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AN APPROXIMATION PROPERTY FOR TEICHMÜLLER POINTS

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Start with a field K of characteristic zero, complete under a discrete valuation and having an algebraically closed residue field k of characteristic $p > 0$. Let R be the valuation ring of K. Assume that we are given a prime element $\pi \in R$ which is algebraic over Q_p. Let q be the cardinality of the residue field of $Q_p(\pi)$ and let ϕ be the unique ring automorphism of R with $\phi(\pi) = \pi$ that lifts the "Frobenius" automorphism $F : k \to k$, $F(x) := x^q$.

Let X be a scheme of finite type over R and assume the Frobenius $x \mapsto x^q$ of the closed fibre $X_0 := X \otimes k$ lifts to a $\phi-$endomorphism $\tilde{\phi}$ of $\hat{X}$, the completion of X with respect to the ideal generated by π. For any point $P \in X(R) = \hat{X}(R)$, $P : Spf\ R \to \hat{X}$ define the point $P^{\tilde{\phi}} \in X(R)$ as the composition

$$Spf\ R \xrightarrow{\phi^{-1}} Spf\ R \xrightarrow{P} \hat{X} \xrightarrow{\tilde{\phi}} \hat{X}$$

Call P a Teichmüller point if $P^{\tilde{\phi}} = P$ and let $T = T(X, \tilde{\phi}) \subset X(R)$ denote the set of Teichmüller points.

Examples

1) Let $X = (G_m)^N = \text{Spec } R[x_1, x_1^{-1}, ..., x_N, x_N^{-1}]$ be a torus over R and let $\tilde{\phi}$ be the unique lifting of ϕ to $\hat{X}$ such that $\tilde{\phi}(x_i) = x_i^q + \pi g_i$, where $g_i \in R[x_1, ..., x_N]$. So T consists of all points $r = (r_1, ..., r_N) \in R^\times \times ... \times R^\times$ such that $\phi(r_i) = r_i^q + \pi g_i(r)$. So if K is the completion of the maximum unramified extension of Q_p and $g_i = 0$, then T consists all points whose coordinates are roots of unity of order prime to p.

2) Assume for simplicity that K is the completion of the maximum unramified extension of Q_p. Let X be an abelian variety with ordinary reduction X_0 and assume X is the canonical lifting of X_0; by [Ka1], there is a canonical lifting of Frobenius, $\tilde{\phi}$, to $\hat{X}$. Then T contains the prime to p torsion of $X(R)$.

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3) Assume again that K is as in Example 2) above. Let $n \geq 3$ be an integer not divisible by p. Let

$$X = \mathrm{Spec}_{\bar{M}_n \otimes R}(\mathrm{Symm}(\omega^{p-1})/(E_{p-1} - 1))$$

where $\bar{M}_n$ is obtained by "adding cusps" to the modular scheme over $Z[1/n]$ classifying the elliptic curves with level n structure (cf. [Ka2], pp. 81-82), ω is the natural invertible sheaf defined in [Ka2], p. 82, and E_{p-1} is the corresponding Eisenstein series. By [Ka2], p.111, X is an affine scheme. By [Ka2] pp. 122-124, there is a natural lifting of the Frobenius, $\tilde{\phi}$, to $\hat{X}$. Using the construction of $\tilde{\phi}$ via the "canonical subgroup" one can check that T contains all points of $X(R)$ that are not cusps and for which the corresponding elliptic curve E/R is a canonical lift of its closed fibre E_0. One can consider more complicated examples by taking d fold products of the X above. Also it is reasonable to expect that a similar example is obtainable by taking modular varieties corresponding to abelian varieties of higher dimension.

4) Let $X = \mathrm{Proj}\ R[x_0, ..., x_N]$ be the projective space over R and let $\tilde{\phi}$ be the unique lifting of ϕ to $\hat{X}$ such that

$$\tilde{\phi}\left(\frac{x_i}{x_j}\right) = \frac{x_i^q + \pi G_i}{x_j^q + \pi G_j}$$

where $G_0, ..., G_N \in R[x_0, ..., x_N]$ are homogenous polynomials of degree q. So , for instance, if $G_i = 0$ and K is the completion of the maximum unramified extension of Q_p then T consists all points whose projective coordinates have ratios roots of unity of order prime to p.

Let X/R be a scheme, let $Y \subset X$ be a closed subscheme and let $P \in X(R)$. Set $R_m := R/\pi^{m+1}R$ for $m \geq 1$, $X_m := X \otimes R_m$, $Y_m := Y \otimes R_m$, and let $P_m \in X_m(R_m)$, be the image of P. One defines the $p-$adic distance from P to Y as

$$\mathrm{dist}(P, Y) = \inf\{p^{-m}; P_m \in Y_m(R_m)\}.$$

Of course $\mathrm{dist}(P, Y) = 0$ if and only if $P \in Y(R)$. Here is our main result:

Theorem 1. *Let X/R be a scheme of finite type, $Y \subset X$ a closed subscheme, $\tilde{\phi} : \hat{X} \to \hat{X}$ a $\phi-$lifting of the Frobenius of the closed fibre X_0 and $T = T(X, \tilde{\phi}) \subset X(R)$ the set of Teichmüller points. Then there exists a real constant $c = c(X, Y, \tilde{\phi}) > 0$ such that for any $P \in T$ with $dist(P, Y) \leq c$ we must have $P \in Y(R)$.*

Remark. Theorem 1 applied to Examples 1, 2, 3 above answers special cases of a question posed to the author by F.Voloch. Cf. [TV] for the case of Example 1.

For the case of curves in projective varieties we can supplement the above Theorem as follows:

Theorem 2. *Assume we are in the situation of Theorem 1 and assume moreover that K is absolutely unramified, X/R is projective and Y/R is a smooth curve of genus ≥ 2. Then the set of points $\{P \in T;\ \mathrm{dist}(P, Y) < 1\}$ is finite.*

The proofs of these two results will be an easy consequence of a construction made in [B1] whose properties we now recall. For any $R-$algebra B we denote by $W_2^\pi(B)$ the ring of "ramified Witt vectors of length two", whose underlying set is $B \times B$ and whose addition and multiplication are given by:

$$(b_0, b_1) + (c_0, c_1) = (b_0 + c_0, b_1 + c_1 - (p/\pi)C_q(b_0, c_0))$$

$$(b_0, b_1) \cdot (c_0, c_1) = (b_0 c_0, b_0^q c_1 + c_0^q b_1 + \pi b_1 c_1)$$

where $C_q(X, Y) = ((X+Y)^q - X^q - Y^q)/p \in Z[X, Y]$. Let $f : A \to B$ be an $R-$algebra homomorphism. By a $\pi-$derivation of f we shall understand a map of sets $\delta : A \to B$ such that the induced map

$$(f, \delta) : A \to B \times B = W_2^\pi(B), \quad x \mapsto (f(x), \delta(x))$$

is a ring homomorphism. For instance the map $\delta_* : R \to R$ defined by $\delta_* x = (\phi(x) - x^q)/\pi$ is a $\pi-$derivation. There is an obvious notion of $\pi-$derivation of a map of sheaves of $R-$algebras on a topological space. For any scheme of finite type X/R we constructed, in [B1], a projective system

$$\ldots \to X^n \xrightarrow{f_n} X^{n-1} \to \ldots \to X^1 \xrightarrow{f_1} X^0 = \hat{X}$$

of $\pi-$formal schemes (where "$\pi-$formal scheme" means "formal scheme for which the ideal generated by π is an ideal of definition"), and $\pi-$derivations δ_n (extending δ_*) of $\mathcal{O}_{X_{n-1}}$ into the direct image of $\mathcal{O}_{X_n}$, such that each δ_{n+1} prolongs δ_n and such that the following universality property is satisfied. For any morphism of $\pi-$formal schemes $g : S \to X^n$ and for any $\pi-$derivation δ of $\mathcal{O}_{X^n}$ into $g_* \mathcal{O}_S$, prolonging δ_n, there is a unique morphism of $\pi-$formal schemes $f : S \to X^{n+1}$ such that $f^* \circ \delta_{n+1} = \delta$ and $f_{n+1} \circ f = g$. This universality property induces natural maps

$$\nabla^n : X(R) \to X^n(R)$$

which induce bijections

$$\nabla_0^n : X_n(R_n) \to X_0^n(k)$$

where, as usual, $X_0^n := X^n \otimes k$. (In case K is absolutely unramified, but only in this case, the $k-$schemes X_0^n are the "Greenberg transforms" of X.)

Now we are in a position to prove Theorem 1. We may assume $\hat{X}/R$ is flat, hence we may define a $\pi-$derivation of the structure sheaf of $\hat{X}$ by the formula $\delta x = (\tilde{\phi}(x) - x^q)/\pi$. By the universality property of X^n, since the δ_n's extend δ_*, there exist induced sections $s^n : \hat{X} \to X^n$ of the projections $X^n \to \hat{X}$ such that $f_n \circ s^n = s^{n-1}$ for all n. Tensorizing with k we get a system of sections $s_0^n : X_0 \to X_0^n$ of the projections $X_0^n \to X_0$. Consider the closed subschemes $Y^n \subset X^n$ and their reductions modulo p, $Y_0^n \subset X_0^n$. Consider the closed subschemes

$$Z(n) = (s_0^n)^{-1}(Y_0^n \cap s_0^n(X_0)) \subset X_0.$$

They form a descending sequence so there exists an index n_0 such that

$$(*) \quad Z(n) = Z(n_0), \quad n \geq n_0.$$

Now let $P \in T$, viewed as a morphism $P : Spf\ R \to \hat{X}$. We claim that $\nabla^n(P) : Spf\ R \to X^n$ factors through $s^n(\hat{X})$, i.e. that $\nabla^n(P) = s^n \circ P$. (Indeed we proceed by induction on n. Assume $\nabla^{n-1}(P) = s^{n-1} \circ P$. Since $P \in T$ we have $P^* \circ \delta = \delta_* \circ P^*$. On the other hand, by the construction of the s^n's we have $s^{n,*} \circ \delta_n = \delta \circ s^{n-1,*}$. We get

$$(s^n \circ P)^* \circ \delta_n = P^* \circ s^{n,*} \circ \delta_n = P^* \circ \delta \circ s^{n-1,*}$$

$$= \delta_* \circ P^* \circ s^{n-1,*} = \delta_* \circ \nabla^{n-1}(P)^*.$$

On the other hand, by the definition of ∇^n we have

$$\nabla^n(P)^* \circ \delta_n = \delta_* \circ \nabla^{n-1}(P)^*.$$

The two equations above plus the universality property of X^n impliy that $\nabla^n(P) = s^n \circ P$, and the induction step is proved.) We conclude that $\nabla_0^n(P_n) \in s_0^n(X_0)(k)$ for all n. In particular if $P \in T$ and $\text{dist}(P, Y) \leq p^{-n_0}$ we get $\nabla_0^{n_0}(P_{n_0}) \in Y_0^{n_0}(k) \cap s_0^{n_0}(X_0)(k)$, hence, by $(*)$, $\nabla_0^n(P_n) \in Y_0^n(k) \cap s_0^n(X_0)(k)$ for all $n \geq n_0$, hence $P \in Y(R)$ and we are done.

To prove Theorem 2 it is enough to prove that $Y_0^1 \cap s_0^1(X_0)$ is a finite set. But $s_0^1(X_0)$ is a projective variety while, by [B2], Proposition 1.10, Y_0^1 is an affine variety. Since both these varieties are closed in X_0^1, their intersection must be finite.

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