Mathematical Research Letters **3**, 449–452 (1996)

SZEGÖ THEOREMS FOR ZOLL OPERATORS

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The purpose of this note is to announce some new results about multi-dimensional Szegö estimates in the spirit of [O]. The setting for these results is a compact d-dimensional manifold, X, and a self-adjoint first order elliptic pseudodifferential operator, $Q: \; L^2(X) \longrightarrow L^1(X)$. This operator is called a *Zoll operator* if its spectrum consists of the integers, $1, 2, \dots$. For such an operator the bicharacteristic flow on the cotangent bundle of X associated with the leading symbol of Q has to be periodic of period 2π; and to simplify the statements of our results we will assume that this flow is *simply periodic* of period 2π, i.e. each bicharacteristic returns for the first time to its initial position at time $t = 2\pi$.[1] Let π_n be projection onto the n-th eigenspace of Q and let $P_n = \pi_1 + \cdots + \pi_n$. Our main result is the following:

Theorem 1. *Let B be a zeroth order pseudodifferential operator for which the symbolic norm of $I - B$ is sufficiently small. Then*

$$\operatorname{Log}\det\left(P_n B P_n\right) \sim b \operatorname{Log} n + \sum_{k=d}^{-\infty} b_k n^k \tag{1}$$

as n tends to infinity.

The constants, b and $b_k, k \neq 0$, are *local* invariants of B, i.e. only depend on the symbol of B. In particular the first two are given by the formulas

$$b_d = \frac{1}{d} \operatorname{Res}\left(Q^{-d} \operatorname{Log} B\right) \tag{2}$$

Received June 14, 1996.

The first author is supported by NSF grant DMS–890771.

The second author is supported by NSF grant DMS–9506057.

[1]If we drop this assumption the right hand side of the formula (1) below becomes the sum of a finite number of terms of the form:

$$\omega_i^n \left(b_i \operatorname{Log} n + \sum_{k=d}^{-\infty} b_{i,k} n^k \right),$$

the ω_i's being roots of unity.

and

$$b_{d-1} = \frac{1}{2}\mathrm{Res}\left(Q^{-d}\,\mathrm{Log}\,B\right) + \frac{1}{d-1}\mathrm{Res}\left(Q^{-(d+1)}\,\mathrm{Log}\,B\right) + \sum_{j=1}^{\infty} j\ \mathrm{Res}\left(Q^{-d}(\mathrm{Log}\,B)_j(\mathrm{Log}\,B)_{-j}\right) \tag{3}$$

"Res" being the Wodzicki residue.[2] The most interesting term in this expansion is the non-local term, b_0, since the exponential of it can be viewed as a regularized determinant of B.

We will give a brief sketch of how to prove this result. Expanding $\mathrm{Log}\,P_nBP_n$ in a Taylor series about the identity it suffices to prove:

Theorem 2. *For every zeroth order pseudodifferential operator, A, there exists an asymptotic expansion*

$$\mathrm{trace}\,(P_nAP_n)^r \sim a\,\mathrm{Log}\,n + \sum_{k=d}^{-\infty} a_k n^k \tag{4}$$

as n tends to infinity.

Proof. (Sketch) Let $U(t) = \exp itQ$. The operator $U(-t)AU(t)$ is a periodic function of t of period 2π; so one can expand it into a Fourier series

$$\sum A_k e^{-ikt}$$

with

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} e^{ikt}U(-t)A\ U(t)dt = \sum_n \pi_{k+n}A\pi_n,$$

and Egorov's theorem says that if A is a zeroth order pseudodifferential operator, the A_k's are as well. To prove the theorem, we'll first assume that all but finitely many of them are zero. Letting $\sigma(j) = \max(0, j_1, \dots, j_1 + \cdots + j_r)$,

$$\mathrm{trace}\,(P_nAP_n)^r = \sum_{j_1+\cdots+j_r=0} \mathrm{trace} \sum_{k+\sigma(j)\le n} \pi_k A_{j_r}\dots A_{j_1}\pi_k, \tag{5}$$

the number of summands in j being finite. The asymptotic expansion (4) now follows from the following result of Colin de Verdiere:

[2] The operators $(\mathrm{Log}\,B)_j$ will be defined below.

Lemma 3. *For any zeroth order pseudodifferential operator, A,*

$$\operatorname{trace} \pi_n A \pi_n \sim \sum_{k=d-1}^{-\infty} c_k(A) n^k \tag{6}$$

as n tends to infinity.

The $c_k(A)$'s, incidentally, are given by Wodzicki residues

$$c_k(A) = \operatorname{Res}\left(Q^{-(k+1)} A\right) \tag{7}$$

(c.f. [GO]) and hence the terms in the asymptotic expansion (4) are given by Wodzicki residues of products of the Fourier coefficients of A.

Finally we prove (4) for arbitrary A by showing that the terms in the Fourier series above are rapidly decreasing as k tends to infinity and hence that A is well-approximated by the finite sum, $\sum_{k=-N}^{N} A_k$.

Theorem 4. *For every integer, N, the operator norm of A_k is bounded by $C_N k^{-N}$, C_N being a positive constant not depending on k.*

Proof. Let $ad\ Q$ be the operation

$$(ad\ Q)\, A = QA - AQ.$$

Then

$$(ad\ Q)^N A = \sum k^N\ A_k$$

so the operator norm of $k^N A_k$ is bounded by the operator norm of $(ad\ Q)^N A$. □

We will conclude by mentioning some other types of multi-dimensional Szegö theorems which can be proved by the methods of this paper:

1. Let B be a zeroth order pseudodifferential operator on $\mathbb{R}^d$ whose Weyl symbol is polyhomogeneous with respect to the homotheties, $(x, \xi) \longrightarrow (\lambda^{\frac{1}{2}} x, \lambda^{\frac{1}{2}} \xi)$ and whose spectrum doesn't contain zero in its convex hull. Letting P_N be projection onto the subspace of $L^2(\mathbb{R}^d)$ spanned by the Hermite functions of degree $\leq n$, trace Log $(P_n B P_n)$ admits a complete asymptotic expansion of the form (1). Moreover, the first two terms in this expansion are given by the formula (2) with Q equal to the harmonic oscillator.

2. Let B^d be the closed unit ball in $\mathbb{C}^d$ and let H^2 be the space of L^2 holomorphic functions on the interior of B^d. Given $f \in C^\infty(B^d)$ let T_f be the contraction to H^2 of the operator, multiplication by f, and let P_n be the orthogonal projection of H^2 onto the space spanned by the monomials

$$z_1^{n_1} \ldots z_d^{n_d}, \qquad n_1 + \cdots + n_d \leq n.$$

Then if f has an unambiguously defined logarithm, trace log $(P_n T_f P_n)$ admits a complete asymptotic expansion of the form (1).

Some generalizations of this result involving Toeplitz operators on strictly pseudoconvex domains [3] will be discussed in a future article.

3. The Zoll operator, Q, which figures in this paper can be an operator on sections of a vector bundle (providing its leading symbol is of the form, qI.)

References

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[3]See [BG]. In this reference its shown that the Zoll operators considered here are special examples of such Toeplitz opperators.