.

Lemma 3.2. *Let X be a smooth closed oriented 4-manifold with $H_1(X,Z) = 0$, $b_2^+(X) = 1$ and $b_2^-(X) \leq 9$. Then for every characteristic element $K \in H^2(X,Z)$, pair of riemannian metrics g_1, g_2 and small enough perturbing 2-forms h_1, h_2 we have*

$$SW_X(K,g_1,h_1) = SW_X(K,g_2,h_2).$$

Proof. Let $K \in H^2(X,Z)$ be a characteristic element for which $d \geq 0$. Then $2e(X) + 3sign(X) = 4 + 5b_2^+(X) - b_2^-(X) \geq 0$, implies $K^2 \geq 0$. As a corollary we have that $K \cdot \omega^+(g_1)$, $K \cdot \omega^+(g_2)$ are non-zero and have the same signs. Now Lemma 3.2 follows from Lemma 3.1. □

From now on we denote the invariant described in Lemma 3.2 by $SW_X(K)$. Our first result in this section is the following.

Theorem 3.3. *Let Y_n, for $n \geq 0$, be defined as in Section 2. Let SW_{Y_n} be defined according to Lemma* 3.2. *Then we have*

- $SW_{Y_n}(\pm L) = \pm n$, *where* $L = PD[T_1]$
- $SW_{Y_n}(L') = 0$ *for all* $L' \neq \pm L$.

The main input in the proof of Theorem 3.3 is a surgery formula that relates SW_M, SW_{Y_0} and SW_{Y_n}. This result is a special case of the more general surgery formulas in [MMSz].

Lemma 3.4. (See[MMSz], cf. also[Sz2]). *For a characteristic element $K \in H^2(M, Z)$ that satisfies $\langle K, [T_2] \rangle = 0$, let $\overline{K}$ denote the corresponding characteristic element in Y_n. Then we have*

$$SW_{Y_n}(\overline{K}) = SW_{Y_0}(\overline{K}) + n \sum_{i=-\infty}^{\infty} SW_M(K + 2iF),$$

where $F = PD[T_2]$, SW_{Y_n} is defined according to Lemma 3.2 *and SW_M is well-defined since $b_2^+(M) = 2$.* □

Proof of Theorem 3.3. We compute SW_M, SW_{Y_0} and then apply Lemma 3.4. Note first that the symplectic sum construction of Gompf, see [G], implies that M has a symplectic structure where the canonical class of the symplectic structure is equal to $PD[T_1]$. It follows from [T1] that

$$SW_M(\pm PD[T_1]) = \pm 1.$$

On the other hand using the generalized adjunction formula, see [KM], [MMSz], it is an easy exercise to show that $SW_M(L') = 0$ for all $L' \neq \pm PD[T_1]$.

It is not hard to show, cf. [Sz2], that Y_0 contains a smoothly embedded torus with self-intersection 1. Applying the generalized adjunction formula to the $b_2^+ = 1$ case, it follows that SW_{Y_0} vanishes.

Now applying Lemma 3.4, we get

$$SW_{Y_n}(\pm PD[T_1]) = \pm n$$

and $SW_{Y_n}(L') = 0$ for all $L \neq \pm PD[T_1]$. □

Proof of Theorem 1.1. Suppose that there exists $n \geq 2$ such that Y_n is not irreducible, i.e $Y_n = X \# Z$ with neither X nor Z being a homotopy S^4. Since $\pi_1(Y_n) = 1$, $b_2^+(Y_n) = 1$, X or Z is negative definite with $b_2 > 0$. Now Lemma 3.2 and the blow-up formula of [FS2] for Seiberg-Witten invariants contradicts Theorem 3.3 and this proves (i).

In order to prove (ii), (iii) we need to study the chamber structure of $Y_n \# k\overline{CP}^2$. For a smooth closed oriented 4-manifold X with $b_2^+(X) = 1$ and $H_1(X, Z) = 0$ we define the set of chambers in the following way.

Fix an orientation of $H^2_+(X,R)$. Let $\Omega = \{x \in H^2(X,R) | x^2 = 1\}$. Let Ω^+ denote the positive component of Ω. If $K \in H^2(X,Z)$ is a characteristic element, i.e $K \equiv w_2(TX)$ $(mod\ 2)$, and $K^2 \geq 2e(X) + 3sign(X)$ then we define a wall

$$w(K) = \{x \in \Omega^+ | x \cdot K = 0\}.$$

The union of these walls W is locally compact in Ω^+. We define the set of chambers of X as the set of connected components of $\Omega^+ \setminus W$. Note that the chambers are open.

For every chamber C we define $SW^C_X(K)$ to be equal to $SW_X(K,g,h)$ where $[\omega^+(g)] \in C$ and h is small enough. It follows from Lemma 3.1, that if $K \cdot C_1$, $K \cdot C_2$ have opposite signs then

$$SW^{C_1}_X(K) = SW^{C_2}_X(K) \pm 1$$

and if $K \cdot C_1$, $K \cdot C_2$ have the same signs then

$$SW^{C_1}_X(K) = SW^{C_2}_X(K).$$

K is called a basic class of C if $SW^C_X(K) \neq 0$. Let $dist(C)$ denote the maximum of $A \cdot B$ where A, B are basic classes of C. We claim the following.

Lemma 3.5. *Let $n \geq 1$ and $k \geq 0$. Then every chamber C of $Y_n \# k\overline{CP}^2$ has at least one basic class K with $SW^C_{Y_n \# k\overline{CP}^2}(K) = \pm n$, and there exists a chamber C_0 satisfying that*

$$SW^{C_0}_{Y_n \# k\overline{CP}^2}(K') = \pm n$$

for all basic classes K' of C_0. Furthermore if a chamber C of $Y_n \# k\overline{CP}^2$ have $dist(C) = k$, then all basic classes A of C satisfies

$$A = (2l+1)L + \sum_{i=1}^{k} (-1)^{\delta_i} E_i$$

with some $l \in Z$, $\delta_i = 0,1$ for $i = 1, ..., k$, where $L = PD[T_1]$ and E_i is the exceptional class of the i-th copy of $\overline{CP}^2$.

Proof. Let us fix the orientation of $H^2_+(Y_n \# k\overline{CP}^2, R)$ in such a way that $L \cdot \omega > 0$ for all $\omega \in \Omega^+$. There is a unique chamber C_0 of $Y_n \# k\overline{CP}^2$ for which $C_0 \cap Im(i)$ is not empty, where $i : H^2(Y_n, R) \to H^2(Y_n \# k\overline{CP}^2, R)$ is the obvious inclusion. Let us fix $\omega_0 \in C_0 \cap Im(i)$. It follows from

Theorem 3.3 and the blow-up formula that all basic classes of C_0 are given by $\pm L \pm E_1 \cdots \pm E_k$ and

$$SW^{C_0}_{Y_n \# k\overline{CP}^2}(\pm L \pm E_1 \cdots \pm E_k) = \pm n.$$

Now let C be another chamber of $Y_n \# k\overline{CP}^2$ and fix $\omega \in C$. Then ω decomposes as $\omega = \omega_1 + \sum_{i=1}^k (-1)^{\epsilon_i} l_i E_i$, where ω_1 lies in $Im(i)$, $\epsilon_i = 0, 1$ and $l_i \geq 0$. Let $K = L + \sum_{i=1}^k (-1)^{\epsilon_i + 1} E_i$. It is easy to se that $K \cdot \omega > 0$, $K \cdot \omega_0 > 0$. It follows that

$$SW^{C}_{Y_n \# k\overline{CP}^2}(K) = SW^{C_0}_{Y_n \# k\overline{CP}^2}(K) = \pm n.$$

Now suppose $dist(C) = k$ and there is a basic class A of C that is not a basic class of C_0. A decomposes as

$$A = A_0 + \sum_{i=1}^{k} (-1)^{\delta_i} t_i E_i,$$

where $A_0 \in Im(i)$, $\delta_i = 0, 1$, $t_i \geq 1$ and odd. After multiplying A by -1 if necessary, we can assume that $A \cdot \omega_0 > 0$. Note that since $A_0^2 \geq 0$ we have $A_0 \cdot L \geq 0$, where equality implies that A_0 is an odd multiple of L.

Since A is a basic class of C but not a basic class of C_0, it follows that $A \cdot \omega_1 < 0$. Let

$$A' = A_0 + \sum_{i=1}^{k} (-1)^{\epsilon_i} t_i E_i.$$

It is easy to see, that $A' \cdot \omega_1 < 0 < A' \cdot \omega_0$ and so A' is a basic class of C. Now

$$A' \cdot K = A_0 \cdot L + \sum_{i=1}^{k} t_i \geq A_0 \cdot L + k \geq k,$$

where $A' \cdot K = k$ implies that $t_i = 1$ for all $i = 1, ..., k$ and A_0 is an odd multiple of L. This finishes the proof of Lemma 3.5. □

Now suppose that contrary to (ii) of Theorem 1.1 there is a diffeomorphism $f : Y_m \# k\overline{CP}^2 \to Y_n \# k\overline{CP}^2$, with $n \neq m$. It is clear that f has to be orientation preserving. Let us fix the chamber C_0 of $Y_n \# k\overline{CP}^2$ as in Lemma 3.5, and let C be the pullback of C_0 under f^*. Then for all characteristic elements K of $Y_n \# k\overline{CP}^2$ we have

$$SW^{C_0}_{Y_n \# k\overline{CP}^2}(K) = SW^{C}_{Y_m \# k\overline{CP}^2}(f^* K).$$

this contradicts the first part of Lemma 3.5 and the contradiction proves (ii).

Note that a simply-connected Kähler surface with $b_2^+ = 1$ is either rational, a surface of general type or a non-rational elliptic surface, in which case it is equal to one of $E_{p,q}\#k\overline{CP}^2$ where $p > 1$, $q > 1$, $(p,q) = 1$ and $k \geq 0$.

Since $CP^2\#l\overline{CP}^2$ has a chamber where the Seiberg-Witten invariant vanishes, it follows from Lemma 3.5, that $Y_n\#k\overline{CP}^2$ with $n \geq 1$ is not diffeomorphic to any rational surface.

The Seiberg-Witten invariants of surfaces of general type are known. It is proved for example in [M], that any surface of general type S with $b_2^+(S) = 1$ has a chamber C, in which $SW_X^C(K) = \pm 1$ for all basic classes of C. It follows now from Lemma 3.5, that if $n \geq 2$, then $Y_n\#k\overline{CP}^2$ is not diffeomorphic to S.

Now we deal with $E_{p,q}$, where $p > 1$, $q > 1$, $(p,q) = 1$. Since $b_2^+(E_{p,q}) = 1$, $b_2^-(E_{p,q}) = 9$, it follows from Lemma 3.2 that $E_{p,q}$ has a unique chamber. Let K denote the canonical class of $E_{p,q}$. It is proved in [M], that $SW_{E_{p,q}}(\pm K) = \pm 1$ and all basic classes K' of $E_{p,q}$ satisfy $K' = tK$, where $|t| \leq 1$.

Suppose that there is a diffeomorphism $f : Y_n\#k\overline{CP}^2 \to E_{p,q}\#k\overline{CP}^2$, with $n \geq 2$. After fixing a homology orientation for $Y_n\#k\overline{CP}^2$, f induces an orientation on $H^2_+(E_{p,q}\#k\overline{CP}^2, R)$. Let C_1 be the unique chamber of $E_{p,q}\#k\overline{CP}^2$ for which $C_1 \cap Im(i)$ is not empty, where $i : H^2(E_{p,q}, R) \to H^2(E_{p,q}\#k\overline{CP}^2, R)$ is the obvious inclusion. The blow-up formula shows that every basic class of C_1 can be written as $tK + \sum_{i=1}^k (-1)^{\delta_i} D_i$ with some $|t| \leq 1$, $\delta_i = 0, 1$, where D_i denotes the exceptional class of the i-th copy of $\overline{CP}^2$. Furthermore

$$SW^{C_1}_{E_{p,q}\#k\overline{CP}^2}(\pm K \pm D_1 \cdots \pm D_k) = \pm 1.$$

It follows that $dist(C_1) = k$.

Let C denote the image of C_1 under f^*. Then C is a chamber of $Y_n\#k\overline{CP}^2$ with $dist(C) = k$. It follows then from the second part of Lemma 3.5, that $f^*(V_1) = V_0$, where $V_0 = \langle L, E_1, ..., E_k\rangle$ and $V_1 = \langle K, D_1, ..., D_k\rangle$. Since $L^2 = K^2 = 0$, it follows that $f^*(K)$ is a multiple of L. Just as in the proof of Lemma 3.5, we have a basic class K' of C such that $K' = L + \sum_{i=1}^k (-1)^{\epsilon_i} E_i$ with $SW^C_{Y_n\#k\overline{CP}^2}(\pm K') = \pm n$, and for all $j > 0$ we have

$$(1) \quad SW^C_{Y_n\#k\overline{CP}^2}(K' + 2jL) = 0.$$

Let A be the unique characteristic element of $E_{p,q}\#k\overline{CP}^2$ with $f^*(A) =$

K'. It follows that

$$SW^{C_1}_{E_{p,q}\# k\overline{CP}^2}(A) = \pm n,$$

which implies $A = tK + \sum_{i=1}^k (-1)^{\delta_i} E_i$, with $|t|$ strictly less than 1. Now $A + (1-t)K$, $A - (1+t)K$ are basic classes of C_1 and consequently $f^*(A + (1-t)K) = K' + 2j_1 L$, $f^*(A - (1+t)K) = K' + 2j_2 L$ are basic classes of C. Since one of j_1, j_2 is positive, this contradicts (1). This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We first need a result of Taubes on symplectic 4-manifolds.

Lemma 3.6. (See[T1], [T2]). *Let X be an oriented symplectic 4-manifold with $H_1(X, Z) = 0$ and $b_2^+(X) = 1$. For all characteristic element L of X and symplectic form ω with $\omega^2 = 1$ we define*

$$SW^\omega_X(L) = SW_X(L, g, -r\omega),$$

where $\omega^+(g) = \omega$ and r is large enough. Then we have

$$SW^\omega_X(-K) = \pm 1,$$

where K is the canonical class of the symplectic structure. Furthermore for all characteristic element K' with $SW^\omega_X(K') \neq 0$ we have

$$-K \cdot \omega \leq K' \cdot \omega,$$

where equality implies $-K = K'$.

Now suppose that there exists an $n \geq 2$ for which Y_n has a symplectic structure. By multiplying with (-1) if necessary, we can assume that $L \cdot \omega > 0$, where $L = PD[T_1]$. Lemma 3.1 shows that if a characteristic element K' of Y_n satisfies $K' \cdot \omega < 0$, then we have

$$SW_{Y_n}(K') = SW^\omega_{Y_n}(K').$$

Now it follows from Theorem 3.3 that

$$(2) \quad SW^\omega_{Y_n}(-L) = \pm n$$

and for all K' with $SW^\omega_{Y_n}(K') \neq 0$ we have

$$-L \cdot \omega \leq K' \cdot \omega.$$

It follows from Lemma 3.6, that L is the canonical class of the symplectic structure. On the other hand (2) and the first part of Lemma 3.6 contradicts the assumption $n \geq 2$. This proves Theorem 1.2. □

Final remark

Starting with any smooth closed four-manifold X that contains a smoothly embedded torus $T \hookrightarrow X$ with self-intersection 0 and satisfies $\pi_1(X \setminus nd(T)) = 1$, one can define a family of simply-connected 4-manifolds Z_n by making the fiber sum of X and the Kodaira-Thurston manifold W along T, T_1 and then using ϕ_n to make a logarithmic transformation along T_2. In this way one can construct interesting simply-connected 4-manifolds. For example one can start with the $K3$ surface which contains three disjoint Gompf nuclei, see [GM]. By using the above construction repeatedly along the three fibers contained in the different nuclei we get a three parameter family of homotopy $K3$ surfaces, $Z_{n,m,k}$. It easily follows from Theorem 3.3 and [Sz2] that if $n \geq 2, m \geq 2, k \geq 2$, then $Z_{n,m,k}$ is non-symplectic.

As another generalization of [Sz2] Fintushel and Stern recently constructed a surprisingly rich family of non-symplectic homotopy $K3$ surfaces, and also proved that $Z_{n,m,k}$ arise as a special case of their construction.

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