

**A NON-ADDITIVE THERMODYNAMIC
FORMALISM AND DIMENSION THEORY
OF HYPERBOLIC DYNAMICAL SYSTEMS**

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ABSTRACT. We develop a non-additive version of the thermodynamic formalism. This allows us to unify and extend many results of dimension theory of hyperbolic dynamical systems. We obtain lower and upper bounds for the dimension of a broad class of Cantor-like sets and hyperbolic sets.

Introduction

We begin with the description of a geometric construction in the real line. We consider p positive numbers $\lambda_1, \dots, \lambda_p < 1$, and choose p disjoint closed intervals $\Delta_1, \dots, \Delta_p$ with length $\lambda_1, \dots, \lambda_p$. For each $k = 1, \dots, p$, we choose again p disjoint closed intervals $\Delta_{k1}, \dots, \Delta_{kp} \subset \Delta_k$ with length $\lambda_k \lambda_1, \dots, \lambda_k \lambda_p$. Iterating this procedure, for each integer n we obtain p^n disjoint closed intervals $\Delta_{i_1 \dots i_n}$ with length $\prod_{k=1}^n \lambda_{i_k}$. We define the set

$$(1) \quad F = \bigcap_{n=1}^{\infty} \bigcup_{(i_1 \dots i_n)} \Delta_{i_1 \dots i_n}.$$

In [M], Moran showed that the Hausdorff dimension of F is the unique root s of the equation

$$(2) \quad \sum_{k=1}^p \lambda_k^s = 1.$$

This result does not depend on the location of the intervals $\Delta_{i_1 \dots i_n}$.

The next step is to consider intervals $\Delta_{i_1 \dots i_n}$ with length $r_{i_1 \dots i_n}$ that may satisfy no asymptotic behavior, and may depend on all the symbolic past.

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Let $\sigma(i_1 i_2 \dots) = (i_2 i_3 \dots)$ be the shift map. If the sequence of functions $\varphi_n(i_1 i_2 \dots) = \log r_{i_1 \dots i_n}$ is sub-additive, that is, if $\varphi_{n+m} \leq \varphi_n + \varphi_m \circ \sigma^n$ for every $n, m \geq 1$, we can prove, under mild additional assumptions, that the Hausdorff and box dimensions of F satisfy

$$(3) \quad \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = s,$$

where s is the unique root of the equation

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{(i_1 \dots i_n)} r_{i_1 \dots i_n}^s = 0.$$

It is remarkable that this holds for any location of the intervals $\Delta_{i_1 \dots i_n}$.

We can generalize the above constructions to the case of arbitrary symbolic dynamics. Let Σ_p^+ be the compact set of one-sided sequences on p symbols, and let $Q \subset \Sigma_p^+$ be a compact shift-invariant subset. We define a set $F = F(Q)$ in a way similar to that in (1), where the union is taken only over the Q -admissible tuples $(i_1 \dots i_n)$, that is, those tuples such that $(i_1 \dots i_n) = (j_1 \dots j_n)$ for some $(j_1 j_2 \dots) \in Q$. In [PeW], Pesin and Weiss showed that if the intervals $\Delta_{i_1 \dots i_n}$ have length $\prod_{k=1}^n \lambda_{i_k}$, then the identities in (3) hold when we replace s by the unique root of Bowen's equation

$$(4) \quad P(s\varphi) = 0.$$

Here φ is the function defined on Q by $\varphi(i_1 i_2 \dots) = \log \lambda_{i_1}$, and P is the topological pressure with respect to the shift map. We recall that for an arbitrary continuous function φ on Q , we have

$$(5) \quad P(s\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\substack{(i_1 \dots i_n) \\ Q\text{-admissible}}} \exp \sup \left(s \sum_{k=0}^{n-1} \varphi \circ \sigma^k \right),$$

where the supremum is taken over the sequences $(j_1 j_2 \dots) \in Q$ such that $(j_1 \dots j_n) = (i_1 \dots i_n)$. As this example illustrates, we can use the thermodynamic formalism in complicated problems of dimension theory.

In [Bo], Bowen introduced equation (4). It has a rather universal character: most equations used to compute or estimate dimensions are particular cases of it. For example, if $Q = \Sigma_p^+$ then equation (4) is equivalent to equation (2).

We can also consider geometric constructions modeled by arbitrary symbolic dynamics, with intervals $\Delta_{i_1 \dots i_n}$ with length $r_{i_1 \dots i_n}$ that may satisfy

no asymptotic behavior, and may depend on all the symbolic past. If the sequence $\{\varphi_n\}$ defined above is sub-additive, we can prove, under mild additional assumptions, that the identities in (3) hold when we replace s by the unique root of the equation $P(s\{\varphi_n\}) = 0$, where P is an appropriate generalization of the topological pressure. This new pressure may become non-additive, in the sense that the function φ_n may substitute the sum $\sum_{k=0}^{n-1} \varphi \circ \sigma^k$ in (5).

In [Ba1], we develop a non-additive version of the topological pressure, for arbitrary sequences of functions $\{\varphi_n\}$ that may neither satisfy additivity nor even sub-additivity. We follow the approach of Pesin in [P1], to define pressure as a Carathéodory dimension characteristic. The new pressure generalizes the well-known notion of topological pressure on a compact set, introduced by Ruelle in [R1] for expansive maps, and by Walters in [W] in the general case.

Using the non-additive thermodynamic formalism, we unify and extend many results of dimension theory of hyperbolic dynamical systems. In particular, we establish lower and upper dimension estimates for repellers of continuous expanding maps, and for the intersection of basic sets of Axiom A[#] homeomorphisms with stable and unstable sets. The main elements of our approach are the use of Markov partitions to reproduce the structure of the invariant sets, and a version of Lyapunov exponents for continuous maps, that in the smooth case coincide with the Lyapunov exponents.

In the special case of smooth dynamical systems, we provide a straightforward approach, with proofs entirely based on the thermodynamic formalism. In particular, we recover the formula giving the Hausdorff dimension of basic sets of Axiom A surface diffeomorphisms, established by McCluskey and Manning in [MM]. When we establish the dimension estimates along a stable or unstable set, we need no information about the symbolic dynamics in the transverse direction.

Non-additive thermodynamic formalism

Non-additive topological pressure. Let (X, ρ) be a compact metric space, and $f: X \rightarrow X$ a continuous map. Let \mathcal{U} be a finite open cover of X . We denote by $\mathcal{W}_n(\mathcal{U})$ the collection of strings $\mathbf{U} = U_0 \cdots U_n$ of sets $U_0, \dots, U_n \in \mathcal{U}$, with length $m(\mathbf{U}) = n$. For each $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, we define the open set $X(\mathbf{U}) = \bigcap_{k=0}^n f^{-k}U_k$. We say that the collection of strings $\Gamma \subset \bigcup_{n \geq 1} \mathcal{W}_n(\mathcal{U})$ covers the set $Z \subset X$ if $\bigcup_{\mathbf{U} \in \Gamma} X(\mathbf{U}) \supset Z$.

We consider a sequence of functions $\varphi_n: X \rightarrow \mathbb{R}$ and denote it by Φ . For each positive integer n , we define

$$\gamma_n(\Phi, \mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x, y \in X(\mathbf{U}) \text{ for some } \mathbf{U} \in \mathcal{W}_n(\mathcal{U})\}.$$

We denote by $|U| = \sup\{\rho(x, y) : x, y \in U\}$ the *diameter* of the set $U \subset X$, and by $|\mathcal{U}| = \sup\{|U| : U \in \mathcal{U}\}$ the *diameter* of the cover \mathcal{U} . We assume that the following property holds:

$$(6) \quad \overline{\lim}_{|\mathcal{U}| \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n} = 0.$$

We follow the approach in [P1] to define pressure as a Carathéodory dimension characteristic. For each string $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, we write $\varphi(\mathbf{U}) = \sup_{X(\mathbf{U})} \varphi_n$ when $X(\mathbf{U}) \neq \emptyset$, and $\varphi(\mathbf{U}) = -\infty$ otherwise. For each set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$(7) \quad M(Z, \alpha, \Phi, \mathcal{U}) = \lim_{n \rightarrow \infty} \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} \exp(-\alpha m(\mathbf{U}) + \varphi(\mathbf{U})),$$

where the infimum is taken over all $\Gamma \subset \bigcup_{k \geq n} \mathcal{W}_k(\mathcal{U})$ that covers Z . Likewise, we define

$$(8) \quad \underline{M}(Z, \alpha, \Phi, \mathcal{U}) = \underline{\lim}_{n \rightarrow \infty} \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} \exp(-\alpha n + \varphi(\mathbf{U})),$$

$$(9) \quad \overline{M}(Z, \alpha, \Phi, \mathcal{U}) = \overline{\lim}_{n \rightarrow \infty} \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} \exp(-\alpha n + \varphi(\mathbf{U})),$$

where the infimum is now taken over all $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ that covers Z . When α goes from $-\infty$ to $+\infty$, each of the quantities in (7), (8), and (9) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define

$$\begin{aligned} P_Z(\Phi, \mathcal{U}) &= \inf\{\alpha : M(Z, \alpha, \Phi, \mathcal{U}) = 0\}, \\ \underline{CP}_Z(\Phi, \mathcal{U}) &= \inf\{\alpha : \underline{M}(Z, \alpha, \Phi, \mathcal{U}) = 0\}, \\ \overline{CP}_Z(\Phi, \mathcal{U}) &= \inf\{\alpha : \overline{M}(Z, \alpha, \Phi, \mathcal{U}) = 0\}. \end{aligned}$$

Theorem 1. *The following limits exist:*

$$\begin{aligned} P_Z(\Phi) &\stackrel{\text{def}}{=} \lim_{|\mathcal{U}| \rightarrow 0} P_Z(\Phi, \mathcal{U}), \\ \underline{CP}_Z(\Phi) &\stackrel{\text{def}}{=} \lim_{|\mathcal{U}| \rightarrow 0} \underline{CP}_Z(\Phi, \mathcal{U}), \quad \overline{CP}_Z(\Phi) \stackrel{\text{def}}{=} \lim_{|\mathcal{U}| \rightarrow 0} \overline{CP}_Z(\Phi, \mathcal{U}). \end{aligned}$$

We call $P_Z(\Phi)$ the *non-additive topological pressure*, and $\underline{CP}_Z(\Phi)$ and $\overline{CP}_Z(\Phi)$ the *non-additive lower and upper capacity topological pressures* of the sequence of functions Φ on the set Z (with respect to f). We emphasize that the set Z need not be compact nor f -invariant.

In the *classical thermodynamic formalism*, we consider a continuous function φ on X , and define the sequence of functions $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$. Using the uniform continuity of φ , one can show that property (6) holds. We recover the notion of topological pressure introduced by Pesin and Pitskel' in [PP], and the notions of lower and upper capacity topological pressures introduced by Pesin in [P1] for arbitrary sets. When $Z = X$, we obtain the topological pressure for compact sets introduced by Ruelle in [R1] in the case of expansive maps, and by Walters in [W] in the general case.

In [Ba1], we establish a non-additive version and generalization of the classical variational principle for the topological pressure.

Bowen's equation. For each real number s we denote by $s\Phi$ the sequence of functions $\{s\varphi_n\}$. Let $h(f)$ be the topological entropy of f .

Theorem 2. *If $K_1n \leq \varphi_n \leq K_2n$ for all sufficiently large n , where K_1, K_2 are negative constants, and $h(f) < \infty$, then there are unique roots $s_P, s_{\underline{CP}},$ and $s_{\overline{CP}}$ of the equations $P_Z(s\Phi) = 0, \underline{CP}_Z(s\Phi) = 0,$ and $\overline{CP}_Z(s\Phi) = 0.$*

Each equation in Theorem 2 is a non-additive version of *Bowen's equation* $P(s\varphi) = 0$. Here P is the classical topological pressure with respect to some continuous map $f: X \rightarrow X$, and φ is a continuous function on X . This equation establishes the connection between the thermodynamic formalism and dimension theory: its unique root often gives the exact value or a good estimate for dimension. A good way to think about the relation between dimension and pressure is that both are Carathéodory dimension characteristics (see [P1]).

Applications to dimension theory

A *symbolic geometric construction* in \mathbb{R}^m is defined by:

- (a) a compact shift-invariant set $Q \subset \Sigma_p^+$ for some $p \geq 1$;
- (b) a decreasing sequence of compact sets $\Delta_{i_1 \dots i_n} \subset \mathbb{R}^m$ for each sequence $(i_1 i_2 \dots) \in Q$, with diameter $|\Delta_{i_1 \dots i_n}| \rightarrow 0$ as $n \rightarrow \infty$.

The *limit set* of the construction is the compact set $F = \bigcap_{n=1}^{\infty} \bigcup \Delta_{i_1 \dots i_n}$, where the union is taken over all Q -admissible n -tuples, that is, all tuples $(i_1 \dots i_n)$ such that $(i_1 \dots i_n) = (j_1 \dots j_n)$ for some $(j_1 j_2 \dots) \in Q$.

We notice that the sets $\Delta_{i_1 \dots i_n}$ need not be disjoint for any fixed n . Our constructions include as trivial examples any iterated function scheme defined by a family of contraction maps. Our setting is much more general and the limit sets that we consider may not be obtained using a family of contraction maps.

Generalized Moran constructions are constructions such that there exist balls $\underline{B}_{i_1 \dots i_n}$ and $\overline{B}_{i_1 \dots i_n}$ for each $(i_1 i_2 \dots) \in Q$ and $n \geq 1$ such that:

- (a) $\underline{B}_{i_1 \dots i_n} \subset \Delta_{i_1 \dots i_n} \subset \overline{B}_{i_1 \dots i_n}$;
- (b) the interiors of $\underline{B}_{i_1 \dots i_n}$ and $\underline{B}_{j_1 \dots j_m}$ are disjoint for any $(i_1 \dots i_n) \neq (j_1 \dots j_m)$ and $m \geq n$;
- (c) $\underline{B}_{i_1 \dots i_n}$ and $\overline{B}_{i_1 \dots i_n}$ have radii $C_1 r_{i_1 \dots i_n}$ and $C_2 r_{i_1 \dots i_n}$, for some positive number $r_{i_1 \dots i_n}$ and positive constants $C_1 \leq C_2$.

We notice that the numbers $r_{i_1 \dots i_n}$ may depend on all the symbolic past, and may satisfy no asymptotic behavior. A special class of generalized Moran constructions is the class of Moran constructions, introduced in [M] when $Q = \Sigma_p^+$ (see [PeW]).

We define the sequence of functions $\varphi_n(i_1 i_2 \dots) = \log r_{i_1 \dots i_n}$ on Q , and denote it Φ . Then property (6) holds for the shift map. We assume that there exist $K > 1$ and $\delta > 0$ such that, for each $(i_1 i_2 \dots) \in Q$ and $n \geq 1$,

$$(10) \quad r_{i_1 \dots i_n} \leq K^{-n} \quad \text{and} \quad r_{i_1 \dots i_{n+1}} \geq \delta r_{i_1 \dots i_n}.$$

By (10) and Theorem 2, there exist unique roots s_P , $s_{\underline{CP}}$, and $s_{\overline{CP}}$ of the equations $P_Q(s\Phi) = 0$, $\underline{CP}_Q(s\Phi) = 0$, and $\overline{CP}_Q(s\Phi) = 0$.

Theorem 3. *For a generalized Moran construction satisfying (10), we have:*

- (a) $s_P \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq s_{\overline{CP}}$;
- (b) *if Φ is sub-additive then $\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = s_P = s_{\underline{CP}} = s_{\overline{CP}} = s$, where s is the unique root of the equation*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\substack{(i_1 \dots i_n) \\ Q\text{-admissible}}} r_{i_1 \dots i_n}^s = 0.$$

In [Ba1], we construct an example which demonstrates that we may have strict inequalities in Theorem 3a if the sequence Φ is not sub-additive. In [Ba2], we establish extensions of Theorem 3 to classes of limit sets with more complicated geometry.

Dimension of hyperbolic sets

Repellers of continuous expanding maps. Let (X, d) be a compact metric space. A continuous map $h: X \rightarrow X$ is called *expanding* if it is a local homeomorphism at every point, and there exist constants $a \geq b > 1$ and $r_0 > 0$ such that, for each $x \in X$ and $0 < r < r_0$, we have

$$(11) \quad B(hx, br) \subset h(B(x, r)) \subset B(hx, ar).$$

We then say that X is a *repeller* of h . One can show that a continuous expanding map is locally Lipschitz (see (12) below).

Let R_1, \dots, R_p be a Markov partition of X with transfer matrix A , and Σ_A^+ the associated topological Markov chain. For each $\omega = (i_1 i_2 \dots) \in \Sigma_A^+$ and integers $n, k \geq 1$, we define the *lower* and *upper ratio coefficients* respectively by

$$\underline{\lambda}_k(\omega, n) = \min \inf \left\{ \frac{d(x, y)}{d(h^n x, h^n y)} : x, y \in R_{j_1 \dots j_{n+k}}, x \neq y \right\},$$

$$\bar{\lambda}_k(\omega, n) = \max \sup \left\{ \frac{d(x, y)}{d(h^n x, h^n y)} : x, y \in R_{j_1 \dots j_{n+k}}, x \neq y \right\},$$

where the minimum and maximum are taken over the Σ_A^+ -admissible tuples $(j_1 \dots j_{n+k})$ with $(j_1 \dots j_n) = (i_1 \dots i_n)$, and $R_{i_1 \dots i_n} = \bigcap_{k=1}^n h^{-k+1} R_{i_k}$.

The ratio coefficients play the role of Lyapunov exponents for continuous maps that may not be differentiable. Since the maps $k \mapsto \underline{\lambda}_k(\omega, n)$ and $k \mapsto \bar{\lambda}_k(\omega, n)$ are respectively non-decreasing and non-increasing for each fixed $\omega \in \Sigma_A^+$ and $n \geq 1$, we can define $\underline{\lambda}(\omega, n) = \lim_{k \rightarrow \infty} \underline{\lambda}_k(\omega, n)$ and $\bar{\lambda}(\omega, n) = \lim_{k \rightarrow \infty} \bar{\lambda}_k(\omega, n)$. One can show that $\log \underline{\lambda}(\omega, n)$ and $\log \bar{\lambda}(\omega, n)$ are respectively sup-additive and sub-additive cocycles (see [Ba1]). Hence, if μ is a shift-invariant probability measure on Σ_A^+ , and $\log^- \underline{\lambda}(\cdot, 1)$ and $\log^+ \bar{\lambda}(\cdot, 1)$ are μ -integrable on Σ_A^+ , then the limits

$$\underline{\lambda}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \underline{\lambda}(\omega, n) \quad \text{and} \quad \bar{\lambda}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{\lambda}(\omega, n)$$

exist for μ -almost every $\omega \in \Sigma_A^+$. This follows from Kingman's sub-additive ergodic theorem. When h is smooth, the numbers $\underline{\lambda}(\omega)$ and $\bar{\lambda}(\omega)$ respectively coincide with the inverse of the largest and smallest Lyapunov exponents of h at the point $\bigcap_{n=1}^\infty h^{-n+1} R_{i_n}$, where $\omega = (i_1 i_2 \dots)$. Related notions were first considered by Kifer in [K].

We define the sequences of functions

$$\underline{\varphi}_n^k(\omega) = \log \underline{\lambda}_k(\omega, n) \quad \text{and} \quad \bar{\varphi}_n^k(\omega) = \log \bar{\lambda}_k(\omega, n)$$

on Σ_A^+ , and denote them $\underline{\Phi}^k$ and $\bar{\Phi}^k$. One can show that

$$(12) \quad a^{-n} \leq \underline{\lambda}_k(\omega, n) \leq \bar{\lambda}_k(\omega, n) \leq b^{-n}$$

for each $\omega \in \Sigma_A^+$, $n \geq 1$, and all sufficiently large $k \geq 1$.

By (12) and Theorem 2, there exist unique roots \underline{s}^k and \bar{s}^k of the equations $\overline{CP}_{\Sigma_A^+}(s \underline{\Phi}^k) = 0$ and $P_{\Sigma_A^+}(s \bar{\Phi}^k) = 0$.

Theorem 4. *If X is a repeller of a topologically mixing expanding map then $\underline{s}^k \leq \dim_H X \leq \underline{\dim}_B X \leq \overline{\dim}_B X \leq \overline{s}^k$.*

The map h is called *asymptotically conformal* if for some $k \geq 1$, given $\varepsilon > 0$ we have $\overline{\lambda}_k(\omega, n) \leq e^{\varepsilon n} \underline{\lambda}_k(\omega, n)$ for all $\omega \in \Sigma_A^+$ and all large n .

Theorem 5. *If X is a repeller of a topologically mixing expanding map which is asymptotically conformal, then $\dim_H X = \underline{\dim}_B X = \overline{\dim}_B X = s$, where s is the unique root of the equation*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\substack{(i_1 \dots i_n) \\ \Sigma_A^+ \text{-admissible}}} |R_{i_1 \dots i_n}|^s = 0.$$

Repellers of smooth expanding maps. Let g be a C^1 map on the manifold M , and J a compact subset of M . We call g *expanding* and J a *repeller* of g if:

- (a) there exists $a > 1$ such that $\|d_x g u\| \geq a \|u\|$ for all $x \in J$ and $u \in T_x M$ (for some Riemannian metric on M);
- (b) $J = \bigcap_{n \geq 0} g^{-n} V$ for some open neighborhood V of J ;
- (c) g is topologically mixing on J .

Then $gJ = J$ and $g: J \rightarrow J$ is expanding in the former sense.

A smooth map g is called *conformal* if $d_x g$ is a multiple of an isometry for every point $x \in M$. Examples of repellers of conformal smooth expanding maps are hyperbolic Julia sets and dynamically defined Cantor sets (see [T]).

We show in [Ba1] that every conformal C^1 expanding map is asymptotically conformal. We define the function $\varphi(x) = -\log \|d_x g\|$ on M , and let P_g be the classical topological pressure with respect to g .

Theorem 6. *Let J be a repeller of the conformal C^1 expanding map g . Then $\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s$, where s is the unique root of Bowen's equation $P_g(s\varphi) = 0$.*

In [R2], Ruelle proved that the Hausdorff dimension of a repeller of a conformal $C^{1+\alpha}$ expanding map is given by Bowen's equation. In [F], Falconer showed that the Hausdorff and box dimensions of these repellers coincide. The statement of Theorem 6 was also established by Gatzouras and Peres in [GP], and by Takens in [T] for dynamically defined Cantor sets.

Basic sets of Axiom $A^\#$ homeomorphisms. Let f be a homeomorphism of the compact metric space (X, d) . Given $\varepsilon > 0$, we define the *local*

stable and unstable sets at the point $x \in X$ respectively by

$$W^s(x) = \{y \in X : d(f^n y, f^n x) \leq \varepsilon \text{ for all } n \geq 0\},$$

$$W^u(x) = \{y \in X : d(f^n y, f^n x) \leq \varepsilon \text{ for all } n \leq 0\}.$$

We say that f is an *Axiom A[#] homeomorphism* (see [AY]) if there exist constants $\lambda \in (0, 1)$ and $\varepsilon, \delta > 0$ such that:

- (a) for each $x \in X$ and $n \geq 0$, we have $d(f^n y, f^n z) \leq \lambda^n d(y, z)$ for all $y, z \in W^s(x)$, and $d(f^{-n} y, f^{-n} z) \leq \lambda^n d(y, z)$ for all $y, z \in W^u(x)$;
- (b) for every $x, y \in X$ with $d(x, y) < \delta$, the set $W^s(x) \cap W^u(y)$ consists of a single point, which we denote $[x, y]$. We require the map $(x, y) \mapsto [x, y]$ to be continuous.

Then the set of non-wandering points is a finite union of disjoint closed f -invariant sets, called *basic sets*.

Let f be an Axiom A[#] homeomorphism and $\Lambda \subset X$ a basic set. We consider a Markov partition R_1, \dots, R_p of Λ with transfer matrix A , and write $R_{i_0 \dots i_n} = \bigcap_{k=0}^n f^{-k} R_{i_k}$. Fix $x \in \Lambda$. For each $\omega = (i_0 i_1 \dots) \in \Sigma_A^+$ and integers $n, k \geq 1$, we define

$$\underline{\Delta}_k^u(\omega, n) = \min \inf \left\{ \frac{d(y, z)}{d(f^n y, f^n z)} : y, z \in W^u(x) \cap R_{j_0 \dots j_{n+k}}, y \neq z \right\},$$

$$\overline{\lambda}_k^u(\omega, n) = \max \sup \left\{ \frac{d(y, z)}{d(f^n y, f^n z)} : y, z \in W^u(x) \cap R_{j_0 \dots j_{n+k}}, y \neq z \right\},$$

where the minimum and maximum are taken over the Σ_A^+ -admissible tuples $(j_0 \dots j_{n+k})$ such that $(j_0 \dots j_n) = (i_0 \dots i_n)$. We assume that

(13) the map $f|W^u(x)$ is expanding for every $x \in \Lambda$,

that is, the map $f|W^u(x): W^u(x) \rightarrow fW^u(x)$ satisfies the inclusions in (11), with h substituted by f , for some constants $a(x) \geq b(x) > 1$. Furthermore, we assume that $\overline{\lim}_{n \rightarrow \infty} (a(f^n x) \dots a(x))^{1/n}$ is finite for every $x \in \Lambda$.

Fix $x \in \Lambda$. We define the sequences of functions $\underline{\varphi}_n^k(\omega) = \log \underline{\Delta}_k^u(\omega, n)$ and $\overline{\varphi}_n^k(\omega) = \log \overline{\lambda}_k^u(\omega, n)$ on Σ_A^+ . By Theorem 2 and (13), there exist unique roots $\underline{s}^k(x)$ and $\overline{s}^k(x)$ of the equations $\overline{CP}_{\Sigma_A^+}(s \underline{\Phi}^k) = 0$ and $P_{\Sigma_A^+}(s \overline{\Phi}^k) = 0$.

Theorem 7. *Let Λ be a basic set of a topologically mixing Axiom A[#] homeomorphism. Then $\underline{s}^k(x) \leq \dim_H W^u(x) \cap \Lambda \leq \underline{\dim}_B W^u(x) \cap \Lambda \leq \overline{\dim}_B W^u(x) \cap \Lambda \leq \overline{s}^k(x)$.*

Property (13) is automatically satisfied when f is of class C^1 , but it may not hold for arbitrary Axiom A[#] homeomorphisms. More generally, if the map f is locally Lipschitz, then property (13) holds.

In a similar way, we can obtain dimension estimates for the stable sets. **Basic sets of Axiom A diffeomorphisms.** Let f be a C^1 diffeomorphism on the manifold M . An f -invariant set $\Lambda \subset M$ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ splits into stable and unstable bundles E^s and E^u , invariant under df . We say that f is an *Axiom A diffeomorphism* if the set of non-wandering points is hyperbolic and is the closure of the set of periodic points of f . We define the functions $\varphi^s(x) = \log \|d_x f|E^s\|$ and $\varphi^u(x) = -\log \|d_x f|E^u\|$ on Λ . Let t^s and t^u be the unique roots of the equations $P_f(t \varphi^s) = 0$ and $P_f(t \varphi^u) = 0$.

Theorem 8. *Let Λ be a basic set of a C^1 Axiom A surface diffeomorphism, with $\dim E^s = \dim E^u = 1$. Then $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t^s + t^u$.*

In [MM], McCluskey and Manning proved that $\dim_H \Lambda = t^s + t^u$. The coincidence of the Hausdorff and box dimensions was previously shown by Takens in the case of C^2 diffeomorphisms (see [T]), and by Palis and Viana in [PV] for arbitrary C^1 diffeomorphisms. The technique in [PV] differs from the one in [Ba1] and is not based on the thermodynamic formalism. In higher dimensional manifolds, the Hausdorff and box dimensions of basic sets may not coincide (see [PoW] for examples).

In [Ba1], we also obtain lower and upper dimension estimates for basic sets in higher dimensional manifolds. In [P2], Pesin gives a straightforward treatment of the case where one of the maps $df|E^u$ and $df|E^s$ is conformal.

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