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THE LIMIT SET INTERSECTION THEOREM FOR FINITELY GENERATED KLEINIAN GROUPS

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1. Introduction

The purpose of this paper is to show that the limit set intersection theorem holds for a pair of finitely generated subgroups of a purely loxodromic Kleinian group with non-empty domain of discontinuity. This result is of interest, as we make no assumption about whether the groups involved are topologically tame. Specifically, we prove the following.

Theorem 5.4. *Let Γ be a purely loxodromic Kleinian group with non-empty domain of discontinuity. If Φ_1 and Φ_2 are finitely generated subgroups of Γ, then $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.*

The proof of the Theorem proceeds by showing that it holds in some special cases involving Kleinian groups with connected limit sets, and then extending to the general case by using a decomposition argument based on the Klein-Maskit combination theorems and a careful tracking of the limit points resulting from this decomposition. We discuss various well-behaved classes of limit points in Section 2. We describe the decomposition results taken from Klein-Maskit combination theory in Section 3. We apply the decomposition results to a special subclass of groups in Section 4. We then complete the proof in Section 5. In Section 6, we discuss the extension of Theorem 5.4 to groups with torsion, the difficulty with extending to groups with parabolics, and make note of a reduction of the Ahlfors measure conjecture.

A general *limit set intersection theorem* for Kleinian groups gives a description of $\Lambda(\Phi_1 \cap \Phi_2)$ in terms of $\Lambda(\Phi_1)$ and $\Lambda(\Phi_2)$, where Φ_1 and Φ_2 are subgroups of a Kleinian group Γ. Ideally, such a theorem has the form that

$$\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2) \cup P(\Phi_1, \Phi_2),$$

where $P(\Phi_1, \Phi_2)$ are those parabolic fixed points $x \in \Lambda(\Gamma)$ for which the stabilizers $st_{\Phi_1}(x)$ and $st_{\Phi_2}(x)$ of x are both rank one and generate a rank two subgroup of Γ. It is usually necessary to impose some finiteness condition on Φ_1, Φ_2, and/or Γ for such a result to hold.

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Such a limit set intersection theorem has been shown to hold under various hypotheses. Maskit [17] shows that it holds for pairs of analytically finite component subgroups of a Kleinian group, and that the set P is empty in this case. Susskind [25] shows that it holds for pairs of geometrically finite subgroups of a Kleinian group; this was generalized by Susskind and Swarup [26] to hold in all dimensions. Soma [24] shows that it holds for pairs of function groups in a Kleinian group. Anderson [6] and Soma [24] show that it holds for pairs of topologically tame subgroups of a Kleinian group, modulo certain exceptional cases involving hyperbolic manifolds which fiber over the circle. Anderson [5] shows that it holds for Φ_1 analytically finite and Φ_2 geometrically finite, under the additional assumption that $\Omega(\Gamma)$ is non-empty. Anderson and Canary [7] show that, if Φ_1 is finitely generated, if Φ_2 is a precisely embedded quasifuchsian or extended quasifuchsian group, if $\Lambda(\Phi_2)$ is contained in and separates $\Lambda(\Phi_1)$, and if $\Phi_1 \cap \Phi_2$ is finitely generated, then $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$.

We close the introduction by giving a few basic definitions. For an account of Kleinian group basics, the reader is referred to [18]. Given a set $X \subset \overline{\mathbb{C}}$ and a Jordan curve c, say that c *separates* X if X is disjoint from one of the components of $\overline{\mathbb{C}} - c$; that is, X lies in one of the closed discs determined by c. We allow the possibility that $c \cap X$ is non-empty.

A group G is *freely indecomposible* if it does not admit a non-trivial free product decomposition, and is *freely decomposible* otherwise.

A *Kleinian group* is a discrete subgroup Γ of $PSL_2(\mathbb{C})$, which may be viewed as acting either on the Riemann sphere $\overline{\mathbb{C}}$ by conformal homeomorphisms or on hyperbolic 3-space $\mathbb{H}^3$ by isometries. A Kleinian group is *purely loxodromic* if every non-trivial element is loxodromic.

The action of Γ partitions $\overline{\mathbb{C}}$ as the union $\overline{\mathbb{C}} = \Omega(\Gamma) \cup \Lambda(\Gamma)$, where the *domain of discontinuity* $\Omega(\Gamma)$ is the largest open set in $\overline{\mathbb{C}}$ on which Γ acts properly discontinuously, and the *limit set* $\Lambda(\Gamma)$ is the smallest non-empty closed set invariant under Γ. A Kleinian group is *non-elementary* if its limit set contains at least three points, and is *elementary* otherwise. A torsion-free Kleinian group is elementary if and only if it is free abelian of rank at most two.

Given a Kleinian group Γ and a set X in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$, define the *stabilizer* of X in Γ to be $st_\Gamma(X) = \{\gamma \in \Gamma : \gamma(X) = X\}$. A *component subgroup* of Γ is the stabilizer of a connected component of the domain of discontinuity $\Omega(\Gamma)$ of Γ.

Let Γ be a finitely generated non-elementary Kleinian group with non-empty domain of discontinuity, and let Φ be a finitely generated non-elementary subgroup of Γ. There exists a canonical metric, the *Poincaré metric*, on $\Omega(\Gamma)$ of curvature -1 so that Γ acts on $\Omega(\Gamma)$ by isometries. The Ahlfors finiteness theorem [3] states that $\Omega(\Gamma)/\Gamma$ has finite area in this metric. If $\Lambda(\Phi) = \Lambda(\Gamma)$, then $\Omega(\Gamma)/\Phi$ is a finite cover of $\Omega(\Gamma)/\Gamma$, and so Φ has finite index in Γ. In particular, if Φ has infinite index in Γ, then $\Lambda(\Phi)$ is a proper subset of $\Lambda(\Gamma)$.

Given a Kleinian group Γ, let $CH(\Gamma) \subset \mathbb{H}^3$ be the *convex hull* of $\Lambda(\Gamma)$, which is the smallest convex set in $\mathbb{H}^3$ invariant under Γ. The quotient $C(\Gamma) = C(N)$ of $CH(\Gamma)$ in $N = \mathbb{H}^3/\Gamma$ is the *convex core* of Γ (or N). In the case that Γ is finitely generated and purely loxodromic, the boundary of the convex core $\partial C(\Gamma)$ is homeomorphic to $\Omega(\Gamma)$ [12], and so it follows from the Ahlfors finiteness theorem [3] that $\partial C(\Gamma)$ is a finite union of compact surfaces.

Suppose that Γ is a purely loxodromic, finitely generated Kleinian group and let $N = \mathbb{H}^3/\Gamma$. A *compact core* for N is a compact submanifold M of N whose inclusion is a homotopy equivalence. It follows from the core theorem of Scott [22] that a hyperbolic 3-manifold with finitely generated fundamental group always has a compact core.

There are several classes of finitely generated Kleinian groups of special interest. A *quasifuchsian* group Γ is a finitely generated Kleinian group whose limit set is a Jordan curve and which contains no element interchanging the components of $\overline{\mathbb{C}} - \Lambda(\Gamma)$. An *extended quasifuchsian* group Γ is a finitely generated Kleinian group whose limit set is a Jordan curve and which contains some element interchanging the components of $\overline{\mathbb{C}} - \Lambda(\Gamma)$. Note that an extended quasifuchsian group contains a canonical quasifuchsian subgroup of index two, consisting of those elements which do not interchange the components of its domain of discontinuity. A *degenerate* group is a finitely generated Kleinian group whose domain of discontinuity and limit set are both non-empty and connected. A *web* group is a finitely generated Kleinian group Γ whose domain of discontinuity contains infinitely many components and the stabilizer of each is a quasifuchsian group; in particular, the boundary of each component of $\Omega(\Gamma)$ is a Jordan curve. Note that, as each component of the domain of discontinuity of a web group is simply connected, the limit set of a web group is necessarily connected.

A *function* group is a finitely generated Kleinian group which has an invariant component in its domain of discontinuity. Quasifuchsian and degenerate groups are function groups, while extended quasifuchsian and web groups are not.

We now state two useful Lemmas which are implicit in the literature.

Lemma 1.1. *Let Γ be a purely loxodromic, finitely generated Kleinian group with non-empty domain of discontinuity. Then, Γ has connected limit set if and only if Γ is either quasifuchsian, extended quasifuchsian, degenerate, or web.*

Proof. By definition, a quasifuchsian, extended quasifuchsian, degenerate, or web group has connected limit set. Conversely, suppose that Γ has connected limit set. If $\Omega(\Gamma)$ is connected, then Γ is degenerate. If $\Omega(\Gamma)$ has two components, then Γ is either quasifuchsian or extended quasifuchsian [19]. If $\Omega(\Gamma)$ has more than two components, it has countably many. Let Δ be any component of $\Omega(\Gamma)$, and let Φ be its stabilizer in Γ. Since Δ is simply connected and Γ contains no parabolics, Φ is either quasifuchsian or degenerate [19]. If Φ were degenerate, then $\Omega(\Phi) = \Delta$ would be an open dense subset of $\overline{\mathbb{C}}$, and so $\Omega(\Gamma)$ could contain no other component, a contradiction. This gives that Φ is quasifuchsian, and hence that Γ is a web group. $\square$

Lemma 1.2. *Let Γ be a purely loxodromic Kleinian group with non-empty domain of discontinuity which is isomorphic to the fundamental group of a closed orientable surface of genus at least two. Then, Γ is either quasifuchsian or degenerate.*

Proof. Since Γ is the fundamental group of a closed orientable surface, it is freely indecomposible. Theorem 3.1 then implies that $\Lambda(\Gamma)$ is connected.

Suppose that Γ is a web group, let Δ be a component of $\Omega(\Gamma)$ and let Φ be its stabilizer in Γ. Since $\Lambda(\Phi)$ is a proper subset of $\Lambda(\Gamma)$, we see that Φ has infinite index in Γ. However, this implies that Γ contains a finitely generated, infinite index subgroup which is not free, which cannot occur.

If Γ is an extended quasifuchsian group, then the 3-manifold $M = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is compact and has connected boundary. However, it is known [15] that, since Γ is isomorphic to the fundamental group of a closed orientable surface S, it must be that M is homeomorphic to the product $S \times I$, a contradiction. $\square$

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2. Limit points

In this Section, we discuss various useful classes of limit points, and state the results we use concerning them.

There is a particular type of limit point which plays an important role in what follows. A limit point x of a Kleinian group Γ is a *point of approximation* if there exists a hyperbolic ray $r \subset \mathbb{H}^3$ ending at x, a compact set $K \subset \mathbb{H}^3$, and a sequence $\{\gamma_n\}$ of distinct elements of Γ so that $\gamma_n(K) \cap r$

is non-empty for all n. Equivalently, a limit point is not a point of approximation if and only if, for each ray r in $\mathbb{H}^3$ ending at x, the image $\pi(r)$ of r under the covering map $\pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ exits every compact subset of $\mathbb{H}^3/\Gamma$. These points are also commonly referred to as *conical limit points.*

A Kleinian group is *geometrically finite* if there exists a finite sided fundamental polyhedron for its action on $\mathbb{H}^3$. By way of example, quasifuchsian and extended quasifuchsian groups are geometrically finite, while degenerate groups are not [13]. Geometrically finite groups are always finitely generated. It is a result of Beardon and Maskit [8] that a purely loxodromic Kleinian group is geometrically finite if and only if every limit point is a point of approximation. A Kleinian group which is not geometrically finite is *geometrically infinite.*

The following Theorem combines results from Proposition 5.1 and Theorem 5.2 from [5], adapted to the situation addressed in this paper.

Theorem 2.1. *Let Γ be a purely loxodromic Kleinian group with non-empty domain of discontinuity, and let Φ_1 and Φ_2 be non-elementary finitely generated subgroups of Γ. If $x \in \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ is a point of approximation for Φ_1, then $x \in \Lambda(\Phi_1 \cap \Phi_2)$ and is a point of approximation for $\Phi_1 \cap \Phi_2$. Moreover, if Φ_1 is geometrically finite, then $\Phi_1 \cap \Phi_2$ is geometrically finite and $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.*

A torsion-free Kleinian group Γ is *topologically tame* if its corresponding hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold, possibly with boundary. It is a result of Bonahon [9] that every finitely generated, freely indecomposible Kleinian group is topologically tame. It is also known that geometrically finite groups are topologically tame [15], as are function groups [24], while freely decomposible web groups are not yet known to be.

We make use of the fact that the limit set intersection theorem holds for topologically tame groups. The result stated below combines information from Theorem A, Lemma 5.1, and Theorem C from [6].

Theorem 2.2. *Let Γ be a purely loxodromic, co-infinite volume Kleinian group. Let Φ_1 be a topologically tame subgroup of Γ, possibly cyclic, and let Φ_2 be a non-elementary topologically tame subgroup of Γ. Then, $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$. Moreover, if Φ_3 is a non-elementary subgroup of Γ and if $x \in \Lambda(\Phi_2) \cap \Lambda(\Phi_3)$ is a point of approximation for Φ_3, then $x \in \Lambda(\Phi_2 \cap \Phi_3)$ and is a point of approximation for $\Phi_2 \cap \Phi_3$.*

Let Γ be a purely loxodromic, topologically tame Kleinian group, and let M be a compact core for $N = \mathbb{H}^3/\Gamma$. Each component of ∂M then faces an end of N. An end E of N is *geometrically finite* if it has a neighborhood which is disjoint from $C(N)$, and is *geometrically infinite* otherwise. Note that a purely loxodromic, finitely generated Kleinian group Γ is geometrically finite if and only if all the ends of $\mathbb{H}^3/\Gamma$ are geometrically finite.

A major tool used in handling topologically tame Kleinian groups is the *covering theorem*, due to Thurston [27] for surface groups and generalized by Canary [10] to all topologically tame groups. In the case of no parabolics, a rough statement is that, if Γ is a Kleinian group and if Φ is a topologically tame subgroup of Γ, then either $\mathbb{H}^3/\Gamma$ is closed or the covering map $\mathbb{H}^3/\Phi \to \mathbb{H}^3/\Gamma$ is finite-to-one on a neighborhood of every geometrically infinite end of $\mathbb{H}^3/\Phi$.

Let S be a component of ∂M facing a geometrically infinite end of N. The inclusion of S into N gives rise to a conjugacy class of subgroups of Γ, and we refer to each such subgroup of Γ as a *geometrically infinite maximal peripheral subgroup* of Γ. As ∂M has only finitely many components, there are only finitely many conjugacy classes of geometrically infinite maximal peripheral subgroups of Γ.

In the case that Γ is freely indecomposible and has non-empty domain of discontinuity, each component of ∂M is incompressible. Each geometrically infinite maximal peripheral subgroup is then isomorphic to the fundamental group of a closed, orientable surface, and so by Lemma 1.2 is a degenerate group. We remark that Corollary C of [10] implies that a finitely generated, infinite index subgroup of a degenerate group is geometrically finite.

We make use of the following characterization of the limit points of a purely loxodromic, finitely generated, freely indecomposible Kleinian group.

Lemma 2.3. *Let Γ be a purely loxodromic, finitely generated, freely indecomposible Kleinian group. Each point of $\Lambda(\Gamma)$ either lies in the limit set of a geometrically infinite maximal peripheral subgroup or is a point of approximation of Γ.*

Proof. Let $N = \mathbb{H}^3/\Gamma$ and let $\pi : \mathbb{H}^3 \to N$ be the covering map. We use a theorem of McCullough [21] to choose a compact core M for N so that $M \subset C(N)$ and so that $\partial C(N) \subset \partial M$. Let $x \in \Lambda(\Gamma)$ be a limit point which is not a point of approximation, and let r be any ray in $CH(\Gamma)$ which ends at x. As $\pi(r)$ exits every compact subset of N, there exists a sub-ray of r, again called r, so that $\pi(r)$ is disjoint from M. Let E be the end of N containing $\pi(r)$, let S be the component of ∂M facing E, and let Φ be a choice of conjugacy class of the image of $\pi_1(S)$ in Γ. Note that Φ is geometrically infinite.

There are a couple of special cases. It may be that N is closed, in which case Γ is geometrically finite, and so every point of $\Lambda(\Gamma)$ is a point of approximation. It may be that Φ has finite index in Γ, in which case Φ and

Γ have the same limit set, and so x lies in the limit set of a geometrically infinite maximal peripheral subgroup.

Suppose now that N has infinite volume, and that Φ has infinite index in Γ. The assumption that Γ is freely indecomposible implies that S is incompressible, and so Φ is isomorphic to the fundamental group of S. In particular, this implies that Φ is freely indecomposible, and hence is topologically tame [9]. If Φ has empty domain of discontinuity, then every end is geometrically infinite, and the covering theorem [10] implies that N is closed, a contradiction. Hence, the domain of discontinuity of Φ must be non-empty. In particular, we see by Lemma 1.2 that Φ is a degenerate group.

Let $P = \mathbb{H}^3/\Phi$, and let $\alpha : P \to N$ be the covering map. Since Φ is degenerate, P has one geometrically infinite end F. Since P is topologically tame and has infinite volume, the covering theorem [10] implies that α is finite-to-one on some neighborhood U of F. By construction, $\alpha(U)$ lies in E, and so there is a subray of r, again called r, so that $\pi(r)$ lies in $\alpha(U)$. In particular, some lift of $\pi(r)$ to P lies in U. Let $\beta : \mathbb{H}^3 \to P$ be the covering map. The image of some ray in $\Gamma(r) = \{\gamma(r) : \gamma \in \Gamma\}$ then lies in U. Hence, there is a conjugate Φ' of Φ so that $\beta(r)$ exits the geometrically infinite end of $\mathbb{H}^3/\Phi'$, and so x lies in $\Lambda(\Phi')$. This completes the proof. $\square$

3. Decomposition

In this Section, we give a brief description of the portion of Klein-Maskit combination theory which we make use of. We remark that we do not give the most general statements from the literature of combination theorems; instead we make statements sufficiently strong for their use here. We begin with a few useful definitions.

A subset $X \subset \overline{\mathbb{C}}$ is *precisely invariant* under a subgroup Φ of a Kleinian group Γ if $st_\Gamma(X) = \Phi$ and if $\gamma(X) \cap X$ is empty for all $\gamma \in \Gamma - \Phi$. A subgroup Φ of a Kleinian group Γ is *precisely invariant* if $\Lambda(\Phi)$ is precisely invariant under Φ in Γ.

In the special case that X is a Jordan curve, say that X is *precisely embedded* under Φ in Γ if $st_\Gamma(X) = \Phi$ and if no translate of X separates X. That is, while a translate $\gamma(X)$ of X is allowed to intersect X, no translate of X can cross X. A quasifuchsian or extended quasifuchsian subgroup Φ of a Kleinian group Γ is *precisely embedded* if the Jordan curve $\Lambda(\Phi)$ is precisely embedded under Φ in Γ. As an example, each component subgroup of a web group Γ is a precisely embedded quasifuchsian subgroup of Γ.

The form of the Klein-Maskit combination theorems we make use of gives information about the behavior of a Kleinian group in terms of subgroups of certain form, obtained by decomposition along a Jordan curve. We now give a brief description of this decomposition. The material here can be found largely in Chapter VII of [18].

Let Γ be a purely loxodromic, finitely generated Kleinian group, and let c be a Jordan curve which separates $\Lambda(\Gamma)$. Set $\Phi = st_\Gamma(c)$, and suppose that Φ is finitely generated, that c is precisely embedded under Φ in Γ, that $c \cap \Lambda(\Gamma) = \Lambda(\Phi)$, and that $c - \Lambda(\Phi)$ is precisely invariant under Φ in Γ. Then, there exists a properly embedded disc $D \subset \mathbb{H}^3$ which is precisely invariant under Φ in Γ and which extends to a closed disc $\overline{D}$ in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$ with $\overline{D} \cap \overline{\mathbb{C}} = c$. In particular, $\overline{D} - \Lambda(\Phi)$ projects to a compact, properly embedded surface S in $P = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.

If S separates P, let P_1 and P_2 be the components of $P - S$, and let Γ_j be the image of the fundamental group of P_j in $\pi_1(P) = \Gamma$. Then, both Γ_1 and Γ_2 are finitely generated, every limit point of Γ either is a limit point of a conjugate of either Γ_1 or Γ_2 or is a point of approximation of Γ, and Γ is the amalgamated free product of Γ_1 and Γ_2 along their common subgroup $\Phi = \Gamma_1 \cap \Gamma_2$. We refer to Γ_1 and Γ_2 as the *factor subgroups* of the decomposition.

If S does not separate P, let $P_1 = P - S$ and let Γ_1 be the the image of the fundamental group of P_1 in Γ. Then, Γ_1 is finitely generated, every limit point of Γ either is a limit point of a conjugate of Γ_1 or is a point of approximation of Γ, and Γ is the HNN extension of Γ_1 by some loxodromic element $\gamma \in \Gamma$. We refer to Γ_1 and $\langle\gamma\rangle$ as the *factor subgroups* of the decomposition.

The first of the decomposition results involves the canonical decomposition of a Kleinian group Γ with non-connected limit set. In this case, one can find a Jordan curve in $\Omega(\Gamma)$ which separates the limit set and which is precisely invariant under the identity in Γ. If one of the factor subgroups of this decomposition is non-elementary and has non-connected limit set, the decomposition can be carried out for the factor subgroup. This process terminates after finitely many steps. This argument is carried out in detail in [2].

Theorem 3.1. *Let Γ be a non-elementary, purely loxodromic, finitely generated Kleinian group whose limit set is not connected. Then, there exists a non-trivial free product splitting $\Gamma = \Xi * \Gamma_1 * \cdots * \Gamma_p$, into precisely invariant subgroups, where Ξ is a finitely generated free group and each Γ_k is finitely generated and has connected limit set. If Z is a continuum in $\Lambda(\Gamma)$ containing more than one point, then the stabilizer of Z in Γ is conjugate to some Γ_k. Every limit point of Γ either is a limit point of a conjugate of some Γ_k or is a point of approximation of Γ. In the case that Γ is a* *function group, each Γ_k is either quasifuchsian or degenerate.*

Note that Theorem 3.1 implies that freely indecomposible finitely generated Kleinian groups have connected limit set.

The second of the decomposition results involves groups containing precisely embedded quasifuchsian or extended quasifuchsian subgroups.

Theorem 3.2. *Let Γ be a purely loxodromic, finitely generated Kleinian group with non-empty domain of discontinuity, let Φ be a finitely generated subgroup of Γ, and let Θ be a precisely embedded quasifuchsian or extended quasifuchsian subgroup of Γ. Suppose that $\Lambda(\Theta)$ separates $\Lambda(\Phi)$. Then, there exist finitely generated, infinite index subgroups Φ_1 and Φ_2 of Φ so that every limit point of Φ either is a limit point of a conjugate of some Φ_j or is a point of approximation of Φ.*

All that need be checked is that the assumptions of Theorem 3.2 imply that the hypotheses of the combination theorems as discussed above hold. Let $c = \Lambda(\Theta)$. Since c is assumed to be precisely embedded under Θ in Γ, it is precisely invariant under $st_\Phi(c) = \Phi \cap \Theta$ in Φ. Theorem 2.1 implies that $\Phi \cap \Theta$ is finitely generated and that $c \cap \Lambda(\Phi) = \Lambda(\Theta) \cap \Lambda(\Phi) = \Lambda(\Theta \cap \Phi) = \Lambda(st_\Phi(c))$.

The third of the decomposition results is similar to Theorem 3.2, though the hypotheses are slightly different.

Theorem 3.3. *Let Γ be a purely loxodromic, finitely generated, infinite co-volume Kleinian group, let $N = \mathbb{H}^3/\Gamma$, and let M be a compact core for N. Let $\mathcal{S} = \{S_1, \ldots, S_p\}$ be a collection of disjoint, incompressible, closed, embedded, orientable surfaces in M, so that no two surfaces in $\mathcal{S}$ are parallel and so that no surface in $\mathcal{S}$ is parallel to a component of ∂M. Let $N_1, \ldots, N_t$ be the components of $N - \mathcal{S}$, and let Γ_j be the image of the fundamental group of N_j in $\pi_1(N) = \Gamma$. Then, every limit point of Γ either is a limit point of a conjugate of some Γ_j or is a point of approximation of Γ.*

In order to apply the combination theorems, it remains only to check, if we let Φ_k be the image of the fundamental group of S_k in Γ, that Φ_k is a precisely embedded quasifuchsian subgroup of Γ. The assumption that S_k is incompressible implies that Φ_k is isomorphic to the fundamental group of S_k. The covering theorem [10] implies that Φ_k must be geometrically finite, and so is either quasifuchsian or extended quasifuchsian. Let D_k be the lift of S_k to $\mathbb{H}^3$ which is invariant under Φ_k, and note that D_k extends to a closed disc $\overline{D_k}$ in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$ with boundary $\Lambda(\Phi_k)$. Since S_k is embedded, D_k is precisely invariant under Φ_k, and so $\Lambda(\Phi_k)$ is precisely embedded under Φ_k in Γ. If Φ_k were extended quasifuchsian, then S_k would be non-orientable. Theorem 3.3 follows by applying Theorem 3.2 successively to Φ_1 through Φ_p.

We remark that, in general, a precisely embedded extended quasifuchsian subgroup Φ of a Kleinian group Γ corresponds to a non-orientable surface S in $N = \mathbb{H}^3/\Gamma$ which is the core of a twisted I-bundle. A regular neighborhood of this core surface has boundary an orientable surface, which corresponds to the quasifuchsian component subgroup of the extended quasifuchsian group.

We close this section with the following Corollary to Theorem 3.2. It is a slight variation on the known fact [1] that degenerate groups cannot be constructed from cyclic groups by Klein-Maskit combination.

Corollary 3.4. *Let Γ be a purely loxodromic, finitely generated Kleinian group with non-empty domain of discontinuity, let Φ be a degenerate subgroup of Γ, and let Θ be a precisely embedded quasifuchsian or extended quasifuchsian subgroup of Γ. Then, $\Lambda(\Theta)$ cannot separate $\Lambda(\Phi)$.*

Proof. Suppose that $\Lambda(\Theta)$ separates $\Lambda(\Phi)$. Theorem 3.2 implies that there exist finitely generated, infinite index subgroups Φ_1 and Φ_2 of Φ so that every limit point of Φ either is a limit point of a conjugate of some Φ_j or is a point of approximation of Φ.

Since Φ is isomorphic to the fundamental group of a closed, orientable surface of genus at least two, every finitely generated, infinite index subgroup is free. As both Φ_1 and Φ_2 are finitely generated, free, purely loxodromic Kleinian groups with non-empty domain of discontinuity, they are Schottky groups [16]. In particular, both Φ_1 and Φ_2 are geometrically finite, and hence, for both j, every point of $\Lambda(\Phi_j)$ is a point of approximation.

This implies that every limit point of Φ is a point of approximation, and hence that Φ is geometrically finite [8]. However, it is known [13] that degenerate groups are not geometrically finite, a contradiction. □

4. Web groups

In this Section, we describe two useful classes of web groups. We begin with the following Lemma.

Lemma 4.1. *Let Γ be a purely loxodromic, finitely generated, freely decomposible Kleinian group, and let $\Gamma = \Gamma_1 * \cdots * \Gamma_p$ be a maximal free product splitting of Γ. Then, each Γ_j is a precisely invariant subgroup of Γ.*

Proof. Suppose that there exists $\gamma \in \Gamma$ so that $\gamma(\Lambda(\Gamma_j)) \cap \Lambda(\Gamma_k)$ is non-empty; we need to show that $j = k$ and that $\gamma \in \Gamma_j$. As finitely generated, freely indecomposible Kleinian groups are topologically tame [9], Theorem 2.2 implies that $\Lambda(\Gamma_j) \cap \gamma(\Lambda(\Gamma_k)) = \Lambda(\Gamma_j \cap \gamma\Gamma_k\gamma^{-1})$. Since $\Lambda(\Gamma_j) \cap \gamma(\Lambda(\Gamma_k))$ is assumed to be non-empty, we see that $\Gamma_j \cap \gamma\Gamma_k\gamma^{-1}$ is non-trivial. However, if $j \neq k$, or if $j = k$ and $\gamma \notin \Gamma_j$, this violates the existence of unique normal forms in free products [14]. Hence, $j = k$ and $\gamma \in \Gamma_j$. □

The following topological lemma will be useful.

Lemma 4.2. *Let Φ_1 and Φ_2 be Kleinian groups with connected limit sets, and assume that $\Gamma = \langle \Phi_1, \Phi_2 \rangle$ is a purely loxodromic Kleinian group. If $\Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ is non-empty, then $\Lambda(\Gamma)$ is connected.*

In addition, suppose Γ contains a precisely embedded quasifuchsian or extended quasifuchsian subgroup Θ whose limit set $\Lambda(\Theta)$ separates $\Lambda(\Gamma)$. If no translate of $\Lambda(\Theta)$ separates either $\Lambda(\Phi_1)$ or $\Lambda(\Phi_2)$, then there is a conjugate Θ' of Θ so that $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) \subset \Lambda(\Theta')$.

Proof. To see that $\Lambda(\Gamma)$ is connected, note that both $\Lambda(\Phi_1)$ and $\Lambda(\Phi_2)$ lie in $\Lambda(\Gamma)$. Define the *length* of an element $\gamma \in \Gamma$ to be the minimal number of elements in the set $\Phi_1 \cup \Phi_2$ whose product is γ, and let Γ_n be the elements of Γ of length at most n for $n \in \mathbb{N}$. Note that $\Gamma_1 = \Phi_1 \cup \Phi_2$. A simple inductive argument on n shows that $X_n = \bigcup_{\gamma \in \Gamma_n} \gamma(\Lambda(\Phi_1) \cap \Lambda(\Phi_2))$ is connected for each $n \in \mathbb{N}$. So, the set $X = \bigcup_{\gamma \in \Gamma} \gamma(\Lambda(\Phi_1) \cap \Lambda(\Phi_2))$, which is the union of the nested sets $X_1, \ldots, X_n, \ldots$, is connected, and so $\overline{X}$ is connected. Since $\overline{X}$ is a closed, non-empty subset of $\Lambda(\Gamma)$ which is invariant under Γ, we see that $\overline{X} = \Lambda(\Gamma)$.

Since $\Lambda(\Theta)$ separates $\Lambda(\Gamma)$, we may choose points a and b in $\Lambda(\Gamma) - \Lambda(\Theta)$ which are separated by $\Lambda(\Theta)$. Since X is dense in $\Lambda(\Gamma)$ and since X is the union of the nested sets $X_1 \subset X_2 \subset \ldots$, we see that $\Lambda(\Theta)$ separates X_n for all n sufficiently large. If no translate of $\Lambda(\Theta)$ separates either $\Lambda(\Phi_1)$ or $\Lambda(\Phi_2)$, then there is exists a conjugate Θ' of Θ so that $\Lambda(\Theta')$ separates $\Lambda(\Phi_1)$ from $\Lambda(\Phi_2)$, and so $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) \subset \Lambda(\Theta')$. □

We need to describe two related subclasses of web groups. A purely loxodromic, finitely generated web group Γ is *extreme* if it admits a non-trivial free product splitting $\Gamma = \Gamma_1 * \cdots * \Gamma_s$, where each Γ_j is either quasifuchsian, degenerate, or infinite cyclic, and each quasifuchsian subgroup which is a free factor is a component subgroup. An example of such a group is given by Maskit [17].

Lemma 4.3. *A finitely generated, infinite index subgroup of an extreme web group is a function group.*

Proof. Let Γ be an extreme web group, and let $\Gamma = \Gamma_1 * \cdots * \Gamma_s$ be a maximal free product splitting of Γ. Let Φ be a finitely generated, infinite index subgroup of Γ. Since $\Lambda(\Phi)$ is a proper subset of $\Lambda(\Gamma)$, there exists a component Δ of $\Omega(\Phi)$ which is not a component of $\Omega(\Gamma)$, and hence contains a point of $\Lambda(\Gamma)$.

Consider the stabilizer $st_\Phi(\Delta)$ of Δ in Φ. Since $st_\Phi(\Delta)$ is finitely generated [4], it is a function group with invariant component Δ. If $st_\Phi(\Delta) \neq \Phi$, there exists an element $\varphi \in \Phi - st_\Phi(\Delta)$ so that $\varphi(\Delta) \neq \Delta$. In particular, there exists a quasifuchsian subgroup Ξ of Φ whose limit set separates Δ from $\varphi(\Delta)$ [17]. We now make use of the Kurosh subgroup theorem (see, for example, [14]), which states that a freely indecomposible subgroup of a free product is conjugate into a free factor. This implies that Ξ is conjugate to a finite index subgroup of some Γ_j, which contradicts the assumption that each quasifuchsian group which is a free factor of Γ is in fact a component subgroup. Hence, $st_\Phi(\Delta) = \Phi$ and so Φ is a function group. $\square$

A purely loxodromic, finitely generated web group Γ is *simple* if it does not contain a precisely embedded extended quasifuchsian subgroup, and if every precisely embedded quasifuchsian subgroup is a component subgroup. Note that an extreme web group is necessarily simple, but the converse need not hold. However, we have the following characterization of simple web groups.

Lemma 4.4. *A simple web group either is freely indecomposible and topologically tame, or is extreme.*

Proof. Let Γ be a simple web group. If Γ is freely indecomposible, it is topologically tame [9]. Otherwise, it admits a non-trivial maximal free product splitting $\Gamma = \Gamma_1 * \cdots * \Gamma_p$. Since each Γ_j is freely indecomposible, Theorem 3.1 implies that it is either loxodromic cyclic or has connected limit set, and Lemma 1.1 then implies that it is either loxodromic cyclic, degenerate, quasifuchsian, extended quasifuchsian, or a web group.

Note that $\Lambda(\Gamma_j)$ is a proper subset of $\Lambda(\Gamma)$, and that, by Lemma 4.1, each free factor Γ_j is a precisely invariant subgroup of Γ. Suppose that Γ_j is a web group. Then, there exists a component subgroup Φ of Γ_j which is not a component subgroup of Γ_j, so that $\Lambda(\Phi)$ separates $\Lambda(\Gamma)$. Since Γ_j is a precisely invariant subgroup of Γ and since Φ is a precisely embedded quasifuchsian subgroup of Γ_j, we see that Φ is a precisely embedded quasifuchsian subgroup of Γ, violating the assumption of simplicity of Γ.

If the free factor Γ_j is an extended quasifuchsian subgroup of Γ, note that $\Lambda(\Gamma_j)$ must separate $\Lambda(\Gamma)$. To see this, choose any point $x \in \Lambda(\Gamma) - \Lambda(\Gamma_j)$. If $\gamma \in \Gamma_j$ is an element interchanging the components of $\overline{\mathbb{C}} - \Lambda(\Gamma_j)$, then $\Lambda(\Gamma_j)$ separates x and $\gamma(x)$. Again, this violates the simplicity of Γ. If the free factor Γ_j is a quasifuchsian subgroup which is not a component subgroup of Γ, then $\Lambda(\Gamma_j)$ must separate $\Lambda(\Gamma)$, again violating the simplicity of Γ. This completes the proof. $\square$

The final result of this section is an immediate consequence of Theorem 3.3.

Lemma 4.5. *Let Γ be a purely loxodromic, finitely generated web group. Then, there exist subgroups $\Gamma_1, \dots, \Gamma_p$ of Γ, where each Γ_j is either a simple web group or an extended quasifuchsian group, such that every limit point of Γ either is a translate of a limit point of some Γ_j or is a point of approximation of Γ.*

Proof. Let $N = \mathbb{H}^3/\Gamma$ and let M be a compact core for N. Since Γ is a web group, all the components of ∂M facing geometrically finite ends of N are incompressible, as they correspond to quasifuchsian component subgroups of Γ.

Let $\mathcal{S} = \{S_1, \dots, S_p\}$ be a maximal collection of disjoint, incompressible, closed, embedded, orientable surfaces in M, so that no surface in $\mathcal{S}$ is parallel to a component of ∂M and so that no pair of surfaces in $\mathcal{S}$ are parallel. Write $N - \mathcal{S} = N_1 \cup \dots \cup N_t$, and let Γ_j be the image of the fundamental group of N_j in $\pi_1(N) = \Gamma$. Note that the component subgroups of Γ_j are all quasifuchsian, as they correspond either to the quasifuchsian component subgroups of Γ or to the quasifuchsian subgroups Φ_k corresponding to the S_k.

No Γ_j can be quasifuchsian or degenerate, as the surfaces in $\mathcal{S}$ are not parallel to each other or to components of ∂M. Since $\mathcal{S}$ is a maximal collection, no N_j contains an essential, closed, incompressible, orientable surface, and so no Γ_j can contain a precisely embedded quasifuchsian subgroup other than a quasifuchsian component subgroup. In particular, each Γ_j is either an extended quasifuchsian group or is a simple web group.

The statement about limit points follows immediately from Theorem 3.3. $\square$

5. The proof

The purpose of this section is to present the proof of Theorem 5.4. We begin with a series of three Lemmas, each handling a special case, which we combine to complete the proof of the Theorem. Note that the inclusion $\Lambda(\Phi_1 \cap \Phi_2) \subset \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ follows from the fact that $\Phi_1 \cap \Phi_2$ is a subgroup of both Φ_1 and Φ_2. Hence, we need only show the opposite inclusion to establish the equality $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$.

Lemma 5.1. *Let Γ be a purely loxodromic Kleinian group with non-empty domain of discontinuity. If Φ_1 is a degenerate subgroup of Γ and Φ_2 is a simple web subgroup of Γ, then $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.*

Proof. Let $x \in \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ be any point. If x is a point of approximation of either Φ_1 or Φ_2, Theorem 2.1 immediately implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$, and is in fact a point of approximation for $\Phi_1 \cap \Phi_2$. Henceforth, we assume that x is not a point of approximation of either Φ_1 or Φ_2. By replacing Γ by $\langle \Phi_1, \Phi_2 \rangle$, we may assume by Lemma 4.2 that $\Lambda(\Gamma)$ is connected, and hence that Γ is a web group.

Suppose that Φ_2 has finite index in Γ, so that $\Lambda(\Phi_2) = \Lambda(\Gamma)$. Then, $\Phi_1 \cap \Phi_2$ has finite index in Φ_1, and so $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1)$. Since $\Phi_1 \subset \Gamma$, we have that $\Lambda(\Phi_1) \subset \Lambda(\Gamma) = \Lambda(\Phi_2)$, and so $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$.

Suppose that Φ_2 is freely indecomposible, and hence topologically tame [9]. As degenerate groups are topologically tame [9], Theorem 2.2 implies that $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$.

Hence, we may assume that Φ_2 is freely decomposible, and we may use Lemma 4.4 to see that Φ_2 is an infinite index extreme web subgroup of Γ. In particular, Lemma 4.3 implies that Γ cannot be extreme. If Γ is freely indecomposible, then it is topologically tame [9]. As infinite index subgroups of co-infinite volume topologically tame groups are themselves topologically tame [11], both Φ_1 and Φ_2 are topologically tame, and Theorem 2.2 implies that $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$.

If Γ is freely decomposible, the proof of Lemma 4.4 implies that Γ contains a precisely embedded quasifuchsian or extended quasifuchsian subgroup Θ whose limit set $\Lambda(\Theta)$ separates $\Lambda(\Gamma)$. By Corollary 3.4, no translate of $\Lambda(\Theta)$ can separate $\Lambda(\Phi_1)$. There are now two cases.

Suppose that some translate of $\Lambda(\Theta)$ separates $\Lambda(\Phi_2)$. Since x is not a point of approximation of Φ_2, Theorem 3.2 implies that there exists a finitely generated, infinite index subgroup Φ_2^0 of Φ_2 so that $x \in \Lambda(\Phi_2^0)$. By Lemma 4.3, Φ_2^0 is a function group, and so is topologically tame [24]. Hence, $x \in \Lambda(\Phi_1 \cap \Phi_2^0)$ by Theorem 2.2. Since $\Lambda(\Phi_1 \cap \Phi_2^0) \subset \Lambda(\Phi_1 \cap \Phi_2)$, we see that $x \in \Lambda(\Phi_1 \cap \Phi_2)$.

Suppose now that no translate of $\Lambda(\Theta)$ separates $\Lambda(\Phi_2)$. Lemma 4.2 then implies that $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) \subset \Lambda(\Theta)$. Since Θ is geometrically finite, Theorem 2.1 implies that both $\Theta \cap \Phi_1$ and $\Theta \cap \Phi_2$ are geometrically finite and that $\Lambda(\Phi_j) \cap \Lambda(\Theta) = \Lambda(\Phi_j \cap \Theta)$ for both j. Hence,

$\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2) \cap \Lambda(\Theta)$. Since geometrically finite groups are topologically tame [15], Theorem 2.2 implies that $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) \cap \Lambda(\Theta) = \Lambda(\Phi_1 \cap \Theta) \cap \Lambda(\Phi_2 \cap \Theta)$, and so $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2 \cap \Theta)$. Since $\Lambda(\Phi_1 \cap \Phi_2 \cap \Theta) \subset \Lambda(\Phi_1 \cap \Phi_2)$, we are done. This completes the proof. $\square$

Lemma 5.2. *Let* Γ *be a purely loxodromic Kleinian group with non-empty domain of discontinuity. If* Φ_1 *and* Φ_2 *are extreme web subgroups of* Γ*, then* $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.

Proof. As in the proof of Lemma 5.1, we may assume that both Φ_1 and Φ_2 are infinite index subgroups of Γ, that $\Gamma = \langle \Phi_1, \Phi_2 \rangle$ and is a web group, and that $x \in \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ is not a point of approximation of either Φ_1 or Φ_2.

Since both Φ_1 and Φ_2 have infinite index in Γ, Lemma 4.3 implies that Γ cannot be extreme. If Γ is freely indecomposible, then it is topologically tame [9]. As infinite index subgroups of co-infinite volume topologically tame groups are themselves topologically tame [11], both Φ_1 and Φ_2 are topologically tame, and Theorem 2.2 implies that $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$.

Otherwise, Γ is freely decomposible, and so there exists a precisely embedded quasifuchsian or extended quasifuchsian subgroup Θ of Γ whose limit set $\Lambda(\Theta)$ separates $\Lambda(\Gamma)$. If some translate of $\Lambda(\Theta)$ separates $\Lambda(\Phi_1)$, Lemma 4.3 implies that Φ_1 contains a function group Φ_1^0 whose limit set contains x. Theorem 3.1 then implies that Φ_1^0 contains a degenerate subgroup Φ_1^{00} whose limit set contains x, as x is not a point of approximation of Φ_1^0. Lemma 5.1 then implies that $x \in \Lambda(\Phi_1^{00} \cap \Phi_2) \subset \Lambda(\Phi_1 \cap \Phi_2)$. The same argument holds if some translate of $\Lambda(\Theta)$ separates $\Lambda(\Phi_2)$.

If no translate of $\Lambda(\Theta)$ separates either $\Lambda(\Phi_1)$ or $\Lambda(\Phi_2)$, we may argue as in the last paragraph of Lemma 5.1 that $\Lambda(\Phi_1 \cap \Phi_2) = \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$. This completes the proof. $\square$

Lemma 5.3. *Let* Γ *be a purely loxodromic Kleinian group with non-empty domain of discontinuity. If* Φ_1 *and* Φ_2 *are simple web subgroups of* Γ*, then* $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.

Proof. If both Φ_1 and Φ_2 are freely indecomposible, they are both topologically tame [9], and Theorem 2.2 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$. If both Φ_1 and Φ_2 are extreme, Lemma 5.2 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$.

Suppose that Φ_1 is freely indecomposible and Φ_2 is extreme. As in the proof of Lemma 5.1, we may assume that $x \in \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ is not a point of approximation of Φ_1. Lemma 2.3 implies that there exists a degenerate subgroup Φ_1^0 of Φ_1 whose limit set contains x. Lemma 5.1 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$. The same argument holds if Φ_1 is extreme and Φ_2 is freely indecomposible. This completes the proof. □

We are now ready to proceed with the proof of the main theorem.

Theorem 5.4. *Let Γ be a purely loxodromic Kleinian group with non-empty domain of discontinuity. If Φ_1 and Φ_2 are finitely generated subgroups of Γ, then $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.*

Proof. Let $x \in \Lambda(\Phi_1) \cap \Lambda(\Phi_2)$ be any point. If x is a point of approximation of either Φ_1 or Φ_2, Theorem 2.1 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$. So, we assume that x is not a point of approximation of either Φ_1 or Φ_2. In particular, both Φ_1 and Φ_2 are non-elementary.

For either $j = 1$ or $j = 2$, since x is not a point of approximation of Φ_j, Theorem 3.1 implies that there exists a finitely generated subgroup Φ_j^0 of Φ_j whose limit set is connected and contains x. If Φ_j^0 is a web group, Lemma 4.5 implies that there exists either a simple web group or an extended quasifuchsian subgroup of Φ_j^0 whose limit set contains x. Hence, we may assume, for both j, that Φ_j is either quasifuchsian, extended quasifuchsian, degenerate, or a simple web group. If either Φ_1 or Φ_2 is quasifuchsian or extended quasifuchsian, it is geometrically finite, and so x is a point of approximation of Φ_j, contrary to assumption.

If both Φ_1 and Φ_2 are degenerate, then Theorem 2.2 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$. If Φ_1 is degenerate and Φ_2 is a simple web group (or vice versa), Lemma 5.1 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$. If both Φ_1 and Φ_2 are simple web groups, Lemma 5.3 implies that $x \in \Lambda(\Phi_1 \cap \Phi_2)$. This completes the proof. □

6. Closing remarks

We begin by noting that Theorem 5.4 holds for groups with torsion.

Corollary 6.1. *Let Γ be a Kleinian group without parabolics and with non-empty domain of discontinuity. If Φ_1 and Φ_2 are finitely generated subgroups of Γ, then $\Lambda(\Phi_1) \cap \Lambda(\Phi_2) = \Lambda(\Phi_1 \cap \Phi_2)$.*

Proof. If Γ is not finitely generated, replace Γ by $\langle \Phi_1, \Phi_2 \rangle$. Selberg's lemma [23] implies that there exists a finite index torsion-free subgroup Γ^0 of Γ. Let $\Phi_j^0 = \Phi_j \cap \Gamma^0$. Since Φ_j^0 has finite index in Φ_j, their limit sets are equal and $\Phi_1^0 \cap \Phi_2^0$ has finite index in $\Phi_1 \cap \Phi_2$. The Corollary now follows immediately from Theorem 5.4. □

It is well known, for a Kleinian group with non-empty domain of discontinuity, that the set of points of approximation has measure zero; this can be obtained, for example, by using the alternate definition of point of approximation, given in Proposition VI.B.9 of [18], to cover the set of points of approximation by a set of measure ε for each $\varepsilon > 0$. It is also well known [11] that the limit set of a topologically tame Kleinian group with non-empty domain of discontinuity has measure zero. Hence, Theorem 3.1, Lemma 4.4, and Lemma 4.5 together imply that, if the limit set of an extreme web group could be shown to have measure zero, then the Ahlfors measure conjecture would be established for purely loxodromic, finitely generated Kleinian groups. We note that a variant of this remark is contained in Maskit [20].

We close with a few remarks concerning groups with parabolic elements. Unfortunately, many of the techniques used in this paper do not generalize to groups with parabolics. In particular, the notion of an extreme web group becomes more complicated, as it is no longer possible to check algebraically. Also, the statements of the decomposition results in Section 3, in which all limit points are either limit points of conjugates of factor subgroups or points of approximation, are not known to generalize to the case of groups with parabolic elements. For those familiar with the proofs of the combination theorems, the difficulty is that limit points which are limits of nested sequences of axes of accidental parabolic elements are not known to be points of approximation.

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